11-1-2021

Different Concepts, Similar Computational Complexity: Nguyen's Results about Fuzzy and Interval Computations 35 Years Later

Hung T. Nguyen
New Mexico State University, hunguyen@nmsu.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Comments:

Technical Report: UTEP-CS-21-89

Different Concepts, Similar Computational Complexity: Nguyen's Results about Fuzzy and Interval Computations 35 Years Later

Hung T. Nguyen and Vladik Kreinovich

Abstract When we know for sure which values are possible and which are not, we have crisp uncertainty – of which interval uncertainty is a usual case. In practice, we are often not 100% sure about our knowledge, i.e., we have fuzzy uncertainty – i.e., we have fuzzy knowledge, of which crisp is a particular case. Usually, general problems are more difficult to solve that most of their particular cases. It was therefore expected that processing fuzzy data is, in general, more computationally difficult than processing interval data – and indeed, Zadeh's extension principle – a natural formula for fuzzy computations – looks very complicated. Unexpectedly, Zadeh-motivated 1978 paper by Hung T. Nguyen showed that fuzzy computations can be reduced to a few interval ones – and in this sense, fuzzy and interval computations have, in effect, the same computational complexity. In this paper, we remind the readers about the motivations for (and proof of) this result, and show how and why in the last 35 years, this result was generalized in various directions.

1 Crisp, Interval, and Fuzzy Uncertainty: A Brief Reminder

Emergence of modern science. In the ancient times, there was no clear separation between speculations, prejudices, feeling, and confirmed scientific facts. For example, Johannes Kepler made great discoveries in astronomy – in particular, he discovered that planets follow elliptical orbits – but he also described horoscopes predicting people's fate. Actually, his salaried position required him to deal both

Hung T. Nguyen
Department of Mathematical Sciences, New Mexico State University
Las Cruces, New Mexico 88003, USA, e-mail: hunguyen@nmsu.edu, and
Faculty of Economics, Chiang Mai University, Thailand

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, El Paso, Texas 79968, USA
e-mail: vladik@utep.edu

with astronomy and astrology – which at time, were not clearly separated. Chemists were analyzing chemical reactions – and at the time, tried different magical incantations that would help them turn matter into gold – and were paid for doing both chemistry and alchemy.

The situation gradually changed when the need for reliable practical applications necessitates a clear separation between science – that studies well-established facts and relations – and semi-poetic imprecise speculations and feeling. The great Newton was interpreting the Bible, the grest Goethe came up with his theory of vision – but these activities were clearly outside what was then considered as science.

Traditional scientific approach to uncertainty. Of course, usually, no matter what object we study, we do not have a complete knowledge of this object. In science, knowledge about objects is usually described in terms of numbers. For example, in mechanics, each object is characterized by its mass, the coordinates of its spatial location, the components of its velocity, and – if this object is rotating – by a unit vector describing its rotation axis and by the angular velocity. The fact that we do not have a complete knowledge means that we do not know the exact values of each of these quantities x; at best, we know the bounds $\underline{x}$ and $\overline{x}$ on the actual value. In such situations, all we know about the actual (unknown) value x is that this value belongs to the interval $[\underline{x}, \overline{x}]$.

In some cases, we know that some values inside this interval are not possible; in this case, the set of all possible values of x has a more general form than an interval. But even in this case, for some real numbers, we are, at this stage, 100% sure that the value of the physical quantity x cannot be equal to this number, which for other numbers, we are 100% sure that the value of the quantity x can be equal to this number. We may have an additional gut feeling that some numbers from this interval are more possible than others, but such gut feeling was not taken into account in the traditional scientific paradigm.

Comment. Sometimes, in addition to the interval (or, more generally, set) of possible values of x, we also have some information about the frequency (probability) with different numbers from this interval appear in different situations. But again, what traditional science considered was only guaranteed knowledge about these probabilities.

