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How to elicit complex-valued fuzzy degrees

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Abstract In the traditional fuzzy logic, an expert's degree of certainty in a statement is described by a single number from the interval $[0,1]$. However, there are situations when a single number is not sufficient: e.g., a situation when we know nothing and a situation in which we have a lot of arguments for a given statement and an equal number of arguments against it are both described by the same number 0.5. Several techniques have been proposed to distinguish between such situations. The most widely used is interval-valued technique, where we allow the expert to describe his/her degree of certainty by a subinterval of the interval $[0,1]$. Eliciting an interval-valued degree is straightforward. On the other hand, in many practical application, another technique has been useful: the technique complex-valued fuzzy degrees. For this technique, there is no direct way to elicit such degrees. In this paper, we explain a reasonable natural approach to such an elicitation.

1 Formulation of the Problem

Need for fuzzy logic: a brief reminder. In many cases, people do not have 100% certainty in their statements. To describe such uncertainty in human reasoning, Lotfi Zadeh proposed to describe a person's degree of certainty by a number from the interval $[0,1]$, where:

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- 1 corresponds to absolute cerainty in a statement,
- 0 corresponds to absolute certainty in its negation, and
- intermediate values between 0 and 1 correspond to intermediate degrees of certainty.

This is one of the main ideas behind fuzzy logic; see, e.g., [3, 6, 10, 12, 13, 15].

Need to go beyond traditional fuzzy logic: reminder. It is known that a single degree does not always provide an adequate description of a person's uncertainty. An example is when we have two scenarios:

1. we do not know anything about a statement S, and
2. we have equal number of arguments for and against the statement S.

In both cases, we have the same number of reasons to believe in S and in its negation $\neg S$, so the natural idea is to assign 0.5 as the degree of certainty in the statement. However, these two cases are completely different.

A possible solution to this problem: interval-valued degrees. One way to distinguish these two scenarios is to allow the person to describe his/her degree of certainty not only by a number, but also by an interval of possible values; see, e.g., [10].

- In a situation when a person knows nothing about the statement, he/she does not have any reason to select one of the degrees, so a natural idea is to return all possible degrees – i.e., the whole interval $[0,1]$.
- In the second case, when a person has an equal number of arguments for and against, it is reasonable to describe the person's certainty by the midpoint 0.5 of the interval $[0,1]$.

Alternative solution: complex-valued degrees. An alternative approach – which also led to useful applications – is to use complex-valued degrees; see, e.g., [1, 4, 5, 7, 8, 9, 11, 14]. Many operations of fuzzy logic can be naturally extended to complex numbers z_i:

- we can extend the usual negation operation $f_\neg(a) = 1-a$ to $f_\neg(z) = 1-z$;
- we can extend one of the most widely used "and"-operations (t-norms) $f_\&(a_1,a_2) = a\cdot b$ to $f_\&(z_1,z_2) = z_2\cdot z_2$; and
- we can extend one of the most widely used "or"-operations (t-conorms) $f_\vee(a_1,a_2) = a_1+a_2-a_1\cdot a_2$ to $f_\&(z_1,z_2) = z_1+z_2-z_2\cdot z_2$.

Why not min? In the above description, we use the product "and"-operation $f_\&(a_1,a_2) = a_1\cdot a_2$. What about an even more popular min "and"-operation $\min(a_1,a_2)$? Minimum is not an analytical function and thus, it cannot be directly extended to complex numbers. However, we know that function $\min(a_1,a_2)$ can be represented a limit of analytical functions, e.g., as

$$\min(a_1,a_2) = \lim_{n\to\infty}(a_1^n + a_2^n)^{-1/n}.$$

So why not try a similar limit $\lim_{n\to\infty}(z_1^{-n}+z_2^{-n})^{-1/n}$. for complex values z_i?

This actually works when $|z_1| \neq |z_2|$: in this case, this limit is equal to the value z_i whose modulus is the smaller of the two. However, when $|z_1| = |z_2|$, this sequence, in general, has no limit. For example, when $z_1 = 1$ and $z_2 = -1$:

- for even n, we get $(z_1^{-n} + z_2^{-n})^{-1/n} = 1$, while
- for odd n, we get $(z_1^{-n} + z_2^{-n})^{-1/n} = \infty$.

