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# Commonsense "And"-Operations

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# Commonsense “And”-Operations

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**Abstract** In many practical situations, we need to estimate our degree of belief in a statement $A \& B$ when the only thing we know are the degrees of belief $a$ and $b$ in combined statements $A$ and $B$. An algorithm for this estimation is known as an “and”-operation, or, for historical reasons, a t-norm. Usually, “and”-operations are selected in such a way that if one of the statements $A$ or $B$ is false, our degree of belief in $A \& B$ is 0. However, in practice, this is sometimes not the case: for example, an ideal faculty candidate must satisfy many properties – be a great teacher, *and* be a wonderful researcher, *and* be a great mentor, etc. – but if one of these requirements is not satisfied, this candidate may still be hired. In this paper, we show how to describe the corresponding commonsense “or”-operations.

## 1 Why “and”-operations

In many practical applications, a certain effect appears if several conditions $C_1, C_2, \ldots$ are satisfied. For each of these conditions $C_i$, we can elicit, from the experts, the degree $d_i \in [0,1]$ to which this condition is satisfied.

However, there are many possible conditions. It is not possible to extract, from the experts, a degree to which each possible “and”-combination $C_1 \& C_2 \& \ldots$ is satisfied. Thus, we need to be able:

- given degrees of confidence $a$ and $b$ in statements $A$ and $B$,
- to estimate the degree to which the “and”-combination $A \& B$ is satisfied.

This estimate is usually denoted by $f_\&(a,b)$. The algorithm for computing this estimate is known as an *“and”-operation* or, for historical reason, a *t-norm*.

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## 2 How usual "and"-operations are obtained

In some situations, about each of the combined statements, we are absolute certain either that this statement is true, or that this statement is false. Then, the "and"-operation should return the true value of the corresponding "and"-statement. So we should have $f_\&(0,0) = f_\&(0,1) = f_\&(1,0) = 0$ and $f_\&(1,1) = 1$.

We want to extend these values to all possible combinations of $a \in [0,1]$ and $b \in [0,1]$. A reasonable idea is to use linear interpolation over each variable (see, e.g., [3]), i.e., to assume that:

- for every $a$, the mapping $b \mapsto f_\&(a,b)$ is linear, and
- for every $b$, the mapping $a \mapsto f_\&(a,b)$ is linear.

As a result, we conclude that the desired function is bilinear, i.e., that it has the form

$$f_\&(a,b) = c_0 + c_a \cdot a + c_b \cdot b + c_{ab} \cdot a \cdot b$$

for some coefficients $c_i$.

Taking into account the above conditions for $a,b \in \{0,1\}$, we conclude that $f_\&(a,b) = a \cdot b$. This is indeed one of the most frequently used "and"-operations; [1, 2, 4, 5, 6, 7].

Similarly, linear interpolation enables us to similarly determine that an appropriate "or"-operation (historically also known as t-conorm) has the form

$$f_\vee(a,b) = a + b - a \cdot b.$$

## 3 Need to go beyond the usual "and"-operations

In some cases, when we say "and", we mean exactly the logical "and": all conditions must be absolutely satisfied.

However, in many practical problems, "and" is "softer" than that. For example, if you ask a person who is planning to buy a house what house he/she wants, the person will say:

- not too far away
- *and* spacey
- *and* not very expensive
- *and* reasonably well thermo-isolated
- *and* in a nice neighborhood, etc.

However, this "and" does not mean literal "and". If this person finds a house that satisfied most of these conditions, he/she will gladly buy it.

How can we describe such commonsense "and"-operations?

## 4 Our solution

In this paper, we consider the case when we only have two conditions. For a commonsense “and”-operation $F_\&(a,b)$, it is reasonable to still have $F_\&(0,0)=0$ and $F_\&(1,1)=1$. However:

- if only one of the conditions $A$ and $B$ is satisfied,
- then the statement $A\,\&\,B$ should also be to some extent true.

In other words, we should have $F_\&(0,1)=F_\&(1,0)=\alpha$ for some small $\alpha>0$.

In this case, we get $F_\&(a,b)=\alpha\cdot(a+b)+(1-2\alpha)\cdot a\cdot b$. Equivalently,

$$F_\&(a,b)=(1-\alpha)\cdot a\cdot b+\alpha\cdot(a+b-a\cdot b)=(1-\alpha)\cdot f_\&(a,b)+\alpha\cdot f_\vee(a,b).$$

In other words, this operation is a convex combination of the usual “and”- and “or”-operations.

## 5 Discussion

The usual “and”-operation is associative. Thus, we can define $f_\&(a,b,c)$ as, e.g., $f_\&(a,f_\&(b,c))$ or as $f_\&(f_\&(a,b),c)$ – and the result will not change.

In contrast, the commonsense “and”-operation is not associative. With the commonsense “and”-operation, we will have two different results.

So, e.g., for three inputs, we get a more general formula

$$F_\&(a,b,c)=\alpha\cdot(a+b+c)+\beta\cdot(a\cdot b+b\cdot c+a\cdot c)+(1-3\alpha-3\beta)\cdot a\cdot b\cdot c.$$

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