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# Commonsense-Continuous Dynamical Systems -- Stationary States, Prediction, and Reconstruction of the Past: Fuzzy-Based Analysis

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# Commonsense-Continuous Dynamical Systems – Stationary States, Prediction, and Reconstruction of the Past: Fuzzy-Based Analysis

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**Abstract** Traditional analysis of dynamical systems usually assumes that the mapping is continuous – in precise mathematical sense. However, as many formal definitions, the mathematical definition of continuity does not always adequately capture the commonsense notion of continuity: that small changes in the input should lead to small changes in the output. In this paper, we provide a natural fuzzy-based formalization of this intuitive notion, and analyze how the requirement of commonsense continuity affects the properties of dynamical systems. Specifically, we show that for such systems, the set of fixed points is closed and convex, and that the only such systems for which we can both effectively predict the future and effectively reconstruct the past are linear systems.

## 1 Formulation of the Problem

Many real-life processes are described by *dynamical systems*, in which the state $s(t+1)$ in the next moment time is uniquely determined by the state $s(t)$ at the current moment of time: $s(t+1) = f(s(t))$ for some continuous function $f(s)$.

For a given dynamical system, it is usually important to describe all its stationary states, i.e., states with the following property: once the system reaches such a state, it remains in this state. Applications of stationary states range from self-sustaining ecological systems to stationary satellites that provide reliable communications.

In general, the set if all stationary states can be complicated, difficult to describe and even more difficult to compute. However, in many such examples:

- while from the formal mathematical viewpoint, these dynamical systems are continuous,

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- from the commonsense viewpoint, the corresponding transformation $f(s)$ is not what we would call continuous: it transforms close states $s$ and $s'$ into states $f(s)$ and $f(s')$ which no longer that close.

So, a natural hypothesis is that for dynamical systems which are continuous in the commonsense meaning of this word, we should have a simpler description of stationary states.

In this paper, we show that this hypothesis is indeed true – by proving the corresponding theorem. We also analyze how the property of commonsense continuity affect the possibility to predict the future and to reconstruct the past.

## 2 Mathematical Continuity vs. Commonsense Continuity: Analysis of the Difference

**The difference.** Most of us are so accustomed to the formal mathematical definition of continuity that we tend to forget that – as many formal definitions – it does not fully capture the commonsense idea of continuity. To many of us, the difference between mathematical and commonsense continuity surfaces only when we try to communicate with folks who are not mathematically trained.

This difference can be illustrated on a very simple example. Suppose that we have a function $f(x)$ which is:

- equal to 0 for $x \le 0$, equal to 1 for $x \ge \varepsilon$ for some small $\varepsilon > 0$, and
- equal to $x/\varepsilon$ for values $x$ between 0 and $\varepsilon$.

From the mathematical viewpoint, no matter how small $\varepsilon$ is, this function is continuous. However, from the commonsense viewpoint, when $\varepsilon$ becomes small, the function clearly becomes discontinuous – if we draw its graph, we will see an abrupt transition.

How can we describe the difference between these two notions in general terms? We know the mathematical definition of continuity, but how can we describe the commonsense continuity?

**What is commonsense continuity.** From the commonsense viewpoint, continuity of a function $f(x)$ means that if $a$ is close to $b$, then $f(a)$ is close to $f(b)$. This is not a formal definition: instead of using a precise description in terms of the distance $d(a,b)$, it uses an imprecise word "close".

A natural way to describe such imprecise ("fuzzy") words in precise terms is to use *fuzzy logic* – technique that was designed exactly for such a transition; see, e.g., [1, 2, 3, 4, 5, 6]. In this technique, each imprecise notion – in particular, the notion of being close – is described by a function $\mu(d)$ that assigns, to each value $d = d(a,b)$ of the distance, the degree $\mu(d(a,b))$ to which the states $a$ and $b$ are close.

The farther away from each other are the states, the less close they are, so the function $\mu(d)$ should be strictly decreasing.

In general, implication "if $A$ then $B$" means that $B$ is true whether $A$ is true – and maybe in other cases as well. Thus, our degree of confidence in $B$ should be larger than our degree of confidence in $A$.

In particular, the statement "if $a$ and $b$ are close, then $f(a)$ and $f(b)$ should also be close" implies that our degree of confidence $\mu(d(f(a),f(b)))$ (that $f(a)$ and $f(b)$ are close) should be larger than or equal to our degree of confidence $\mu(d(a,b))$ that $a$ and $b$ are close: $\mu(d(f(a),f(b))) \geq \mu(d(a,b))$. Since the function $\mu(d)$ is strictly decreasing, this inequality is equivalent to

$$d(f(a),f(b)) \leq d(a,b). \tag{1}$$

This inequality – called 1-Lipschitz property in mathematics – is thus an appropriate description of commonsense continuity.

**Definition 1.** *By a* commonsense-continuous dynamical system*, we mean a mapping* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *for which, for all* $a,b \in \mathbb{R}^n$*, we have* $d(f(a),f(b)) \leq d(a,b)$.

*Comment.* It is easy to check:

- that every commonsense-continuous function is also continuous in the usual mathematical sense, and
- that a composition of two commonsense-continuous functions is also commonsense-continuous.

## 3 Main Result

**Definition 2.** *We say that the point* $a \in \mathbb{R}^n$ *is a* fixed point *of the mapping* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *if* $f(a) = a$.

**Proposition 1.** *For each commonsense-continuous dynamical system, its set of fixed points is a closed convex set.*

**Proof.** Closeness is easy to prove: if we have a sequence of fixed points $a_n$ for which $f(a_n) = a_n$ and $a_n \to a$, then, due to mathematical continuity of the mapping $f$, we also have $f(a) = a$. So, the limit of fixed points is also a fixed point.