Need to go beyond traditional scientific paradigm. In the early 1960s, Lotfi Zadeh, one of the world's leading specialists in automatic control, a co-author of the then most widely used book on automatic control, noticed that in many practical situations, automatic controllers – that take into account all scientific information about the object of control – perform much worse than human controllers. He realized that human expert controllers use additional knowledge, knowledge which is – in contrast to what traditional science considered – not precise. The rules describing this knowledge were usually formulated in terms of imprecise ("fuzzy") words from natural language, such as "small", "approximately", etc., words that the traditional scientific approach ignored.

So, instead of ignoring these words, Zadeh proposed to incorporate the corresponding scientific knowledge into the automatic controllers. For this purpose, he came up with a methodology that he called *fuzzy*; see, e.g., [1, 12, 17, 21, 22, 26].

Fuzzy methodology: a brief description. For precise ("crisp") properties like "smaller than 10", every number either satisfies this property or does not. Such properties can be described by describing a set of all the values that satisfy this property, or, equivalently, by a function $\mu(x)$ that assigns, to each possible value x, the value 1 if x has this property and 0 if not. In mathematics, such a function is known as a *characteristic function*.

In contrast, for fuzzy words like "small", for some values x, experts are not 100% sure whether x is small or not, they are sure to some degree. We want to process such information in a computer. Computers were not designed to process words from natural languages, they were designed to process numbers. So, to be able to use computers to process expert information, we need to be able to describe this degree of confidence by a number. A natural way to do it is to ask the expert to mark his/her degree on a scale from 0 to 1, where 1 means absolute confidence, and 0 means no confidence at all. Alternatively, we can use a scale from 0 to 10 or from 0 to 5, and then divide the result, correspondingly, by 10 or by 5. This is how students evaluate their instructors, this is how we evaluate the quality of different services. Thus, to describe a property, we need to describe a function that assigns, to each value x, a degree $\mu(x) \in [0,1]$ to which each value x satisfies the given property (e.g., is small). The corresponding function is known as a *membership function* or, alternatively, as a *fuzzy set*.

An additional complication comes from the fact that many rules describe what happens when several properties are satisfied. For example, a rule may describe what to do if the temperature t in a chemical reactor is slightly below the desired one *and* the pressure p is slightly higher than desired. We can, in principle, ask the expert to mark the degrees $\mu_{\text{temp}}(t)$ corresponding to different values t and the degrees $\mu_{\text{press}}(p)$ corresponding to different values p, but what we real need is, for all possible pairs (t,p), to estimate the degree of the above "and"-statement. It may be still possible to ask the expert about all such pairs, but what if there are 5 inputs? Ten inputs? Even if we consider only 10 different values for each quantity, this would still make 10^5 or even 10^{10} combinations – and we cannot ask that many questions to an expert.

Since we cannot always directly elicit the expert's degree of certainty in an "and"-statement – of the type $A\,\&\,B$ – we need to be able to estimate this degree based on whatever information we have – i.e., based on the degrees of certainty a and b of the statements A and B. The algorithm for this estimation is known as an *"and"-operation* (or, for historical reasons, a *t-norm*); we will denote it by $f_\&(a,b)$. The most widely used "and"-operations are $\min(a,b)$ and $a \cdot b$.

Similarly, we need an algorithm to estimate the degree of certainty of $A \vee B$; such an algorithm is known as an *"or"-operation*, or a *t-conorm*; we will denote it by $f_\vee(a,b)$. The most widely used "or"-operations are $\max(a,b)$ and $a+b-a\cdot b$. To describe the degree of certainty of a negation $\neg A$, we need an algorithm which is known as a *negation operation* $f_\neg(a)$. The mostly widely used negation operation is $f_\neg(a) = 1 - a$. Because of the important role of these logical operations, fuzzy methodology is often called *fuzzy logic methodology*.

2 Data Processing under Interval and Fuzzy Uncertainty: Reminder

Need for data processing. In many real-life situations, we are interesting in quantities which cannot be measured directly – e.g., in future values of some quantities. Since we cannot measure these quantities y directly, we need to estimate y based on available information – i.e., based on the known values $\widetilde{x}_1, \ldots, \widetilde{x}_n$ of related quantities $x_1, \ldots, x_n$. We will denote the estimating algorithm by $y = f(x_1, \ldots, x_n)$.

