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How to Solve the Apportionment Paradox

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Really Good Theorems Are Those That End Their Life as Definitions: Why

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Abstract It is known that often, after it is proven that a new statement is equivalent to the original definition, this new statement becomes the accepted new definition of the same notion. In this paper, we provide a natural explanation for this empirical phenomenon.

1 Formulation of the Problem

Empirical fact. A recent book [1] cites a statement that is widely believed by mathematicians: that really good theorems are those that end their life as definitions.

This is indeed an empirical fact with which many mathematicians are familiar – often:

- after an interesting and useful theorem is proven that provides an equivalent condition to the original definition,
- this equivalent statement eventually becomes a new definition.

A natural why-question. A natural question is: why is this phenomenon really ubiquitous?

What we do in this paper. In this paper, we provide a natural explanation for this natural phenomenon.

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2 Analysis of the Problem and the Resulting Explanation

Which of the equivalent statements should we select? For many mathematical notions – be it continuity, compactness, etc. – there are several different properties equivalent to this notion. We need to select one of these properties as the definition, then all other properties become theorems.

Which of the equivalent formulations should we select?

What do we want from a definition: a natural idea. The main objective of mathematics is to prove theorems. From this viewpoint, the natural role of a definition is to help prove theorems about the corresponding notion.

Thus, we should select a definition which, on average, makes it easier to derive different properties of the corresponding notion.

Let us describe this idea in precise terms. Let $A_1,\ldots,A_n$ denote all the known results about the corresponding notion. For each of possible definitions D, we can describe the complexity of deriving the property A_i – as measured, e.g., by the length of the proof – by $c(D \to A_i)$.

This is the complexity of an individual derivation. To combine these individual complexity into a meaning average, we need to take into account that different statements A_i may have different importance:

- Some of these statement A_i are important, they appear in many different areas.
- On the other hand, some results A_i are obscure, important for maybe a single result about some unusual and rarely appearing object.

To take this difference into account, we need to assign, to each statement A_i, a number w_i describing, e.g., the relative frequency with which this statement is used in mathematical literature. Then, average complexity $c(D)$ of selecting the statement D as the definition can be obtained if we add up all individual complexities multiplied by these weights w_i:

$$c(D) = \sum_{i=1}^{n} w_i \cdot c(D \to A_i). \tag{1}$$

What does it mean that a theorem is good? As we have mentioned, often, there are many equivalent statements describing the same notion, i.e., there are several statements D', D'', ..., which are all equivalent to D:

$$D \leftrightarrow D' \leftrightarrow D'' \ldots$$

What do we mean when we say that a theorem proving the equivalence $D \leftrightarrow D'$ is a good theorem? This usually means that once we know that D' is equivalent to D, it makes it easier to derive several different results A_i.

What does it mean that a theorem is very good? Similarly, when we say that the equivalence $D \leftrightarrow D'$ is a very good theorem, this means that, once we know that D' is equivalent to D, it makes it easier to derive many different results A_i.

So when does a very good theorem become a new definition? So when does the new equivalent formulation D' become a new definition? This happens when the average length of the derivations from the new definition D' become smaller than the average length of deriving statements from the original definition D, i.e., when

$$\sum_{i=1}^{n} w_i \cdot c(D' \to A_i) < \sum_{i=1}^{n} w_i \cdot c(D \to A_i). \tag{2}$$

When does this happen?

- If D' corresponds to a very good theorem, this means that we have a large number of statement A_i for which

  $$c(D' \to A_i) < c(D \to A_i).$$

  Let us denote the set of the indices i of all these statements by G.
- For all other statements A_j, there is no direct and easier derivation from the new definition D'. For such statements A_j, the change from D to D' will actually increase the length of the derivation, since now, to prove these statements A_j, we will first need to prove D. So, the complexity of the new derivation will now increase from the original value $c(D \to A_j)$ to the new value

  $$c(D' \to A_j) = c(D' \to D) + c(D \to A_j).$$

The inequality (2) will be satisfied if the decrease in complexity of deriving statements A_i with $i \in G$ will compensate the increase in complexity caused by the need to add the length $c(D' \to D)$ to the derivation of all other statements, i.e., if

$$\sum_{i \in G} w_i \cdot (c(D \to A_i) - c(D' \to A_i)) > c(D' \to D) \cdot \sum_{j \notin G} w_j. \tag{3}$$

Clearly, when the set G becomes sufficiently large, the left-hand side of the formula (3) indeed becomes larger than the right0hand side, and thus, it becomes reasonable to select D' as the new definition. This explains the phenomenon described in [1].

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