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# The P-Laplacian Problem Via An Euler Equation And The Basis Properties Of Its Eigenfunctions

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THE  $p$ -LAPLACIAN PROBLEM VIA AN EULER EQUATION AND THE BASIS  
PROPERTIES OF ITS EIGENFUNCTIONS

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PROPERTIES OF ITS EIGENFUNCTIONS

by

LUIS R. SUAREZ SALAS, B.S.

THESIS

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## Abstract

The Laplacian operator is used in many fields of science, such as fluidodynamics, mechanics and elasticity. Mathematically, much research has been devoted to develop a theory with which it and other variations can be understood. In this work, we present the  $p$ -Laplacian problem via an Euler equation. We then study the properties of its eigenfunctions which generalize the trigonometric functions sine and cosine. In connection with a Fourier series, we then show the generalized trigonometric functions possess basis properties for  $L^r((0, 1)^d)$ ,  $d = 1, 2, 3$ . Finally, we introduce the spaces of variable exponent and the analogue  $p(x)$ -Laplacian problem which has immense applications such as in image restoration and in the modeling of electrorheological fluids.

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## Definitions and Basic Results

We begin by stating definitions and basic results which will be used in the sequel.

### 1.1 Normed Spaces

A **vector space** over a field  $\mathbb{F}$  is a set  $V$  of objects called vectors along with two operations: addition of vectors  $+ : V \times V \rightarrow V$  and multiplication of vectors by scalars  $\bullet : \mathbb{F} \times V \rightarrow V$ . These operations satisfy the following conditions:

1. Addition is commutative and associative;
2. There is a zero vector  $0$  in  $V$  such that  $v + 0 = v$  for each vector  $v$ ;
3. For each vector  $v$  there is a vector  $-v$  such that  $v + (-v) = 0$ ;
4. For all scalars  $\alpha$  and  $\beta$  and all vectors  $x$  and  $y$ ,  $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ ,  $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$ , and  $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ ;
5. for each vector  $x$ ,  $1 \cdot x = x$ .

A **subspace** of  $V$  is then a subset of  $V$  that satisfies the above conditions under the same two operations.

We can associate to each vector a real number by introducing the concept of norm which generalizes the notion of length and size. A **norm**  $\|\cdot\|$  on  $V$  is a real valued function  $\|\cdot\| : V \rightarrow \mathbb{R}$  which satisfy the following conditions for all  $x$  and  $y$  in  $V$  and each scalar  $\alpha$ :

1.  $\|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x = 0$ ;
2.  $\|\alpha x\| = |\alpha| \|x\|$ ;
3.  $\|x + y\| \leq \|x\| + \|y\|$ .

The resulting ordered pair  $(V, \|\cdot\|)$  is called a **normed space**.

A norm induces the metric  $d$  defined by  $d(x, y) := \|x - y\|$ ,  $x, y \in V$ . In general a **metric**  $d : V \times V \rightarrow \mathbb{R}$  satisfies the following three conditions for all  $x, y$ , and  $z$  in  $V$ :

1.  $d(x, y) \geq 0$ , and  $d(x, y) = 0$  if and only if  $x = y$ ;
2.  $d(x, y) = d(y, x)$ ;
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

One can verify using the norm properties that the formula  $\|x - y\| = d(x, y)$  indeed induces a metric. Similarly, a set  $V$  with a defined metric  $d$  is called a **metric space** and can be denoted as  $(V, d)$ .

The **norm topology** on  $V$  is the topology obtained from the induced metric. Namely, we define the **open ball** for  $x \in V$  and  $r > 0$  as  $\{y : y \in V, d(x, y) < r\}$ . The **closed ball** is then  $\{y : y \in V, d(x, y) \leq r\}$ . Using the norm, we can similarly define the **closed unit ball** in  $V$ , denoted by  $B_V$ , as  $\{y : y \in V, \|y\| \leq 1\}$ . The **unit sphere** of  $V$  is  $\{y : y \in V, \|y\| = 1\}$  and is denoted by  $S_V$ .