Need to take uncertainty into account. The values $\widetilde{x}_i$ come either from measurements or from expert estimates. In both cases, the available value $\widetilde{x}_i$ is somewhat different from the actual (unknown) value x_i of the corresponding quantity. As a consequence, even if the relation $y = f(x_1, \ldots, x_n)$ is exact, the resulting estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ for y is, in general, different from the actual value y of the quantity of interest. To make decisions, we need to know how accurate is this estimate.

Case of interval uncertainty. In the interval case, for each quantity x_i, the only thing we know is the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ that contains x_i. So, the set $\mathbf{y}$ of all possible values of y is the set of all possible values $f(x_1, \ldots, x_n)$ when each x_i is in the corresponding interval:

$$\mathbf{y} = \{f(x_1, \ldots, x_n) : x_i \in \mathbf{x}_i \text{ for all } i\}.$$

In mathematical terms, the right-hand side of this equality is known as the *range* of the function $f(x_1, \ldots, x_n)$; it is usually denoted by $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

For many important classes of problems, there are feasible algorithms which either compute this range or at least compute a reasonable approximation for this range; see, e.g., [11, 16, 19]. The problem of computing this range is known as the problem of *interval computation*.

Comment. While there exist efficient interval computation algorithms for many classes of problems, it should be mentioned that, in general, the problem of computing this range is NP-hard; see, e.g., [15]. This means, in effect, that unless P = NP (which most computer scientists believe to be not true), no feasible algorithm is possible that would *always* compute the exact range.

Case of fuzzy uncertainty. What if for each i and for each x_i, we only know the degree $\mu_i(x_i)$ to which this value x_i is possible? In this case, the value y is possible if and only if for some tuple $(x_1, \ldots, x_n)$ for which $y = f(x_1, \ldots, x_n)$, the value x_1 is possible, *and* the value x_2 is possible, *and* ...

We know the degree $\mu_i(x_i)$ to which x_i is possible. So, to get the degree $\mu(y)$ to which y is possible, we need to apply an "and"-operation for "and" and an "or"-operation for "for some" (which is nothing else but "or"). Thus, we get

$$\mu(y) = f_\vee\{f_\&(\mu_1(x_1),\ldots,\mu_n(x_n)) : y = f(x_1,\ldots,x_n)\}.$$

Here, the "or"-operation is applied to infinitely many values. For most "or"-operations, e.g., for $f_\vee(a,b) = a+b-a\cdot b$, if we apply this operation to infinitely many positive terms, we get 1. The only exception is when we use $f_\vee(a,b) = \max(a,b)$. In this case, the above expression takes the following form:

$$\mu(y) = \sup_{(x_1,\ldots,x_n): y=f(x_1,\ldots,x_n)} f_\&(\mu_1(x_1),\ldots,\mu_n(x_n)). \quad (1)$$

In particular, for the most commonly used "and"-operation $f_\&(a,b) = \min(a,b)$, we get:

$$\mu(y) = \sup_{(x_1,\ldots,x_n): y=f(x_1,\ldots,x_n)} \min(\mu_1(x_1),\ldots,\mu_n(x_n)). \quad (2)$$

The formula (2) was first proposed by Zadeh and is therefore called *Zadeh's extension principle.*

3 Nguyen's Theorem: Brief History, Formulation, and the Main Idea Behind the Proof

What was expected. The formulas (1) and (2) looks much more complex than the corresponding interval formulas – which are, of course, a particular case of the fuzzy formulas when all fuzzy sets are crisp, i.e., when for each i and each x_i, we have $\mu_i(x_i) = 1$ or $\mu_i(x_i) = 0$. This is a known phenomenon – that general computational problems are usually more complex than their particular cases. For example, solving linear and quadratic equations is straightforward, we have explicit formulas for these solutions, but solving general polynomial equations is complicated. Solving systems of linear equations is feasible, but already solving systems of quadratic equations is NP-hard.

It was therefore expected that fuzzy computation – i.e., fuzzy data processing – is much more complex than interval computation. Lotfi Zadeh himself understood the complexity of this problem, and realized that to make fuzzy methodology practically useful, it is important to develop efficient algorithms for at least some cases of fuzzy computing.

