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What Is 1/0 from the Practical Viewpoint: A Pedagogical Note

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Abstract What is 1/0? Students are first taught – in elementary school – that it is undefined, then – in calculus – then it is infinity. In both cases, the answer is usually provided based on abstract reasoning. But what about the practical meaning? In this paper, we show that, depending on the specific practical problem, we can have different answers to this question: in some practical problems, the correct answer is that 1/0 is undefined, in others, the correct answer is that $1/0 = 0$ – and there are probably other practical problems where we can have different answers. Bottom line: there is no unversal answer, the correct answer depends on what practical problem we are considering.

1 Formulation of the Problem

What is 1/0: what students learn. In elementary school, students learn that you cannot divide by 0. This makes sense: by definition, the ratio a/b is a number that, multiplied by b, gives a. Of course, no matter what number you multiple by $b = 0$, you always get 0, so you will never get $a = 1$.

Later, the student learn that in calculus, 1/0 is infinity – since 0 is the limit of, e.g., a sequence $1/n$, we can thus interpret 1/0 as the limit of values $1/(1/n) = n$, i.e., infinity.

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Problem. From the purely mathematical viewpoint, both answers make sense – as well as many other fact and results from mathematics. However, there is a difference between this mathematical fact and other mathematical facts: many other facts makes perfect sense in practical applications. For example, $8/4 = 2$ means that if we equally divide 8 apples between 4 students, each student gets 2 apples. Such practical examples help students better understand the corresponding mathematical facts and results. In contrast, 1/0 seems to be a purely mathematical exercise. Usually, no practical examples are provided to explain the meaning of this ratio. This makes studying this idea too abstract and thus, more complicated to many students.

What we do in this paper. In this paper, we provide one of the possible practical meanings of this ratio, and explain, for this practical example, what will the ratio 1/0 mean in this particular case.

Of course, this is just one possible practical example. We are sure that there can be many other practical examples, and in many of them, the meaning of 1/0 will be different.

2 Practical Problem

General description of the situation. Let us start with the general description of a practical problem that corresponds to computing $1/c$ for any real number c.

This problem is related to signal propagation. It is well known that as a signal travels – be it by wire or through air – its amplitude decreases. As a result, when the sender sends a signal of amplitude a, the receiving agent receives a signal of smaller amplitude $r = c \cdot a$, for some value $c < 1$.

To reconstruct the original signal, the receiving agent thus needs to *amplify* the received signal, i.e., in precise terms, to multiply it by some constant $C > 1$.

What is the problem.

- We know the coefficient $c < 1$ that describes how much the original signal decreased.
- We want to find the amplification coefficient C that allows us to reconstruct the original signal.

3 Idealized Setting

Description of the ideal case. Let us first consider the ideal situation when there is no noise, and the only change in the original signal is that its amplitude decreases, from a to $c \cdot a$.

What is the proper amplification coefficient. In this case:

- we receive the signal $r = c \cdot a$, and

- we multiply it by C, getting $C \cdot r = C \cdot c \cdot a$.

We want to make sure that for all signals a sent by the sender, the resulting signal $C \cdot c \cdot a$ is equal to exactly a, i.e., that $C \cdot c \cdot a = a$.

In particular, for $a = 1$, we get $C \cdot c = 1$, so $C = 1/c$. One can easily check that for this amplification coefficient $C = 1/c$, and for every sender's signal a, we indeed have $C \cdot c \cdot a = (C \cdot c) \cdot a = 1 \cdot a = 1$.

Thus, this practical problem provides a practical interpretation for $1/c$.

In this case, what is 1/0? In this interpretation, the ratio 1/0 is simply not defined: if $c = 0$, then, no matter what amplification coefficient C we select, we will never get $C \cdot c = 1$.

This is exactly what kids learn in school. True, this is exactly what kids learn, that 1/0 is not defined. So far, nothing new, nothing interesting.

But remember that we consider an idealized case, when we assume that there is no noise. In practice, there is always some noise. What happens to this practical problem in this more realistic setting?

4 Realistic Setting

Realistic setting: general idea. Let us now take into account that, in addition to being multiplied by a coefficient $c < 1$, the signal also gets corrupted by noise n. In other words, the received signal r is equal to

$$r = c \cdot a + n, \tag{1}$$

where n denotes the noise, i.e., the additional change in the signal.

In this case, after amplification, you do not get the original signal, you get a signal

$$s = C \cdot r = C \cdot c \cdot a + C \cdot n, \tag{2}$$

which is different from a even when $C \cdot c = 1$.

The goal is to find the amplification coefficient C for which the amplified signal s is the closest to the original signal a.