But how can we elicit these complex-valued degrees? Eliciting interval-valued degrees is straightforward. If we ask people to estimate their degree of certainty in a given statement by an interval, most of them will easily give us a interval.

However, the same cannot be said for the case of complex numbers. The problem with using complex numbers is that people don't remember them, and, even if they do, don't have good intuition about complex numbers. As a result, eliciting complex-valued degrees is a challenge.

In this paper, we describe a natural approach for eliciting such degrees.

2 Analysis of the Problem

What is a natural real-valued analogue of a complex-valued degree?

- As we have mentioned, many people do not have a good intuition about complex-valued degrees z.
- On the other hand, most people can naturally describe their certainty by a real number a.

Thus, for each complex number z that describes a person's actual degree of certainty, there is a corresponding real number that this person will provide if asked about his/her level of certainty. Let us denote this corresponding real number by $a = f(z)$.

What are the natural properties of the corresponding function $f(z)$?

- First, if the complex-valued degree z is actually a real number from the interval $[0,1]$, then it is reasonable to assume that this same real number $z \in [0,1]$ will describe the person's real-valued degree of certainty, i.e., we should have $f(z) = z$.
- Second, small changes in z should not cause big changes in $f(z)$, i.e., the function $f(z)$ must be continuous.
- The third requirement concerns "and"-operations. If we have two statements S_1 and S_2 with complex-valued degrees z_1 and z_2, then we can estimate the real-valued degree of certainty in a combined statement $S_1 \,\&\, S_2$ in two different ways:
  - we can apply the "and"-operation and then transform the result $z_1 \cdot z_2$ into real numbers, thus returning $f(z_1 \cdot z_2)$;
  - alternatively, we can first transform both degree z_1 and z_2 into real numbers, resulting in the values $f(z_1)$ and $f(z_2)$, and then apply an "and"-operation that would produce $f(z_1) \cdot f(z_2)$.

It is reasonable to require that the resulting degree should not depend on how we estimated it, i.e., that we should have $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$.

It turns out that these properties uniquely determine the desired function $f(z)$.

Proposition 1. *For any continuous function from complex numbers to real numbers, the following two conditions are equivalent:*

1. *for every real number $z \in [0,1]$, we have $f(z) = z$, and for every two complex numbers z_1 and z_2, we have $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$;*
2. *we have $f(z) = |z|$, i.e., $f(z)$ is equal to the absolute value (modulus) of a complex number $z = a + b \cdot \mathrm{i}$: $|z| = \sqrt{a^2 + b^2}$.*

Proof.

1°. It is easy to see that the function $f(z) = |z|$ satisfies the first condition. So, to complete the proof, it is sufficient to prove that every function $f(z)$ that satisfies the first condition is equal to the modulus.

2°. Let us first prove that $f(z) = z$ for all non-negative real numbers z.

Indeed, for real numbers $z \in [0,1]$ this is postulated in the first condition. For a real number $z > 1$, the multiplicative property – which is part of the first condition – implies that $f(z) \cdot f(1/z) = f(1)$. Here, $1/z < 1$, so, due to the first condition, we have $f(1/z) = 1/z$ and $f(1) = 1$. Thus, the multiplicative property implies that $f(z) \cdot (1/z) = 1$. Multiplying both sides of this equality by z, we get the desired equality $f(z) = z$.

3°. It is known that every complex number z can be represented as

$$z = \rho \cdot \exp(\alpha \cdot \mathrm{i}),$$

where $\rho = |z|$ is a non-negative real number, and α is also a real number, which can be also chosen to be non-negative (actually, we can always choose $\alpha \in [0, 2\pi)$). Due to the multiplicative property, we have $f(z) = f(\rho) \cdot f(\exp(\alpha \cdot \mathrm{i}))$. From Part 2 of this proof, we know that $f(\rho) = \rho = |z|$. Thus, to complete our proof, it is sufficient to prove that $f(\exp(\alpha \cdot \mathrm{i})) = 1$ for all α.