Nguyen's theorem: unexpected result. In 1975, Professor Zadeh invited Hung T. Nguyen, a promising recent PhD in Mathematics and Statistics from University of Paris, to spend two years at the University of California-Berkeley – to have a mathematician's look at fuzzy theory. For this purpose, he asked Hung T. Nguyen to read his papers and related papers of others – including a paper by a Japanese researcher visiting Berkeley on the computational aspects of fuzzy computing.

Hung T. Nguyen started working on this topic and came up with a general result about fuzzy computation, published in [20]. To explain this result, we need to recall the notion of an α-cut. The notion was known in fuzzy methodology because in many situations, we need to make a decision – e.g., whether to perform a certain action or not. When the satellite deviates a little bit from the desired trajectory, we need to decide whether we should use the precious fuel to correct its trajectory or not yet. If a chemical process starts deviating a little bit from the desired parameter, we need to decide whether to apply an appropriate control – e.g., shut down the reactor.

If we know for sure that a sufficiently large deviation took place, then yes, we should perform the corresponding action. But what if we can only conclude that this deviation occurred with some degree of confidence d? In this case, we need to select some threshold value $\alpha \in (0,1]$, and perform the action if our degree d is larger than or equal to α: $d \geq \alpha$. Correspondingly, when this degree depends on the value of some quantity x ($d = \mu(x)$), then we perform the action if and only if $\mu(x) \geq \alpha$.

For each fuzzy set $\mu(x)$, the corresponding set $\{x : \mu(x) \geq \alpha\}$ is known as the α-*cut* of this fuzzy set. What Nguyen proved was that under reasonable conditions, the α-cut $\mathbf{y}(\alpha)$ of y is equal to the range of the function $f(x_1,\ldots,x_n)$ when each x_i is in the α-cut $\mathbf{x}_i(\alpha)$ of x_i:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha),\ldots,\mathbf{x}_n(\alpha)). \tag{3}$$

The value α corresponds to an expert's degree of confidence in a statement. An expert cannot estimate his/her degree with accuracy higher than 0.1. Thus, it is sufficient to consider only values α differing by 0.1: 0, 0.1, 0.2, ..., 1.0. So, fuzzy computation can reduced to a few cases of interval computation.

This theorem is the main tool behind fuzzy computing. This theorem shows that there is no need to come up with new algorithms for fuzzy computing – it is sufficient to use well-developed interval algorithms, and this is exactly what most practitioners are doing.

Every year, there are sessions on interval computations at fuzzy conferences – and sessions on possible fuzzy applications at interval conferences.

Such situations happen. The fact that a more general case turned out to be no more computationally complex than a particular case was unexpected, but such situations happened before. For example, to come up with equations of General Relativity that describe gravity – i.e., forces caused by masses (= energy) – Einstein came up with a completely new idea of curved space-time; see, e.g., [5, 18, 25]. The general feeling was that without this new physical idea, we cannot come up with a reasonable explanation for these complex nonlinear partial differential equations. However, later, it turned out that the same equations appear if we consider a simple tensor field in flat (not-curved) space time, by making a natural-for-gravity assumption that the source of this field includes both the energy-momentum of other fields and the energy-momentum of the gravity field itself; see, e.g., [8, 9, 10, 18, 13].

In physics. there have been many examples of this type – when a seemingly completely revolutionary theory turned out to be derivable from the previous physics. For example, even the notion of the black hole – which was originally perceived as specific for general relativity – follows already from Newtonian mechanics. Indeed, in Newtonian mechanics, for each celestial body, there is an escape velocity – so that any object travelling slower than that will fall back to the body. If this escape velocity exceeds the speed of light – the largest possible speed – then nothing can leave this body, including light. Even the thresholds for when the body with a given mass and radius becomes a black hole are very similar in General Relativity and in Newton's mechanics; see, e.g., [5, 18, 25].

From this viewpoint, it is not very surprising that the general fuzzy computing was reduced to a simpler interval computing case. Not surprising but still not trivial, since each such reduction requires mathematical and physical ingenuity. It took almost 40 years to show that General Relativity can be derived from field theory, it took several centuries after Newton to conclude that black holes can exist in Newtonain physics, and it took more than 10 years to realize that fuzzy computation can be reduced to interval computations.

