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As Complexity Rises, Meaningful Statements Lose Precision – but Why?

Miroslav Svítek, Olga Kosheleva, and Vladik Kreinovich

Abstract One of the motivations for Zadeh's development of fuzzy logic – and one of the explanations for the success of fuzzy techniques – is the empirical observation that as complexity rises, meaningful statements lose precision. In this paper, we provide a possible explanation for this empirical phenomenon.

1 Formulation of the Problem

Empirical fact. Many researchers are familiar with Lotfi Zadeh's observation that "As complexity rises, precise statements lose meaning and meaningful statements lose precision"; see, e.g., [3], p. 43. This is one of the most cited phrases by Zadeh. This empirical fact served as one of the main motivations for developing fuzzy techniques. This empirical fact also serves as a good explanation for why these techniques have been successful in many applications; see, e.g., [1, 2, 4, 5, 6, 7].

But why? But how can we explain this empirical fact? In this paper, we provide a possible explanation.

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2 Analysis of the Problem

Towards reformulating the question in precise terms. In general, we can have both precise and imprecise ("fuzzy") statements about a system. The empirical fact – as observed by Zadeh – is that:

- when a system is simple, this system is adequately described by precise statements, while
- as the system becomes more complex, its adequate description requires more and more fuzzy statements.

How can we explain this empirical fact?

Towards a corresponding model. Let us consider possible statements $S_1, \ldots, S_n$ that we can make about a system.

In general, for each system, for each statement S_i, we can – following the general fuzzy methodology – describe our degree of confidence in this statement by a number x_i from the interval $[0,1]$. So, our description boils down to a tuple $x = (x_1, \ldots, x_n)$ of numbers from the interval $[0,1]$ – i.e., to a point in an n-dimensional cube $[0,1]^n$.

What we want from a model. We want our model to be consistent with all the different observation patterns characterising the system's behavior. Let us denote the number of such patterns by p, and let us denote the requirement that the tuple x is consistent with the j-th pattern by $f_j(x) = 0$.

Among all the models that are consistent with all the patterns, we should select a model which is the best: this could be the simplest to describe, the simplest to use, the least deviating from the current model, etc. In general, for each model x, let us denote its "degree of quality" by $a(x)$.

In this term, selecting, among all the descriptions for which $f_j(x)$ for all $= j$, the description x which is the best, means selecting the description for which the degree $a(x)$ is the largest possible.

Our descriptions are not ideal. In general, every description is approximate. To get an ideal "most adequate" description, we need to consider more than n statements. In geometric terms, the ideal description is outside our n-dimensional cube $[0,1]^n$.

It is reasonable to assume that the closer we are to this ideal description, the more adequate our model. From this viewpoint, we expect the quality function $a(x)$ to have no local maxima – its only maximum is the global maximum.

Using known facts from calculus. It is known that if a function has no local maxima inside an area, then its maximum in this area is attained on the border of this area.

Let us start with the case when we have no observation patterns at all. Let us first consider the trivial case when we have no observation patterns at all, i.e., in mathematical terms, when we have no constraints. As we have argued, the global maximum of this objective function is attained outside the cube, and there are no local maxima inside the cube. Thus, in line with the above fact from calculus, in this case, the desired maximum of the quality function $a(x)$ is attained on the border of the n-dimensional cube.

This border consists of faces, which are described by the equations $x_i = 0$ or $x_i = 1$. On each of these faces, we also do not expect to have a local maximum, so the optimal description should correspond to the border of each face, i.e., to the set of all points where two of the values x_i are equal to 0 or 1.

Following the same line of reasoning, we conclude that the maximum of the objective function $a(x)$ on the n-dimensional cube is attained at an extreme point of the cube, i.e., at a point where each of the values x_i is equal to 0 or to 1.

So, in the absence of any observation patterns, the best description is a crisp description.

What if we take observation patterns into account. In general, the same argument as in the previous subsection leads us to the conclusion that the maximum of the quality function $a(x)$ is attained at one of the extreme points of the corresponding area.

If we take observation patterns into account, this means that the corresponding area consists of all the tuples x for which $f_j(x) = 0$ for all j from 1 to p, i.e., this area is equal to the following set:

$$S \stackrel{\text{def}}{=} \{x = (x_1, \ldots, x_n) : x_i \ge 0 \text{ for all } i \text{ and } f_j(x) = 0 \text{ for all } j\}.$$

In general, for a set defined by equalities and inequalities, an extreme point is when as many inequalities $g(x) \ge 0$ as possible become equalities, i.e., satisfy the condition $g(x) = 0$. In general:

- if the number of equations is smaller than the number of unknowns, then we have many solutions;
- if the number of equations is equal to the number of unknowns, then we have a unique (or at least locally unique) solution; and
- if the number of equations is larger than the number of unknowns, then the system, in general, does not have a solution.

Thus, for a tuple x consisting of n real values, the largest number of equalities that this tuple can satisfy is n. So, extreme points correspond to the case when n equalities are satisfied.

We already have p equalities $f_j(x) = 0$ that are satisfied. Thus, for an extreme point for which n equalities are satisfied, $n - p$ remaining inequalities become equalities. These remaining inequalities have the form $0 \le x_i \le 1$. Thus, the fact that these inequalities become equalities means that for the corresponding values i, we have:

- either $x_i = 0$
- or $x_i = 1$.

The fact that $x_i = 0$ or $x_i = 1$ means that in this description, the i-th statement is crisp. We therefore conclude that in the best model, out of n statements S_i, $n - p$ of them are crisp.

The remaining truth values are determined by p equations $f_j(x) = 0$. In the general case, all components of a solution of a system of p equations with p unknowns are different from 0 and 1. Thus, in the general case, for the remaining p statements k, we have $0 < x_k < 1$ – i.e., these statements are, in general, not crisp.

Mathematical conclusion. So, in the general case, if we have p observation patterns, then in the best description, we have:

- p fuzzy statements, and
- $n - p$ crisp statements.

How this is related to system complexity. The more complex a system, the more different behavioral patterns it exhibits. This is, in a nutshell, is what we mean by a complex system. For example:

- a pendulum shows the same behavior all the time; in this sense, it is a simple system;
- on the other hand, a human being has many different patterns of behavior and is, thus, a complex system.

In the previous subsection of this section, we presented the conclusion of our analysis: that the more different patterns of behavior a system exhibits, the larger the number of fuzzy statements in this system's best description. So, indeed, as complexity rises, more meaningful statements become fuzzy – i.e., lose precision.

This is exactly Zadeh's observation. Thus, our analysis indeed explains this observation.

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References

1. R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.
2. G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

3. D. McNeill and P. Freiberger, *Fuzzy Logic*, Simon & Schuster, New York, 1993.
4. J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.
5. H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.
6. V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
7. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.