We will prove it in three steps.

3.1°. First, for $\alpha = 2\pi \cdot (1/n)$, the product of n such numbers is equal to 1:

$$\exp\left(2\pi \cdot \frac{1}{n} \cdot \mathrm{i}\right) \cdot \ldots \cdot \exp\left(2\pi \cdot \frac{1}{n} \cdot \mathrm{i}\right) = \exp(2\pi \cdot \mathrm{i}) = 1.$$

Thus, the multiplicative property implies that

$$f\left(\exp\left(2\pi \cdot \frac{1}{n} \cdot \mathrm{i}\right)\right) \cdot \ldots \cdot f\left(\exp\left(2\pi \cdot \frac{1}{n} \cdot \mathrm{i}\right)\right) = f(1) = 1,$$

i.e.,

$$\left(f\left(\exp\left(2\pi\cdot\frac{1}{n}\cdot\mathrm{i}\right)\right)\right)^n=1.$$

Since all the values $f(z)$ are non-negative real numbers, we thus get

$$f\left(\exp\left(2\pi\cdot\frac{1}{n}\cdot\mathrm{i}\right)\right)=1.$$

3.2°. Let us now consider the case when $\alpha=2\pi\cdot(m/n)$ for some rational number m/n. In this case, we have

$$\exp\left(2\pi\cdot\frac{m}{n}\cdot\mathrm{i}\right)=\exp\left(2\pi\cdot\frac{1}{n}\cdot\mathrm{i}\right)\cdot\ldots\cdot\exp\left(2\pi\cdot\frac{1}{n}\cdot\mathrm{i}\right)\ (m\text{ times}).$$

Thus, the multiplicative property implies that

$$f\left(\exp\left(2\pi\cdot\frac{m}{n}\cdot\mathrm{i}\right)\right)=f\left(\exp\left(2\pi\cdot\frac{1}{n}\cdot\mathrm{i}\right)\right)\cdot\ldots\cdot f\left(\exp\left(2\pi\cdot\frac{1}{n}\cdot\mathrm{i}\right)\right).$$

According to Part 3.1 of this proof, all the values in the right-hand side are equal to 1, so we conclude that

$$f\left(\exp\left(2\pi\cdot\frac{m}{n}\cdot\mathrm{i}\right)\right)=1.$$

3.3°. Every non-negative real number α can be represented as $2\pi\cdot\beta$, where $\beta\stackrel{\text{def}}{=}\alpha/(2\pi)$. Each non-negative real number β can be represented as a limit of rational numbers β_k – e.g., numbers obtained if we take, in the decimal representation of the number β, the first k digits after the decimal point: $\beta_k\to\beta$. Due to Part 3.2 of this proof, for each k, we have $f(\exp(2\pi\cdot\beta_k\cdot\mathrm{i}))=1$. Since the function $f(z)$ is continuous, we can thus conclude that in the limit $\beta_k\to\beta$, we also have $f(\exp(2\pi\cdot\beta\cdot\mathrm{i}))=1$, i.e., $f(\exp(\alpha\cdot\mathrm{i}))=1$.

The statement of Part 3 is proven, and thus, the Proposition is proven too.

So how can we elicit complex-valued degrees: analysis of the problem. As we have mentioned, most people cannot provide a complex-valued degree z describing their degree of certainty, but what they can provide is the real value $f(z)=|z|$ corresponding to this degree.