To complete the proof, it is thus sufficient to prove convexity. Let $a$ and $b$ be fixed points, i.e., $f(a) = a$ and $f(b) = b$, and let $c = \alpha \cdot a + (1-\alpha) \cdot b$ for some $\alpha \in (0,1)$. In this case, the points $a$, $b$, and $c$ lie on the same line, with $c$ being in between $a$ and $b$, so

$$d(a,b) = d(a,c) + d(b,c).$$

Let us prove that $f(c) = c$.

Indeed, by the definition of commonsense continuity, we have

$$d(f(a),f(c)) \leq d(a,c).$$

Since $f(a) = a$, this means

$$d(a, f(c)) \le d(a, c). \tag{1}$$

Similarly, we get

$$d(f(c), f(b)) = d(f(c), b) \le d(c, b). \tag{2}$$

By adding up the inequalities (1) and (2), we conclude that

$$d(a, f(c)) + d(f(c), b) \le d(a, c) + d(c, b) = d(a, b).$$

On the other hand, due to the triangle inequality, we have

$$d(a, f(c)) + d(f(c), b) \ge d(a, b),$$

thus

$$d(a, f(c)) + d(f(c), b) = d(a, b). \tag{3}$$

So, $f(c)$ lies on the line segment connecting $a$ and $b$.

We know that $d(a, f(c)) \le d(a, c)$. We cannot have $d(a, f(c)) < d(a, c)$, since then, by adding (2) to this inequality, we would have

$$d(a, f(c)) + f(f(c), b) < d(a, c) + d(c, b) = d(a, b),$$

and we know, from the formula (3), that the sum in the left-hand side is equal to $d(a, b)$. Thus, we must have $d(a, f(c)) = d(a, c)$. So, both points $c$ and $f(c)$ lie on the same line segment connecting $a$ and $b$, at the exact same distance from $a$. Hence, indeed $f(c) = c$.

The proposition is proven.

## 4 Auxiliary Result: Predicting the Future and Reconstructing the Past

**Predicting and reconstructing the past.** By definition of a dynamical system, for each current state $a$, the state at the next moment of time is $f(a)$. Similarly, the state at the previous moment of time is equal to $f^{-1}(a)$, where $f^{-1}$ denotes the inverse function: $f^{-1}(a) = b$ if and only if $f(b) = a$.

**What does effective prediction mean.** Effective prediction of the future means that if we know the current state approximately, i.e., we only know the state $\widetilde{a}$ which is close to the actual state $a$, then the state $f(\widetilde{a})$ that we predict based on this approximate knowledge should be close to the actual future state $f(a)$ – the state that we would have predicted if we had the full knowledge of the current state $a$. In other words, effective prediction of the future means that the mapping $f$ from the current state to the next state must be commonsense continuous.

If the function $f(a)$ predicting the next state is commonsense-continuous, then predicting the next-to-next state – and any future state – is also commonsense-continuous, since the corresponding prediction functions $f(f(a))$, $f(f(f(a)))$, $\ldots$, are compositions of commonsense-continuous functions.

Thus, we arrive at the following definition.

**Definition 3.** *We say that a mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^n$* allows effective prediction *if this mapping is commonsense-continuous.*

**What does effective reconstruction of the past mean.** Effective reconstruction of the past means that if we know the current state approximately, i.e., we only know the state $\widetilde{a}$ which is close to the actual state $a$, then the past state $f^{-1}(\widetilde{a})$ that we reconstruct based on this approximate knowledge should be close to the actual past state $f^{-1}(a)$ – the state that we would have reconstructed if we had the full knowledge of the current state $a$. In other words, effective reconstruction of the past means that the mapping $f^{-1}$ from the current state to the previous state must be commonsense continuous.

If the function $f^{-1}(a)$ reconstructing the previous state is commonsense-continuous, then predicting the previous-to-previous state – and any past state – is also commonsense-continuous, since the corresponding reconstruction functions $f^{-1}\left(f^{-1}(a)\right)$, $f^{-1}\left(f^{-1}\left(f^{-1}(a)\right)\right)$, $\ldots$, are compositions of commonsense-continuous functions.

Thus, we arrive at the following definition.

**Definition 4.** *We say that a mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^n$* allows effective reconstruction of the past *if its inverse $f^{-1}$ is commonsense-continuous.*

**Result.** In the ideal world, we would like to be able both to effectively predict and to effectively reconstruct the past. Unfortunately, as the following simple result shows, for a nonlinear system, we cannot do both.

**Proposition 2.** *If a mapping $f$ allows both effective prediction and effective reconstruction of the past, then this mapping is linear.*

**Proof.** If both $f$ and $f^{-1}$ are commonsense-continuous, this means that we have

$$d(f(a), f(b)) \leq d(a, b) \tag{4}$$

and $d(f^{-1}(a), f^{-1}(b)) \leq d(a,b)$ for all $a$ and $b$. For $a' = f^{-1}(a)$ and $b' = f^{-1}(b)$, we have $a = f(a')$ and $b = f(b')$, thus the second inequality takes the form

$$d(a', b') \leq d(f(a'), f(b')) \tag{5}$$

for all $a'$ and $b'$. From (4) and (5), we conclude that $d(f(a), f(b)) = d(a, b)$ for all $a$ and $b$, i.e., that the mapping $f$ preserves distance. It is known that all such mappings are linear.

The proposition is proven.

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