A normed space,  $V$ , is called complete or a **Banach Space** if every Cauchy sequence in  $V$  converges inside of  $V$  with the induced metric. This means that for a Cauchy sequence  $\{x_n\}$  in  $V$ , there exists an  $x \in V$  for which the following holds

$$\lim_{n \rightarrow \infty} \|x_n - x\| = \lim_{n \rightarrow \infty} d(x_n, x) = 0.$$

We conclude this section with important properties of the norm.

**Proposition.** *Let  $X$  be a normed space.*

1. *The function  $x \rightarrow \|x\|$  is continuous from  $X$  into  $\mathbb{R}$ .*
2. *Addition of vectors is a continuous operation from  $X \times X$  into  $X$ .*
3. *Multiplication of vectors by scalars is a continuous operation from  $\mathbb{F} \times X$  into  $X$ .*

These results follow directly from the three conditions of the norm. We rephrase these results in the following corollary.

**Corollary.** *Let  $X$  be a normed space. If  $x, y \in X$  and  $\alpha_0$  is a nonzero scalar, the maps  $x \rightarrow x + y$  and  $x \rightarrow \alpha_0 x$  are homeomorphisms.*

This results tell us that if  $V$ , a subset of  $X$ , is open, closed, or compact, any translation or dilation will preserve this property i.e. if  $x_0 \in X$ , then  $x_0 + V$  and  $\alpha_0 V$  will be open, closed, or compact when  $V$  is. Using the continuity of the norm we can also obtain the following results. In them, we note the summation is the vector space addition operation defined in the space between elements in the vector space. Also, the convergence is in the norm, i.e. for  $\{x_n\}$  and  $y$  in the vector space  $X$ , the expressions  $\sum_n x_n \rightarrow y$  means  $\|\sum_n x_n - y\|$  tends to zero as  $n$  tends to infinity.

**Proposition.** *Let  $X$  be a normed space.*

1. *If  $\sum_n x_n$  converges in  $X$ , then  $x_n \rightarrow 0$ .*
2. *If  $\sum_n x_n$  and  $\sum_n y_n$  both converge in  $X$ , then so does  $\sum_n (x_n + y_n)$ . Furthermore,  $\sum_n (x_n + y_n) = \sum_n x_n + \sum_n y_n$ .*
3. *If  $\sum_n x_n$  converges in  $X$  and  $\alpha$  is a scalar, then  $\sum_n \alpha x_n$  converges to  $\alpha \sum_n x_n$ .*
4. *If  $\sum_n x_n$  is a sum in  $X$ , then  $\|\sum_n x_n\| \leq \sum_n \|x_n\|$ .*

We close this subsection by introducing the topological notion of convexity. Let  $V$  be a subset of a vector space  $X$ . We say  $V$  is **convex** if  $x, y \in V$  imply  $tx + (1 - t)y \in V$  for any  $t \in (0, 1)$ .

**Proposition.** .

1. *Arbitrary intersections of convex sets are convex.*
2. *Translates and scalar multiples of convex sets are convex.*

3. The set  $V$  is convex if and only if for  $s, t > 0$ ,  $sV + tV = (s + t)V$ .

**Proposition.** Let  $X$  be a normed space.

1. If  $V$  is a convex subset of  $X$  then both the closure and interior of  $V$  are convex.
2. Every ball in  $X$ , open or closed, is convex.

Indeed, for  $x, y$  in the closure of  $V$  it suffices to look at a sequence  $\{x_n\}$  and  $\{y_n\}$  tending to  $x$  and  $y$ , respectively. Then, for  $t \in (0, 1)$  it is clear the sequence  $\{tx_n + (1 - t)y_n\}$  tends to  $tx + (1 - t)y$  and hence it is in the closure. Hence, the closure of  $V$  is convex. For the interior of  $V$ ,  $V^\circ$ , we note for  $t \in (0, 1)$  that  $tV^\circ + (1 - t)V^\circ \subseteq tV + (1 - t)V = V$ . Furthermore,  $tV^\circ + (1 - t)V^\circ$  is an open set, which implies  $tV^\circ + (1 - t)V^\circ \subseteq V^\circ$ , hence the interior of  $V$  is convex. For the second result, we denote by  $B(x, r)$  the open ball centered at  $x \in X$  with radius  $r > 0$ . If  $y, z \in B(x, r)$  and  $t \in (0, 1)$

$$\|ty + (1 - t)z\| = \|ty + (1 - t)z - tx - (1 - t)x\| \leq t\|y - x\| + (1 - t)\|z - x\| < r.$$

Therefore,  $ty + (1 - t)z \in B(x, r)$ , and so it is convex. A similar result follows for a closed ball.