Realistic setting: details. We do not know the value of the noise n – if we knew it, we could simply substract this known value from the received signal r and thus, eliminate the effect of the noise. From the mathematical viewpoint, this means that n is a random variable.

Natural characteristics of a random variable n are its mean value $E[n]$ and its variance $V \stackrel{\text{def}}{=} E[(n - E[n])^2]$ – or, equivalently, its standard deviation $\sigma \stackrel{\text{def}}{=} \sqrt{V}$ for which $V = \sigma^2$; see, e.g., [1]. While we do not know the exact value of the noise, based on the previous experiences, we can estimate both the mean and the standard deviation.

The additive random noise can be both positive and negative. A priori, there is no reason to believe that positive values are more probable or negative values are more probable, so it make sense to assume that both are equally probable, and that the mean value of the noise is 0. Let us denote the standard deviation of noise by σ_n.

Similarly, we do not know what will be the signal that the sender will be sending – if we knew, there would be no need to send anything. Thus, the signal can also be viewed as a random variable. We also do not have any reason to believe that positive values of the signal will be more or less probable than negative values. So, it also makes sense to assume that the mean value of the signal is 0. Let us denote the standard deviation of the signal by σ_a.

How do we gauge which coefficient C is better. We are interested in minimizing the reconstruction error, i.e., the difference $d \stackrel{\text{def}}{=} s - a$ between the reconstructed signal s and the original signal a. Due to (2), we get the following expression for this error:

$$d = (C \cdot c - 1) \cdot a + C \cdot n. \tag{3}$$

Since the mean values of a and n are both 0s $E[a] = E[n] = 0$, the mean value of their linear combination d is also 0: $E[d] = 0$. It is therefore reasonable to gauge the value d by its variance $V[d] = E[d^2]$. Due to (3), we have

$$E[d^2] = (C \cdot c - 1)^2 \cdot E[a^2] + 2(C \cdot c - 1) \cdot C \cdot E[a \cdot n] + C^2 \cdot E[n^2]. \tag{4}$$

Signal a and noise n are clearly independent, so $E[a \cdot n] = E[a] \cdot E[n] = 0 \cdot 0 = 0$. Thus, the formula (4) takes the form

$$E[d^2] = (C \cdot c - 1)^2 \cdot \sigma_a^2 + C^2 \cdot \sigma_n^2. \tag{5}$$

We want to find the amplification C that minimizes this expression.

The resulting optimal value of the amplification coefficient. Differentiating the expression (5) with respect to C and equating the derivative to 0, we conclude that

$$2(C \cdot c - 1) \cdot c \cdot \sigma_a^2 + 2C \cdot \sigma_n^2. \tag{6}$$

If we divide both sides by 2 and move all the terms not containing C to the other side, we get

$$C \cdot (c^2 \cdot \sigma_a^2 + \sigma_n^2) = c \cdot \sigma_a^2, \tag{7}$$

hence the optimal amplification coefficient C is equal to

$$C = \frac{c \cdot \sigma_a^2}{c^2 \cdot \sigma_a^2 + \sigma_n^2}. \tag{8}$$

Of course, this value depends on the noise level. When the noise is small $\sigma_n \approx 0$, the value C is close to the limit value of this expression when $\sigma_n \to 0$:

$$C \approx C_{\lim} = \lim_{\sigma_n \to 0} \frac{c \cdot \sigma_a^2}{c^2 \cdot \sigma_a^2 + \sigma_n^2}. \tag{9}$$

What is this limit?

What if $c \neq 0$**.** In this case, both numerator and denominator have definite limits, so

$$C_{\text{lim}} = \frac{c \cdot \sigma_a^2}{c^2 \cdot \sigma_a^2} = \frac{1}{c}. \tag{10}$$

Thus, this practical problem indeed provides a natural practical interpretation for the value $1/c$ – at least when $c \neq 0$.

But what if we take $c = 0$**?** In this case, the problem also makes sense, but the limit is different: here for all σ_n, we have

$$C = \frac{0}{\sigma_n^2} = 0,$$

and thus,

$$C_{\text{lim}} = \lim_{\sigma_n \to 0} 0 = 0. \tag{11}$$

So, in this case, a practical problem leads to an unexpected conclusion that $1/0 = 0$.

Conclusion. On the example of two practical problems, we got two different answers to the question of what is 1/0: that it is undefined, and that it is equal to 0. We are sure that there may be other practical problems in which the answer is $1/c$ for $c \neq 0$ and for $c = 0$, we get a different value.

Bottom line: what is 1/0 depends on the specific practical problem, we cannot always rely on abstract arguments.

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