What if we collect the real value $d_+\stackrel{\text{def}}{=}|z|$ corresponding to the original statement and the real value $d_-\stackrel{\text{def}}{=}|1-z|$ corresponding to the negation of this statement? For a general complex number $z=a+b\cdot\mathrm{i}$, we have $1-z=(1-a)-b\cdot\mathrm{i}$. Thus, we have

$$d_+{}^2=a^2+b^2 \tag{1}$$

and

$$d_-{}^2=(1-a)^2+b^2=1-2a+1^2+b^2. \tag{2}$$

Subtracting (2) from (1) we get:

$$d_+{}^2 - d_-{}^2 = 2a - 1,$$

thus

$$a = \frac{d_+{}^2 - d_-{}^2 + 1}{2}. \tag{3}$$

Substituting this value a into the formula (1), we get

$$b^2 = d_+^2 - \left(\frac{d_+^2 - d_-^2 + 1}{2}\right)^2 = d_+^2 - \frac{(d_+^2 - d_-^2)^2 + 2d_+^2 - 2d_-^2 + 1}{4} =$$

$$\frac{4d_+^2 - (d_+^2 - d_-^2)^2 - 2d_+^2 + 2d_-^2 - 1}{4} = \frac{2d_+^2 + 2d_-^2 - (d_+^2 - d_-^2)^2 - 1}{4}.$$

Thus,

$$b = \pm\frac{1}{2} \cdot \sqrt{2(d_+{}^2 + d_-{}^2) - (d_+{}^2 - d_-{}^2)^2 - 1} \tag{4}$$

Thus, we arrive at the following way to elicit complex-valued fuzzy degrees.

3 So How to Elicit Complex-Valued Fuzzy Degrees: Algorithm and Discussion

Algorithm. In order to estimate the person's complex-values degree of certainty in a given statement S, we ask a person to provide:

- a real-valued estimate d_+ of this person's certainty in the given statement S, and
- a real-valued estimate d_- of this person's certainty in its negation $\neg S$.

After that, we compute the values a and b by using formulas (3) and (4), and return the complex-valued degree $z = a + b \cdot \mathrm{i}$.

When is such reconstruction possible? If the values d_+ and d_- come for a complex-valued degree of certainty, then the above algorithm reconstructs the original complex-valued degree. But, realistically, complex-valued degrees are an approximate description of uncertainty – just like any other uncertainty formalism. So, a natural question is: if we have two numbers d_+ and d_- form the interval $[0,1]$, when will the above algorithm lead to a complex number?

In general, the triangle inequality implies that $|z| + |1 - z| \geq 1$. Thus, for the above algorithm to work, we need to have $d_+ + d_- \geq 1$. It turns out that this is the only condition: once this inequality holds, the above algorithm works.

Proposition 2. *For every two numbers $d_-, d_+ \in [0,1]$ for which $d_+ + d_- \geq 1$, there exists a complex number $z = a + b \cdot \mathrm{i}$ for which $|z| = d_+$, $|1 - z| = d_-$, and $0 \leq a \leq 1$.*

Proof.

1°. Let us first prove that when the condition $d_+ + d_- \geq 1$ is satisfied, then the value a described by the formula (3) is located between 0 and 1.

1.1°. Since $d_-^2 \geq 0$, from the formula (3), we conclude that

$$a \leq \frac{d_+^2 + 1}{2}.$$

Since $d_+^2 \leq 1$, we get

$$a \leq \frac{1+1}{2} = 1.$$

1.2°. Since $d_-^2 \leq 1$, we have $d_+^2 - d_-^2 + 1 \geq d_+^2 - 1 + 1 = d_+^2 \geq 0$, thus, by the formula (3), we have $a \geq 0$.

2°. Let us now prove that the above expression for b^2, namely, the difference

$$d_+^2 - \left(\frac{d_+^2 - d_-^2 + 1}{2}\right)^2,$$

is always non-negative, and thus, we can always extract a square root from this expression and get a real value b. Indeed, the non-negativity of the above difference is equivalent to

$$d_+^2 \geq \left(\frac{d_+^2 - d_-^2 + 1}{2}\right)^2.$$

On both sides of this inequality, we have squares of non-negative numbers. For non-negative values, the function x^2 is strictly increasing, so the desired inequality is equivalent to a similar inequality between the original numbers:

$$d_+ \geq \frac{d_+^2 - d_-^2 + 1}{2}.$$

Multiplying both sides of this inequality by 2, we get $2d_+ \geq d_+^2 - d_-^2 + 1$. Moving d_-^2 to the left-hand side and $2d_+$ to the right-hand side, we get an equivalent inequality $d_-^2 \geq d_+^2 - 2d_+ + 1$. The right-hand side of this inequality is a full square, so the desired inequality takes the form $d_-^2 \geq (1 - d_+)^2$. This inequality follows from the fact that $d_+ + d_- \geq 1$, thus $d_- \geq 1 - d_+ \geq 0$. Squaring both sides of this inequality, we indeed get $d_-^2 \geq (1 - d_+)^2$, and thus, we get the desired inequality

$$d_+^2 - \left(\frac{d_+^2 - d_-^2 + 1}{2}\right)^2 \geq 0.$$

The proposition is proven.

How is this related to interval-valued degrees? In this approach, the degrees d_+ and d_- of the statements S and $\neg S$ satisfy the inequality $d_+ + d_- \geq 1$. From this viewpoint, the values d_+ and d_- can be viewed as similar to upper endpoints of the intervals.

Indeed, if we estimate the degree of certainty in a statement S by an interval $[\underline{a},\overline{a}]$, then a natural interval-valued degree for $\neg S$ would be the interval formed by all the values $1-a$ when $a \in [\underline{a},\overline{a}]$, i.e., the interval $[1-\overline{a},1-\underline{a}]$. For the upper endpoints of these two intervals, we get $\overline{a}+(1-\underline{a}) = 1+(\overline{a}-\underline{a})$. Since $\overline{a} \geq \underline{a}$, this sum is always greater than or equal to 1.

In this case, $d_+ = \overline{a}$ and $d_- = 1-\underline{a}$. So, if we know d_+ and d_-, we can find $\overline{a} = d_+$ and $\underline{a} = 1-d_-$. Thus, a complex-valued degree z can be represented by an interval $[1-|1-z|,|z|]$.

Vice versa, if we know an interval-valued degree $[\underline{a},\overline{a}]$, then we can find $d_+ = \overline{a}$ and $d_- = 1-\underline{a}$, and, by applying the formulas (3) and (4), get the corresponding complex-valued degree $z = a+b\cdot \mathrm{i}$, where

$$a = \frac{\overline{a}^2-(1-\underline{a})^2+1}{2} = \frac{\overline{a}^2-1+2\underline{a}-\underline{a}^2}{2} = \underline{a}+\frac{\overline{a}^2-\underline{a}^2}{2}$$

and

$$b^2 = d_+^2 - a^2 = \overline{a}^2 - \left(\underline{a}+\frac{\overline{a}^2-\underline{a}^2}{2}\right)^2 =$$

$$\overline{a}^2 - \left(\underline{a}^2+\underline{a}\cdot(\overline{a}^2-\underline{a}^2)+\left(\frac{\overline{a}^2-\underline{a}^2}{2}\right)^2\right) =$$

$$(\overline{a}^2-\underline{a}^2)-\underline{a}\cdot(\overline{a}^2-\underline{a}^2)-\left(\frac{\overline{a}^2-\underline{a}^2}{2}\right)^2 =$$

$$(\overline{a}^2-\underline{a}^2)\cdot\left(1-\underline{a}-\frac{\overline{a}^2-\underline{a}^2}{4}\right).$$

Comment. If instead, we use lower endpoints as a description of interval-valued uncertainty, we will get $d_+ + d_- \leq 1$, i.e., in effect, the case of intuitionistic fuzzy degrees; see, e.g., [2].

Example.

- The above argument leads to a conclusion that in the scenario when we know nothing about a statement, we should take $d_+ = d_- = 1$. In this case, the formula (3) implies that $a = 0.5$, and the formula (4) implies that $b = \pm 0.5$. Thus, this scenario corresponds to the complex-valued degree

$$z = 0.5 \pm \frac{\sqrt{3}}{2}\cdot \mathrm{i}.$$

- On the other hand, in the second scenario, when we have $d_+ = d_- = 0.5$, we get $z = 0.5$ – a different complex-valued degree.

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