## 1.2 Linear Operators

Suppose  $X$  and  $Y$  are vector spaces. A **linear transformation** from  $X$  into  $Y$  is a function  $T : X \rightarrow Y$  which satisfies the following two conditions for  $x, z \in X$  and any  $\alpha \in \mathbb{F}$ :

1.  $T(x + z) = T(x) + T(z)$ ;
2.  $T(\alpha x) = \alpha T(x)$ .

The kernel or null space of a linear operator  $T$ , is defined as  $\ker(T) = \{x : x \in X, Tx = 0\}$ . The rank of a linear operator is the dimension of the range, which is the set  $R(T) = \{T(x) : x \in X\}$ . The range forms a subspace of  $Y$ . A set is called bounded if it is contained inside of

a ball. For a linear operator  $T : X \rightarrow Y$ , we say  $T$  is **bounded** if  $T(A)$  is a bounded subset of  $Y$  whenever  $A$  is a bounded subset of  $X$ . Furthermore, the collection of all bounded linear operators from  $X$  into  $Y$  is denoted by  $B(X, Y)$ . If  $X = Y$ , then just by  $B(X)$ . We say  $T$  is continuous if for any  $x \in X$  and  $\epsilon > 0$  there exists  $\delta > 0$  such that for  $y \in X$

$$\|x - y\|_X < \delta \quad \text{implies} \quad \|T(x) - T(y)\|_Y < \epsilon.$$

The following are important equivalent properties of a linear operator [1],[3],[4].

**Theorem.** *Let  $X$  and  $Y$  be normed spaces and let  $T : X \rightarrow Y$  be a linear operator. Then the following are equivalent.*

1. *The operator  $T$  is continuous at 0.*
2. *The operator  $T$  is continuous.*
3. *The operator  $T$  is bounded.*
4. *There is a nonnegative real number  $M$  such that  $\|Tx\|_Y \leq M\|x\|_X$  for each  $x \in X$ .*
5. *The quantity  $\sup\{\|Tx\|_Y : x \in B_x\}$  is finite.*

This theorem tells us that the space of continuous linear functions from  $X$  into  $Y$  is the same as  $B(X, Y)$ . In addition, it suggests a norm structure on  $B(X, Y)$ . We first treat this set as a vector space with the usual addition of functions and multiplication by scalars. Then, motivated by our last result, we define the **operator norm**, for  $T \in B(X, Y)$ , as

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

Homogeneity follows from the properties of the supremum, namely for scalar  $\alpha$

$$\|\alpha T\| = \{\|\alpha Tx\| : x \in B_X\} = |\alpha|\{\|Tx\| : x \in B_X\} = |\alpha|\|T\|.$$

Applying our last theorem, we have for  $S, T \in B(X, Y)$  and  $x \in B_X$

$$\|(S + T)(x)\| \leq \|S\|\|x\| + \|T\|\|x\| \leq (\|S\| + \|T\|)\|x\|.$$

Thus,

$$\|S + T\| = \sup\{\|(S + T)(x)\| : x \in B_X\} \leq \|S\| + \|B\|$$

showing the triangle inequality. Finally, it is clear that if  $\|T\| = 0$ ,  $T$  is the zero operator since for any  $x \in X/\{0\}$

$$\|x\|^{-1}T(x) = T(\|x\|^{-1}x) = 0$$

implying  $T(x) = 0$ . Hence, this is indeed a norm. In fact, the operator norm can be define in different ways, as the following proposition states [1].

