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Olga Kosheleva

*The University of Texas at El Paso*, [olgak@utep.edu](mailto:olgak@utep.edu)

Vladik Kreinovich

*The University of Texas at El Paso*, [vladik@utep.edu](mailto:vladik@utep.edu)

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## MACROCAUSALITY IMPLIES LORENZ GROUP: A PHYSICS-RELATED COMMENT ON GUTS'S RESULTS

**Olga Kosheleva**

Ph.D. (Phys.-Math.), Associate Professor, e-mail: olgak@utep.edu

**Vladik Kreinovich**

Ph.D. (Phys.-Math.), Professor, e-mail: vladik@utep.edu

University of Texas at El Paso, El Paso, Texas 79968, USA

**Abstract.** It is known that, in the space-time of Special Relativity, causality implies Lorenz group, i.e., if we know which events can causally influence each other, then, based on this information, we can uniquely reconstruct the affine structure of space-time. When the two events are very close, quantum effects, with their probabilistic nature, make it difficult to detect causality. So, the following question naturally arises: can we uniquely reconstruct the affine structure if we only know causality for events which are sufficiently far away from each other? Several positive answers to this question were provided in a recent paper by Alexander Guts. In this paper, we describe a very simple answer to this same question.

**Keywords:** causality, special relativity, Alexandrov-Zeeman theorem.

### 1. Introduction

**Causality in Special Relativity reminder.** According to Special Relativity Theory, an event  $a = (t, x_1, x_2, x_3)$  can causally influence an event  $b = (s, y_1, y_2, y_3)$  (we will denote it by  $a \leq b$ ) if and only if a signal originated in  $a$  can reach  $b$  while traveling with a speed not exceeding the speed of light  $c$ , i.e., if

$$s - t \geq c \cdot \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}. \quad (1)$$

**Causality implies Lorenz group: main result.** The famous result by A. D. Alexandrov (see, e.g., [1, 2, 4]) shows that any bijection  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  of the 4-D space-time that preserves causality – i.e., for which  $a \leq b$  if and only if  $f(a) \leq f(b)$  – is a composition of rotations, shifts, scalings  $a \cdot \lambda \cdot a$ , Lorenz transformations, and, if needed, spatial inversion  $(t, x_1, x_2, x_3) \rightarrow (t, -x_1, -x_2, -x_3)$ .

**Causality implies Lorenz group: physical consequences.** Thus, if we know, for every two events  $a$  and  $b$ , whether  $a$  can causally influence  $b$ , then, based on this information, we can uniquely reconstruct the linear (= affine) structure on the 4-D space-time.

**Quantum effects lead to an additional complexity.** A usual way to test a causal relation between the events is to make some change in  $a$  and observe the resulting effect in  $b$ . This can be done in a deterministic situation. However, when the events  $a$  and  $b$  are very close to each other, with microworld-size differences between their coordinates, then quantum effects prevail, the relation becomes probabilistic, and detecting causality becomes difficult. As a result, we can only observe causal relation when the event  $b$  is sufficiently far away from the event  $a$ .

**Related natural question and Guts’s answers.** Can we still reconstruct the linear structure of space-time based on this observable (macro) causality?

This question was studied in a recent paper [3] by Alexander Guts that provides many positive answers to this question.

**What we do in this paper.** The main objective of this paper is to provide yet another positive answer – an answer that probably follows from Guts’s results, but that is so simple that we believe it is worth describing.

## 2. Analysis of the Problem and the Main Result

**Let us be as general as possible – without abandoning simplicity.** While our main interest is in the causal relation (1), we follow a natural mathematical tendency of formulating this result in the most general form – as long as this desire for generality does not make things more complicated.

So, instead of relation (1), let us consider any relation  $a \leq b$  that is described by a closed convex cone  $F$ :  $a \leq b$  if and only if  $b - a \in F$ .

**Natural requirement on macrocausality relation.** In addition to the “theoretical” causality relation  $a \leq b$ , we assume that there is also an additional observable (= macro) causality relation  $a \ll b$ . Of course, when we can observe that  $a$  causally influence  $b$ , this means that  $a \leq b$  in the theoretical sense as well, i.e., that  $a \ll b$  implies  $a \leq b$ .

If  $a$  observably influences  $b$  ( $a \ll b$ ), this means that  $b$  is sufficiently far away from  $a$ . Thus, if  $b \leq c$ , this means that  $c$  is even further from  $a$  than  $b$  – so we should also be able to detect that  $a$  influences  $c$  as well. In other words, if  $a \ll b$  and  $b \leq c$ , then  $a \ll c$ .

