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How to Describe Relative Approximation Error? A New Justification for Gustafson's Logarithmic Expression

Martine Ceberio
The University of Texas at El Paso, mceberio@utep.edu

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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Martine Ceberio, Olga Kosheleva, and Vladik Kreinovich

Abstract How can we describe relative approximation error? When the value b approximate a value a, the usual description of this error is the ratio $|b-a|/|a|$. The problem with this approach is that, contrary to our intuition, we get different numbers gauging how well a approximates b and how well b approximates a. To avoid this problem, John Gustafson proposed to use the logarithmic measure $|\ln(b/a)|$. In this paper, we show that this is, in effect, the only regular scale-invariant way to describe the relative approximation error.

1 Formulation of the Problem

It is desirable to describe relative approximation error. If we use a value a to approximate a value b, then the natural number of accuracy of this approximation is the absolute value $|a-b|$ of the difference between these two values. This quantity is known as the *absolute approximation error*.

In situations when both a and b represent values of some physical quantity, the absolute error changes when we replace the original measuring unit with the one which is $\lambda > 0$ times smaller. After this replacements, the numerical values describing the corresponding quantities get multiplied by λ: $a \mapsto a' = \lambda \cdot a$ and $b \mapsto b' = \lambda \cdot b$; for example, if we replace meters by centimeters, 1.7 m becomes $100 \cdot 1.7 = 170$ cm. In this case, the numerical value of the absolute approximation error also gets multiplied by λ :

$$|a' - b'| = |\lambda \cdot a - \lambda \cdot b| = \lambda \cdot |a - b|.$$

Martine Ceberio, Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA
e-mail: mceberio@utep.edu, olgak@utep.edu, vladik@utep.edu

However, it is sometimes desirable to provide a measure of approximation error that would not depend on the choice of the measuring unit. Such measures are known as *relative approximation error*.

Traditional description of relative approximation error and its limitations. Usually, the relative approximation errors is described by the ratio

$$\frac{|a-b|}{b}; \tag{1}$$

see, e.g., [3].

The problem with this measure is that intuitively, the value a approximates the value b with exactly the same accuracy as b approximates a. However, with the measure (1), this is not true. For example, 0.8 approximates 1 with relative accuracy

$$\frac{|0.8-1|}{1} = 0.2,$$

while 1 approximates 0.8 with relative accuracy

$$\frac{|1-0.8|}{0.8} = 0.25 \neq 0.2.$$

Logarithmic measure of relative accuracy. To avoid the above-described asymmetry, John Gustafson [2] proposed to use the following alternative expression for relative approximation accuracy

$$|\ln(a/b)|. \tag{2}$$

One can easily check that this expression is indeed symmetric: $|\ln(a/b)| = |\ln(b/a)|$.

Natural question. There can be several different symmetric measures, why logarithmic one?

What we do in this paper. In this paper, we provide a natural explanation for selecting the logarithmic measure.

2 Why Logarithmic Measure: An Explanation

What we want. What we want is, in effect, a metric on the set $\mathbb{R}_+$ of all positive real numbers, i.e., a function $d(a,b) \geq 0$ for which:

- $d(a,b) = 0$ if and only if $a = b$;
- $d(a,b) = d(b,a)$ for all a and b, and
- $d(a,c) \leq d(a,b) + d(b,c)$ for all a, b, and c.

We want this metric to be *scale-invariant* in the following precise sense:

Definition 1. *We say that a metric $d(a,b)$ on the set of all positive real numbers is* scale-invariant *if*

$$d(\lambda \cdot a, \lambda \cdot b) = d(a,b) \tag{3}$$

for all $\lambda > 0$, $a > 0$, and $b > 0$.

It is also reasonable to require that the desired metric – just like the usual Euclidean metric – is uniquely generated by its local properties, in the sense that the distance between every two points is equal to the length of the shortest path connecting these points:

Definition 2. *Let M be a metric space with metric $d(a,b)$.*

- *By a* path *s from a point $a \in M$ to a point $b \in M$, we mean continuous mapping $s : [0,1] \mapsto M$.*
- *We say that a path has* length *L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if we have a sequence $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = 1$ for which $t_{i+1} - t_i \le \delta$ for all i, then*

$$\left| \sum_{i=1}^{n-1} d(t_i, t_{i+1}) - L \right| \le \varepsilon.$$

- *We say that a metric $d(a,b)$ is* regular *if every two points $a, b \in M$ can be connected by a path of length $d(a,b)$.*

Proposition 1.

- *Every scale-invariant regular metric on the set of all positive real numbers has the form $d(a,b) = k \cdot |\ln(b/a)|$ for some $k > 0$.*
- *For every $k > 0$, the metric $d(a,b) = k \cdot |\ln(b/a)|$ is regular and scale-invariant.*

Comment. Thus, we have indeed justified the use of logarithmic metric.