**Proposition.** *Let  $X$  and  $Y$  be vector spaces. Suppose  $T \in B(X, Y)$ , then:*

1.  $\|T\| = \sup\{\|Tx\| : x \in X, \|x\| < 1\}$ ;
2.  $\|T\| = \sup\{\|Tx\| : x \in S_X\}$ ;
3. *if*  $x \in X, \|Tx\| \leq \|T\|\|x\|$ .
4.  $\|T\| = \sup\{\frac{\|Tx\|}{\|x\|} : x \in X, x \neq 0\}$ , *i.e.*  $\|T\|$  *is the smallest nonnegative real number*  $M$  *such that*  $\|Tx\| \leq M\|x\|$  *for*  $x \in X$ .

A linear operator,  $T$ , from a normed space  $X$  into the normed space  $Y$  is called an **isomorphism** into  $Y$  if it is one-to-one and continuous and its inverse mapping  $T^{-1}$  is continuous on the range of  $T$ . It is an **isometric isomorphism** if  $\|Tx\|_Y = \|x\|_X$  whenever  $x \in X$ . We say  $X$  and  $Y$  are **isomorphic** if there is an isomorphism from  $X$  onto  $Y$ , denoted by  $X \cong Y$ . They are **isometrically isomorphic** if there is an isometric isomorphism from  $X$  onto  $Y$ . We conclude this section with a theorem on linear operators of finite dimensional normed spaces and with a theorem characterizing finite dimensional normed spaces [1].

**Theorem.** *Let  $X$  and  $Y$  be normed spaces such that  $X$  is finite dimensional. Then every linear operator from  $X$  into  $Y$  is bounded.*

**Theorem.** *A normed space  $X$  is finite dimensional if and only if it has the Heine-Borel property, which happens if and only if  $S_X$  is compact.*

### 1.3 Dual Space

In this section we introduce a special class of linear operators. Let  $X$  be a vector space, we denote by  $X^\#$  the vector space of all linear functionals, i.e. linear operators from  $X$  to its field of scalars. In normed spaces, we similarly look for the linear functionals but reserve the definition for the bounded ones. Then, if  $X$  is a normed space, the **dual space** of  $X$  is the normed space  $B(X, \mathbb{F})$  with the operator norm defined previously. It is denoted by  $X^*$ . By the previous theorem, we note  $X^\# = X^*$  whenever  $X$  is finite dimensional. From the completeness of a scalar field, we get the following result [1],[3].

**Theorem.** *If  $X$  is a normed space, then  $X^*$  is a Banach space.*

We can also extend results from linear algebra to connect the dimensions between a normed space and its dual space [1],[2]:

**Theorem.** *A normed space is finite dimensional if and only if its dual space is finite-dimensional.*

Suppose  $x \in X$ , a normed space, and  $x^* \in X^*$ . Then  $x^*(x) \in \mathbb{F}$ , hence the norm on  $X^*$  is just the absolute value. Furthermore, computing the operator norm, we find for  $x^* \in X^*$

$$\|x^*\| = \sup\{|x^*x| : x \in B_X\}.$$

By the continuity of the absolute value, we could conclude there is an  $x \in B_X$  for which  $\|x^*\| = |x^*x|$  if  $B_X$  were compact. Such is the case when  $X$  is finite dimensional. Since the supremum is attained for all the elements of  $X^*$ , we call them **norm attaining** functionals. In fact, even in the infinite dimensional case, there are still norm attaining functionals in  $X^*$ . To obtain this result and an analogous expression for the norm of  $x$ , we need the normed space version of the Hahn-Banach Extension Theorem [1],[3],[4]:

**Theorem.** *Suppose that  $f_0$  is a bounded linear functional on a subspace  $Y$  of a normed space  $X$ . Then there is a bounded linear functional  $f$  on  $X$  such that  $\|f\| = \|f_0\|$  and  $f$  agrees with  $f_0$  on  $Y$ .*

Applying this result, it can be shown there are norm attaining functionals. This is obtained as the following corollary which will be refined in the next section [1],[3].

**Corollary.** *If  $x$  is a nonzero element of a normed space  $X$ , then there is a bounded linear functional  $f$  on  $X$  such that  $\|f\| = 1$  and  $f(x) = \|x\|$ .*

Letting  $x \in S_X$  will give norm attaining functionals. The Hahn-Banach Extension Theorem also gives the following analogues result [1],[3].

