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Why Base-20, Base-40, and Base-60 Number Systems?

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Abstract Historically, to describe numbers, some cultures used bases much larger than our usual base 10, namely, bases 20, 40, and 60. There are explanations for base 60, there is some explanation for base 20, but base 40 – used in medieval Russia – remains largely a mystery. In this paper, we provide a possible explanation for all these three bases, an explanation based on the natural need to manage large groups of people. We also speculate why different cultures used different bases.

1 Formulation of the Problem

Historical facts. In the ancient times, in addition to our usual base-10 number system and to systems with a smaller or similar-size base, some cultures used number systems with much larger bases:

- Babylonians used the 60-based system (see, e.g., [3, 6, 7]). We still divide an hour into 60 minutes, a minute into 60 seconds – this idea originated with the ancient Babylonians.
- Ancient Romans used the base-20 system. This can still be traced to how numbers are named in modern French: for example, 80 is quatre-vingts, meaning four-twenties, and 96 is quatre-vingt-seize, meaning four-twenties-sixteen. A similar

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20-based system – with 20 divided into four 5s – was used by the Mayans and by the Aztecs; see, e.g., [3, 4, 5, 6, 7]
- An unusual 40-based system was used in medieval Russia. For example, to describe the (large) number of churches in the medieval Moscow, the Russian chronicle says that there were 40 of 40s (sorok sorokov), i.e.,

$$40 \cdot 40 = 1600.$$

But why? A natural question is: why these bases and not others?

There are answers to some "why" questions, but not to all of them. There is a good explanation of why 60: this is the number that has unusually many divisors: it is divisible by 2, 3, 4, 5, 6, 10, 12, 15, and 20. So:

- 1/3 of a usual 60-minute hour is a whole number of minutes,
- 1/4 of an hour is a whole number of minutes, etc.

This would not have been possible if we divided an hour into 100 minutes; see, e.g., [3, 6, 7].

There is a similar partial explanation of base 20; see, e.g., [1]. However, there is no similar explanation for selecting 40. Moreover, from the viewpoint of the above explanation of the base-60 system, the values 20 and 40 are not good at all: for example, if the Romans selected 24 or 30 instead of 20, they would have had many more divisors.

What we do in this paper. In this paper, we provide a possible explanation for all three number bases – an explanation based on analyzing practical problems that ancient and medieval folks faced.

2 Analysis of the Problem and the Resulting Explanation

Practical problem: management. Ancient and medieval civilizations had many activities involving large groups of people: from army to construction. The possibilities to undertake big construction projects – e.g., in irrigation or in building a protective fortress – and to have a strong army to make peaceful life possible, these possibilities are one of the main reasons why civilizations appeared in the first place.

When you have a large group of people involved in a certain activity, it is important to manage them properly.

- This problem is not as acute in the army, where the soldiers are trained to follow orders – and thus, to be managed.
- However, effective management is crucial in civilian projects, when most workers do not have special training in following orders. These workers need to be organized, and there is a need to have managers (overseers) for overseeing the organized groups of workers.

When the overall number of workers is very large, it is not enough to simply organize workers in groups – there will still be too many groups. So we need to combine groups into groups of higher level – in other words, we need to have a hierarchical organization.

Let us start at the lowest level of the hierarchy. On the lowest level of the hierarchy, we need to combine workers into working groups. How many people can one boss effectively oversee? To answer this question, we need to take into account that, according to psychology, there is a "seven plus minus two" law, according to which a person can only keep between $7-2=5$ and $7+9$ ideas in mind; how many depends on the person:

- some can only keep 5,
- some can keep 9;

see, e.g., [2, 8, 9, 10].

- So, to make sure that any person can serve as a supervisor of such lower-level group, we need to make sure that this group contains no more than 5 people – otherwise people who can only keep 5 ideas in their mind at the same time will not be able to effectively supervise this group.
- On the other hand, everyone can keep 5 ideas, so it will be a waste of resources to make these primary groups with fewer then 5 folks.

Thus, the ideal size of the primary group is 5.

Comment. This argument shows that it is reasonable to expect base-5 number systems. Such systems have actually been used by several cultures; see, e.g., [4].

Second level of the hierarchy. As we have mentioned earlier, even if we divide thousands of workers into groups of 5, we will get many groups. So, to effectively supervise these primary groups, we need to combine them into secondary groups.

How many primary groups should we combine into a secondary one? It is much more difficult to be a boss of bosses than simply a low-level boss of people. Because of this increased difficulty, the number of primary groups combined into a secondary group should be smaller than 5 – the number of people in each primary group. So, we have 3 options:

- we can have 4 groups of 5, making up 20 – which explains the base-20 system; actually, the Mayans explicitly considered 20 as 4 groups of 5;
- we can have 3 groups of 5, making up 15; historically, there is no direct evidence of base-15 systems, but there is an indirect evidence: e.g., Russia used to have 15-kopeck coins, a very unusual nomination;
- we can have 2 groups of 5, making up 10; this is our usual decimal system; its representation as two groups of 5 can be seen, e.g., in the design of the abacus; see, e.g., [3, 5].

Third level. On the next level, it is even more difficult to manage, so the number of secondary groups that form a ternary group must be smaller than the number of primary groups in a secondary group. Here:

- For $10 = 2 \cdot 5$, there is no possibility to have fewer than 2 secondary groups.
- For $15 = 3 \cdot 5$, the only option is having 2 groups of 15 together, making it $2 \cdot 15 = 30$. There does not seem to be any evidence of any culture using base-30 number systems.
- For $20 = 4 \cdot 5$, we have two options:
  - having 3 groups of 20, making it $3 \cdot 20 = 60$; and
  - having 2 groups of 20, making it $2 \cdot 20 = 40$.

The last two options provide an explanation of why 60 and 40 were used as bases.

Why 60 in Babylon, 40 in Russia, and 20 in Europe: brainstorming. The above arguments explain why 20, 40, and 60 were used as bases, but do not explain why different systems appeared in different countries – this requires going beyond mathematics, to history. We are not historians, but we can try to speculate.

Our speculation is based on the natural idea that the more obedient people are, the less they rebel, the easier it is to control them, and thus, the larger ternary groups can be formed.

From this viewpoint:

- Babylonia was ruled by mighty rulers for several centuries, so it could perform a control of the largest number of 20-size groups supervised by one person: 3. This explains why the corresponding value $3 \cdot 20 = 60$ was used in Babylonia.
- Medieval Russia was also ruled with a heavy hand, but there were still many riots and uprisings. So, it could afford only the smaller number of 20-size groups supervised by one person: 2. This explains why the corresponding value $2 \cdot 20 = 40$ was used in Russia.
- Finally, the Roman Empire was the site of many uprisings and revolts. This kind of explains why even combining two 20-size groups under one person was difficult – and this is why the ancient Romans only used base-20 system.

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