So how is this theorem proved: main idea. According to the formula (2), the value $\mu(x)$ is the maximum of several values. When is the maximum of several numbers larger than or equal to α? When one of these numbers if larger than or equal to α:

$$\mu(y) \geq \alpha \Leftrightarrow$$
$$\exists x_1,\ldots,x_n\,(y = f(x_1,\ldots,x_n) = y\,\&\,\min(\mu_1(x_1),\ldots,\mu_n(x_n)) \geq \alpha). \quad (4)$$

Thus,

$$y \in \mathbf{y}(\alpha) \Leftrightarrow \exists x_1,\ldots,x_n\,(y = f(x_1,\ldots,x_n) = y\,\&\,\min(\mu_1(x_1),\ldots,\mu_n(x_n)) \geq \alpha). \quad (5)$$

When is the smallest of n numbers larger than or equal to α? When all of them are larger than or equal to α:

$$\min(\mu_1(x_1),\ldots,\mu_n(x_n)) \geq \alpha \Leftrightarrow \mu_1(x_1) \geq \alpha\,\&\ldots\&\,\mu_n(x_n) \geq \alpha. \quad (6)$$

By definition of the α-cut, this means that

$$\min(\mu_1(x_1),\ldots,\mu_n(x_n)) \geq \alpha \Leftrightarrow x_1 \in \mathbf{x}_1(\alpha)\,\&\ldots\&\,x_n \in \mathbf{x}_n(\alpha). \quad (7)$$

Thus, the value y belongs to the α-cut $\mathbf{y}(\alpha)$ if and only if there exist values $x_1,\ldots,x_n$ for which $y = f(x_1,\ldots,x_n)$ and each x_i belongs to the corresponding α-cut $\mathbf{x}_i(\alpha)$. In other words, the set $\mathbf{y}(\alpha)$ is indeed equal to the range $f(\mathbf{x}_1(\alpha),\ldots,\mathbf{x}_n(\alpha))$. This is exactly what the theorem says.

Important warning. What we described is an idea, but not the full proof. It would be a full proof if we had the maximum of finitely many terms – but in our case, we have infinitely many terms, and it is known that in this case, the supremum may be larger than or equal to α without any of the maximized numbers being larger than or equal to α. For example, the supremum of the values $1-2^{-n}$ corresponding to $n = 0,1,2\ldots$ is equal to $\alpha = 1$, while all the values $1-2^{-n}$ are smaller than 1.

Thus, to have a real proof, we need to guarantee that the supremum is attained for some tuple $(x_1,\ldots,x_n)$. This can be guaranteed, e.g., if all the membership functions $\mu_i(x_i)$ are continuous and all the α-cuts are compact – which for continuous functions of real numbers is equivalent to requiring that all the α-cuts are bounded, a requirement which is true for most practical membership functions.

4 Extensions of Nguyen's Theorem: A Brief Overview

Extensions beyond real numbers. The above proof does not depend on the fact that x_i are real numbers:

- they could be vectors, tuples;
- they could be, more generally, elements of a general metric (or even general topological) space.

Such extensions have indeed been published; see, e.g., [2, 3, 4].

Extensions to interval-valued, type-2, and more general fuzzy sets. In the above description, we implicitly assumed that an expert can always describe his/her degree of confidence ny an exact number. In reality, however, just like people are not 100% confident about their estimates, they are also not 100% confident about their degrees of confidence. A natural idea is to allow an interval of possible degrees (this leads to interval-values fuzzy sets) or even to fuzzy sets describing each degree (this leads to general type-2 fuzzy sets). In both cases, we face a natural computational question: how to propagate such type-2 uncertainty through a data processing algorithm?

It turned out that such computations can also be reduced to interval computations; see, e.g., [14].

Extensions to other t-norms. In the above text, we considered the case when we use $f_\&(a,b) = \min(a,b)$. What if we use a different t-norm? Several extensions of Nguyen's theorem to different t-norms have been proposed; see, e.g., [7].