Finally, since we are considering a homogeneous space-time, it is reasonable to require that the macrocausality relation  $\ll$  is shift-invariant, i.e., that  $a \ll b$  implies  $a + c \ll b + c$ .

Now, we are ready to formulate our result.

*Comment.* To make this paper understandable to as many readers as possible, we add as many definitions as needed – even though they are most probably familiar to many readers.

**Definition 1.** Let  $R$  be a binary relation on a set  $S$ . We say that a bijection  $f : S \mapsto S$  preserves the relation  $R$  if for every  $a, b \in S$ , we have  $aRb$  if and only if  $f(a)Rf(b)$ .

**Definition 2.** A set  $S \subseteq \mathbb{R}^n$  is called a convex cone if for every two elements  $s_1, s_2 \in S$  and for every two non-negative real numbers  $c_1$  and  $c_2$ , the element  $c_1 \cdot s_1 + c_2 \cdot s_2$  also belongs to  $S$ .

**Proposition.** Let  $F \subseteq \mathbb{R}^n$  be a convex cone which is a closed set, let  $a \leq b$  mean  $b - a \in F$ , and let  $\ll$  be a binary relation on  $\mathbb{R}^n$  that is satisfied by at least one pair  $(a_0, b_0)$  and that satisfies the following properties for all  $a, b$ , and  $c$ :

1. if  $a \ll b$ , then  $a \leq b$ ;
2. if  $a \ll b$  and  $b \leq c$ , then  $a \ll c$ ; and
3. if  $a \ll b$ , then  $a + c \ll b + c$ .

Then every bijection that preserves  $\ll$  also preserve  $\leq$ .

**Conclusion.** For the case when  $\leq$  is the Special Relativity causality relation (1) and  $\ll$  is macrocausality, this result means that every bijection that preserves macrocausality is a composition of rotations, shifts, scalings, Lorentz transformations, and, if needed, spatial inversion.

Thus, even if we only know observable causality, we can still uniquely reconstruct linear structure on  $\mathbb{R}^n$ .

**Proof.** To prove the Proposition, let us show that the relation  $\leq$  can be described in terms of  $\ll$ , namely, that for all  $b$  and  $c$ :

$$b \leq c \Leftrightarrow \forall a (a \ll b \Rightarrow a \ll c). \quad (2)$$

This will imply that any bijection that preserved  $\ll$  preserves  $\leq$  as well.

Indeed, the left-to-right part of (2) follows from the second property listed in the Proposition. So, to complete the proof, we need to prove the right-to-left implication. Indeed, let us assume that

$$\forall a (a \ll b \Rightarrow a \ll c). \quad (3)$$

Due to the third property (shift-invariance), we have  $a \ll b$  if and only if  $0 \ll b - a$ , i.e., if and only if  $b - a \in M \stackrel{\text{def}}{=} \{c : 0 \ll c\}$ .

We know that  $a_0 \ll b_0$ .

- Thus, due to shift-invariance, we have  $a_0 + (b - b_0) \ll b$ , and therefore, due to (3), we have

$$a_0 + (b - b_0) \ll c.$$

- Then, due to shift-invariance, we have  $a_0 + (b - b_0) + (b - c) \leq b$  and therefore, due to (3), we have

$$a_0 + (b - b_0) + (b - c) \ll c.$$

- Then, due to shift-invariance, we have  $a_0 + (b - b_0) + 2(b - c) \leq b$  and therefore, due to (3), we have

$$a_0 + (b - b_0) + 2(b - c) \ll c,$$

etc.

By induction, we can prove that for every natural number  $m$ , we have

$$a_0 + (b - b_0) + m \cdot (b - c) \ll c.$$

Due to the first property, this implies that

$$a_0 + (b - b_0) + m \cdot (b - c) \leq c.$$

By definition of the relation  $\leq$ , this means that

$$c - (a_0 + (b - b_0) + m \cdot (b - c)) = (m + 1) \cdot (c - b) + (b_0 - a_0) \in F.$$

Since  $F$  is a convex cone, we also have

$$\frac{1}{m + 1} \cdot ((m + 1) \cdot (c - b) + (b_0 - a_0)) = c - b + \frac{b_0 - a_0}{m + 1} \in F.$$

In the limit  $m \rightarrow \infty$ , these elements tend to  $c - b$ . Since  $F$  is a closed set, we thus have  $c - b \in F$ , which means that  $b \leq c$ .

The proposition is proven.

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