**Theorem.** *Let  $x \in X$ , a normed space, then*

$$\|x\| = \sup\{|x^*x| : x^* \in B_{X^*}\}$$

*where the supremum is attained for some  $x^* \in B_{X^*}$ .*

To make notation more precise, we adopt the bracket notation to express the operation of a linear functional on an element of a normed space. For  $x \in X$  and  $x^* \in X^*$ , we let  $x^*(x) = \langle x, x^* \rangle$ . For  $T \in B(X, Y)$ , we define the map  $T^*$  from  $Y^*$  to  $X^*$ , called the **adjoint** of  $T$ , by the formula

$$\langle x, T^*y^* \rangle = \langle Tx, y^* \rangle,$$

for  $x \in X$  and  $y^* \in Y^*$ . It can be shown that  $T^* \in B(Y^*, X^*)$  and  $\|T^*\| = \|T\|$  [1],[3]. Assuming  $T$  is an isomorphism gives the following theorem.

**Theorem.** *Suppose that  $X$  and  $Y$  are normed spaces and  $T$  is an isomorphism from  $X$  onto  $Y$ . The map  $T^* : Y^* \rightarrow X^*$  given by  $T^*y^* = y^*T$ , is an isomorphism from  $Y^*$  onto  $X^*$ , and  $\|T^*\| = \|T\|$ . If  $T$  is an isometric isomorphism, then so is  $T^*$ .*

If additionally  $X$  and  $Y$  are Banach Spaces, we get an isomorphic relationship between  $T$  and its transpose  $T^*$  [1].

**Theorem.** *Suppose that  $X$  and  $Y$  are Banach spaces and that  $T \in B(X, Y)$ .*

1. *The operator  $T$  maps  $X$  onto  $Y$  if and only if  $T^*$  is an isomorphism from  $Y^*$  onto a subspace of  $X^*$ .*



2. The operator  $T^*$  maps  $Y^*$  onto  $X^*$  if and only if  $T$  is an isomorphism from  $X$  onto a subspace of  $Y$ .

## 1.4 Reflexivity and Compact Operators

The dual space of a normed space  $X$  gives rise to the second dual. We denote by  $X^{**}$  the dual space of  $X^*$ . In the same manner, we inductively define the  $n^{\text{th}}$  dual as  $(X^{(n-1)})^*$ . Letting  $(T(x))(x^*) = x^*x$ , for  $x \in X$  and  $x^* \in X^*$ , we note  $T(x) \in X^{**}$ . In addition,  $T$  is an isometric isomorphism from  $X$  into  $X^{**}$ . The map  $T$  is called the **natural map** from  $X$  into  $X^{**}$ . If the natural map is onto we say the normed space  $X$  is **reflexive**. This condition gives the following results [1].

**Theorem.** *Every reflexive normed space is a Banach space.*

**Proposition.** *Let  $X$  be a reflexive normed space. Then every member of  $X^*$  is norm attaining.*

We present important characterizations of reflexivity that will be used in the next section. First, we define the concept of weak convergence. If  $x$  is an element and  $(x_n)$  a sequence in a normed space  $X$ , we say  $(x_n)$  **converges weakly** to  $x$  if  $\langle x_n, x^* \rangle \rightarrow \langle x, x^* \rangle$  whenever  $x^* \in X^*$ . We use the half arrow to denote weak convergence,  $x_n \rightharpoonup x$ . By the Hahn-Banach theorem, for  $x, y \in X$ ,  $x \neq y$ , there is a bounded linear functional  $f$  on  $X$  such that  $f(x) \neq f(y)$  (see, [1]). Hence, a weak limit is unique. Indeed, suppose  $x_n \rightharpoonup x$  and  $x_n \rightharpoonup y$ . Since this implies  $x^*x = x^*y$  for all  $x^* \in X^*$ , it must hold for  $f$ , hence  $x = y$ . Furthermore, it can be shown that convergence in the norm implies weak convergence. The following result connects weak convergence with reflexivity [1].