It turns out that for other t-norms, we can also have an efficient data processing algorithm [23] – although this time the reduction is not to interval algorithms but to algorithms from convex optimization; see, e.g., [24].

Other extensions. An interesting and promising extension was proposed in [6], where the authors represented a fuzzy number – a fuzzy generalization of an interval – as an interval $[\ell(\alpha), r(\alpha)]$ formed by two what they called *gradual numbers* $\ell(\alpha)$ and $r(\alpha)$: mappings from $(0,1]$ to the real line. The left gradual number $\ell(\alpha)$ is formed by lower endpoints of the α-cut intervals, while the right gradual number $r(\alpha)$ is formed by its right end-points, so that $[\ell(\alpha), r(\alpha)] = \mathbf{x}(\alpha)$.

Each of these gradual numbers may not have a clear meaning, but this subdivision seems to simplify computations – just like in physics, while it is not possible to actually separate, e.g., a proton into three quarks, many computations are simplified if we represent a proton this way [5, 25].

We hope that other fruitful extensions will occur.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes), and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

References

1. R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.
2. C. Carlsson and R. Fuller, "On additions of interactive fuzzy numbers", *Acta Polytechica Hugarica*, 2005, Vol. 2, pp. 59–73.
3. L. C. de Barros, R. C. Bassanezi, and P. A. Tonelli, "On the continuity of the Zadeh's extension", *Proceedings of the 7th World Congress of the International Fuzzy Systems Association IFSA'1997*, Prague, June 25–29, 1997, pp. 22–26.
4. J. C. de Mota Ferreira, R. C. Bassanezi, and A. J. V. Brandão, *On the Nguyen Theorem for topological spaces*, Preprint, available at https://www.researchgate.net/publication/328538986
5. R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.
6. J. Fortin, D. Dubois, and H. Fargier, "Gradual numbers and their application to fuzzy interval analysis", *IEEE Transactions on Fuzzy Systems*, 2008, Vol. 16, No. 2, pp. 388–402.
7. R. Fuller and T. Keresztfalvi, "On generalization of Nguyen's theorem", *Fuzzy Sets and Systems*, 1991, Vol. 41, pp. 371–374.
8. S. N. Gupta, "Quantization of Einstein's gravitational field: general treatment", *Proceedings of Physics Society*, 1952, Vol. A65, pp. 608–619.
9. S. N. Gupta, "Gravitation and electromagnetism", *Physical Review*, 1954, Vol. 96, pp. 1683–1685.
10. S. N. Gupta, "Einstein's and other theories of gravitation", *Reviews of Modern Physics*, 1957, Vol. 29, pp. 334–336.
11. L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2001.
12. G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
13. V. Kreinovich, "Gupta's derivation of Einstein equations," *Soviet Physics Doklady*, 1975, Vol. 20, No. 5, pp. 341–342.
14. V. Kreinovich, "From processing interval-valued fuzzy data to general type-2: towards fast algorithms", *Proceedings of the IEEE Symposium on Advances in Type-2 Fuzzy Logic Systems T2FUZZ'2011, part of the IEEE Symposium Series on Computational Intelligence*, Paris, France, April 11–15, 2011, pp. ix–xii.
15. V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational Complexity and Feasibility of Data Processing and Interval Computations*, Kluwer, Dordrecht, 1998.
16. G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.
17. J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

18. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco, California, 1973.
19. R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.
20. H. T. Nguyen, "A note on the extension principle for fuzzy sets", *Journal of Mathematical Analysis and Applications*, 1978, Vol. 64, pp. 369–380.
21. H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.
22. V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
23. A. Pownuk, V. Kreinovich, and S. Sriboonchitta, "Fuzzy data processing beyond min t-norm", In: C. Berger-Vachon, A. M. Gil Lafuente, J. Kacprzyk, Y. Kondratenko, J. M. Merigo Lindahl, and C. Morabito (eds.), *Complex Systems: Solutions and Challenges in Economics, Management, and Engineering*, Springer Verlag, 2018, pp. 237–250.
24. R. T. Rockafeller, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1997.
25. K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2017.
26. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.