Proof. It is easy to see that the metric $d(a,b) = k \cdot |\ln(b/a)|$ is regular and scale-invariant. Let us prove that, vice versa, every scale-invariant regular metric on the set of all positive numbers has this form.

Let us first note that on the shortest path, each point occurs only once: otherwise, if had a point c repeated twice, we could cut out the part of the path that connect the first and second occurrences of this point, and thus get an even shorter path.

On the set of all positive real line, the only path between two points $a < b$ that does not contain repetitions in a continuous monotonic mapping of the interval $[0,1]$ into the interval $[a,b]$. So, this is the shortest path.

On each shortest path between the point a and c for which $a < c$, for each intermediate point b, we have $d(a,c) = d(a,b) + d(b,c)$. Indeed, the sequence t_i contains a point t_j close to b, thus each sum $\sum d(t_i, t_{i+1})$ is close to the sum of two subsums – before this point and after this point. When $\varepsilon \to 0$, the first subsum – corresponding to the shortest path from a to b – tends to $d(a,b)$, and the second subsum tends to $d(b,c)$. Thus, in the limit, for all $a < b < c$, we indeed have

$$d(a,c) = d(a,b) + d(b,c). \quad (4)$$

By scale-invariance, for $\lambda = 1/a$, we have $d(a,b) = d(1,b/a)$. For all $x > 1$, let us denote $f(x) \stackrel{\text{def}}{=} d(1,x)$. In these terms, for $a < b$, we have $d(a,b) = f(b/a)$ and thus, the equality (4) takes the form $f(c/a) = f(b/a) + f(c/b)$. In particular, for each $x \geq 1$ and $y \geq 1$, we can take $a = 1$, $b = x$, and $c = x \cdot y$, and conclude that $f(x \cdot y) = f(x) + f(y)$. It is known (see, e.g., [1]) that the only non-negative non-zero solutions to this functional equation are $f(x) = k \cdot \ln(x)$ for some $k > 0$. Thus, indeed, for $a < b$, we have $d(a,b) = f(b/a) = k \cdot \ln(b/a)$. Since $d(a,b) = d(b,a)$, for $a > b$, we get $d(a,b) = d(b,a) = k \cdot \ln(a/b)$, i.e., exactly $d(a,b) = k \cdot |\ln(b/a)|$.

The proposition is proven.

3 Related Result

In the previous text, we considered situations in which we can select different measuring units. For some quantities, we can select different starting points: e.g., when we measure time, we can start from any moment of time. In such quantities, if we select a new starting point which is a_0 moments earlier, then the numerical value corresponding to the same moment of time is shifted from a to $a \mapsto a' = a + a_0$. In sich cases, it is reasonable to consider shift-invariant metrics:

Definition 3. *We say that a metric $d(a,b)$ on the set of all real numbers is* shift-invariant *if*

$$d(a + a_0, b_0) = d(a,b) \quad (5)$$

for all a, b, and a_0.

Proposition 2.

- *Every shift-invariant regular metric on the set of all real numbers has the form $d(a,b) = k \cdot |a - b|$ for some $k > 0$.*
- *For every $k > 0$, the metric $d(a,b) = k \cdot |a - b|$ is regular and shift-invariant.*

Comment. So, in this case, we get the usual description of the absolute approximation error.

Proof. It is easy to see that the metric $d(a,b) = k \cdot |a - b|$ is regular and shift-invariant. Let us prove that, vice versa, every shift-invariant regular metric on the set of all real numbers has this form.

Similarly to the proof of Proposition 1, we conclude that for all $a < b < c$, the equality (4) is satisfied.

By shift-invariance, for $a_0 = -b$, we have $d(a,b) = d(0, b - a)$. For all $x > 0$, let us denote $g(x) \stackrel{\text{def}}{=} d(0,x)$. In these terms, for $a < b$, we have $d(a,b) = g(b - a)$ and thus, the equality (4) takes the form $g(c - a) = g(b - a) + g(c - b)$. In particular, for each $x \geq 1$ and $y \geq 1$, we can take $a = 0$, $b = x$, and $c = x + y$, and conclude that $g(x+y) = g(x) + g(y)$. It is known (see, e.g., [1]) that the only non-negative non-zero solutions to this functional equation are $g(x) = k \cdot x$ for some $k > 0$. Thus, indeed, for $a < b$, we have $d(a,b) = g(b/a) = k \cdot (b - a)$. Since $d(a,b) = d(b,a)$, for $a > b$, we get $d(a,b) = d(b,a) = k \cdot (b - a)$, i.e., exactly $d(a,b) = k \cdot |a - b|$.

The proposition is proven.

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