# Decision Making Under Uncertainty: Cases When We Only Know an Upper Bound or a Lower Bound

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# Decision Making Under Uncertainty: Cases When We Only Know an Upper Bound or a Lower Bound

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**Abstract** In situations when we have a perfect knowledge about the outcomes of several situations, a natural idea is to select the best of these situations. For example, among different investments, we should select the one with the largest gain. In practice, however, we rarely know the exact consequences of each action. In some cases, we know the lower and upper bounds on the corresponding gain. It has been proven that in such cases, an appropriate decision is to use Hurwicz optimism-pessimism criterion. In this paper, we extend the corresponding results to the cases when we only know an upper bound or a lower bound.

## 1 Formulation of the problem

In investment, when a person knows the exact monetary consequence of each action, he/she naturally selects an action with the largest possible gain.

In practice, we usually know the consequences only with some uncertainty. For example, instead of the exact gain value, the whole set $S$ of different possible gain values are consistent with our knowledge. How should we then make a decision? What is the equivalent price $v(S)$ that we are willing to pay to participate in the corresponding action?

For example, we may know the lower bound $a$ and the upper bound on the gain. In this case, the set $S$ is the interval $[a,b]$.

Alternatively, we may know:

- only the lower bound, in which case $S = [a,\infty)$ or
- only the upper bound, in which case $S = (-\infty,b]$.

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## 2 How this problem is solved if we know both bounds

**Shift-invariance.** Suppose that we are willing to pay $v(S)$ for the set $S$. Then, for the set $S$ *and* a fixed amount $c$, we are willing to pay $v(S)+c$.

In this joint offer, the set of possible outcomes is

$$S+c \stackrel{\text{def}}{=} \{s+c : s \in S\}.$$

So, a reasonable price to pay for this joint offer is $v(S+c)$.

These are two different descriptions of the same situation. The price that are willing to pay to participate in this situation should not depend on how we describe this situation. So, we should have $v(S+c) = v(S)+c$. This property is called *shift-invariance*.

**Scale-invariance.** Another idea is that the transformation $S \mapsto v(S)$ should not depend on the choice of the monetary unit. For example, if we select pesos instead of dollars, we should get the same equivalent value.

In precise terms, this means $v(\lambda \cdot S) = \lambda \cdot v(S)$, where

$$\lambda \cdot S \stackrel{\text{def}}{=} \{\lambda \cdot s : s \in S\}.$$

This property is known as *scale-invariance.*

**Additivity.** The third idea is that participation in two independence actions, with sets $S_1$ and $S_2$, is equivalent to participation in a single action with the result

$$S_1+S_2 = \{s_1+s_2 : s_1 \in S_1 \,\&\, s_2 \in S_2\}.$$

These are two ways of representing the same situation. So we should have

$$v(S_1+S_2) = v(S_1)+v(S_2).$$

This property is known as *additivity*.

**Known results** (see, e.g., [2]). For interval uncertainty, additivity implies Hurwicz formula $v([a,b]) = \alpha \cdot b + (1-\alpha)\cdot a$ for some $\alpha \in [0,1]$. The same formula emerges if we assume shift- and scale-invariance.

## 3 What if we only know the lower bound

**Description of the case.** Suppose that we only know the lower bound $a$. In this case, the set of possible gains is the infinite interval $[a,\infty)$. What is the price

$$f(a) \stackrel{\text{def}}{=} v([a,\infty))$$

that we should pay for this situation?

**What if we assume additivity.** For infinite intervals,

$$[a,\infty)+[b,\infty)=[a+b,\infty).$$

Thus, additivity implies that $f(a+b)=f(a)+f(b)$, for $f(a)\geq a$.

It is known that this functional equation implies that $f(a)=k\cdot a$; see, e.g., [1]. The condition $a\leq f(a)$ implies that $k\geq 1$.

**What if we assume scale-invariance.** Here,

$$\lambda\cdot[a,\infty)=[\lambda\cdot a,\infty).$$

Thus, scale-invariance means $f(\lambda\cdot a)=\lambda\cdot f(a)$ for all $\lambda>0$ and $a$. In particular:

- for $a=1$, we get $f(\lambda)=k_+\cdot\lambda$, where $k_+\stackrel{\text{def}}{=}f(1)$; and
- for $a=-1$, we similarly get $f(-\lambda)=k_-\cdot\lambda$, i.e., $f(x)=(-k_-)\cdot x$.

**What if we assume shift-invariance.** Here,

$$[a,\infty)+c=[a+c,\infty).$$

Thus, shift-invariance means that $f(a+c)=f(a)+c$. In particular, for $a=0$, we get $f(c)=a_0+c$, where we denoted $a_0\stackrel{\text{def}}{=}f(0)$. Since $f(0)\geq 0$, we have $a_0\geq 0$.

## 4 What if we only know the upper bound

**Description of the case.** Suppose that we only know the upper bound $a$. In this case, the set of possible gains is the infinite interval $(-\infty,a]$. What is the price

$$g(a)\stackrel{\text{def}}{=}v((-\infty,a])$$

that we should pay for this situation?

**What if we assume additivity.** For infinite intervals,

$$(-\infty,a]+(-\infty,b]=)(-\infty,a+b].$$

Thus, additivity implies that $g(a+b)=g(a)+g(b)$, for $g(a)\leq a$.

It is known that this functional equation implies that $g(a)=k\cdot a$; see, e.g., [1]. The condition $g(a)\leq a$ implies that $k\leq 1$.

**What if we assume scale-invariance.** Here,

$$\lambda\cdot(-\infty,a]=(-\infty,\lambda\cdot a].$$

Thus, scale-invariance means $g(\lambda\cdot a)=\lambda\cdot g(a)$ for all $\lambda>0$ and $a$. In particular:

- for $a = 1$, we get $g(\lambda) = k_+ \cdot \lambda$, where $k_+ \stackrel{\text{def}}{=} g(1)$; and
- for $a = -1$, we similarly get $g(-\lambda) = k_- \cdot \lambda$, i.e., $g(x) = (-k_-) \cdot x$.

**What if we assume shift-invariance.** Here,

$$(-\infty, a] + c = (-\infty, a + c].$$

Thus, shift-invariance means that $g(a+c) = g(a) + c$. In particular, for $a = 0$, we get $g(c) = a_0 + c$, where we denoted $a_0 \stackrel{\text{def}}{=} g(0)$. Since $g(0) \le 0$, we have $a_0 \le 0$.

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