**Theorem.** *A normed space is reflexive if and only if each of its bounded sequences has a weakly convergent subsequence.*

We now define the notion of a compact operator between two Banach spaces. Let  $X$  and  $Y$  be two Banach spaces. A linear operator  $T$  from  $X$  into  $Y$  is **compact** if the

closure of  $T(A)$  is a compact subset of  $Y$  whenever  $A$  is a bounded subset of  $X$ . We denote by  $K(X, Y)$  the collection of all compact linear operators from  $X$  into  $Y$ . Note that in general, a compact operator is bounded, hence  $K(X, Y) \subset B(X, Y)$ . We have the following equivalent definition for a compact operator [23], [24].

**Theorem.** *Let  $T$  be a linear operator from a Banach space  $X$  into a Banach space  $Y$ . Then,  $T$  is compact if and only if every bounded sequence  $(x_n)$  in  $X$  has a subsequence  $(x_{n_j})$  such that the sequence  $(Tx_{n_j})$  converges.*

Furthermore, the collection of compact linear operators is closed in  $B(X, Y)$ .

**Corollary.** *If  $X$  and  $Y$  are Banach space, then  $K(X, Y)$  is a closed subspace of  $B(X, Y)$ .*

The following result will be used in the sequel [26].

**Theorem.** *A compact operator  $T$  maps weakly convergent sequences into norm convergent sequences, that is if  $x_n \rightharpoonup x$  then  $T(x_n) \rightarrow T(x)$ .*

## 1.5 Strict Convexity and the Gâteaux derivative

The unitary ball of a normed space is related to different properties of the space. It also plays an important role when defining the operator norm. In this section we introduce the notion of a strictly convex space by imposing additional conditions on the unit sphere. This will lead to the concept of smoothness which is connected to the Gâteaux derivative. A normed space  $X$  is **strictly convex** if for  $t \in (0, 1)$

$$\|tx_1 + (1 - t)x_2\| < 1$$

whenever  $x_1, x_2 \in S_X$  and  $x_1 \neq x_2$ . This condition is equivalent to saying there are no non trivial line segments in the unit sphere [5], [20]. In fact, we can equivalently define strict convexity by the condition

$$\left\| \frac{1}{2}(x_1 + x_2) \right\| < 1$$

whenever  $x_1, x_2 \in S_X$  and  $x_1 \neq x_2$ . Indeed, letting  $t = \frac{1}{2}$  in the first definition gives the second one. For the converse direction, we note

$$\|(1 - 2t)x_2 + 2t\frac{x_1 + x_2}{2}\| = \|tx_1 + (1 - t)x_2\| = \|(2t - 1)x_1 + (2 - 2t)\frac{x_1 + x_2}{2}\|.$$

Applying the triangle inequality to the left side when  $t \in (0, \frac{1}{2})$  or to the right side when  $t \in (\frac{1}{2}, 1)$  shows the second definition implies the first one. From this reformulation, it can be shown that this property is conserved under an isometrically isomorphism. It is in general not simply isomorphic invariant. To link strictly convexity to smoothness we first need the concept of a support hyperplane. Let  $A$  be a subset of a normed space  $X$ . A nonzero  $x^*$  in  $X^*$  is a **support functional** for  $A$  if there is an  $x_0$  in  $A$  such that  $\operatorname{Re}\langle x_0, x^* \rangle = \sup\{\operatorname{Re}\langle x, x^* \rangle : x \in A\}$ , in which case  $x_0$  is a **support point** of  $A$ . The set  $\{x : x \in X, \operatorname{Re}\langle x, x^* \rangle = \operatorname{Re}\langle x_0, x^* \rangle\}$  is a **support hyperplane** for  $A$ , and the functional  $x^*$  and the support hyperplane are both said to **support**  $A$  at  $x_0$  [1], [5]. For example, in  $\mathbb{R}^2$  with the Euclidean norm, we look at the disk of radius 2,  $A = \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$ . Defining the functional, for  $x \in \mathbb{R}^2$ , by the formula  $f_1(x) = x_1$ , we note it is a support functional for  $A$  since for  $x_0 = (2, 0)$ ,

$$f_1(x_0) = \sup\{f_1(x) : x \in A\}.$$

Thus,  $(2, 0)$  is a support point of  $A$  and the set  $\{(2, \lambda) : \lambda \in \mathbb{R}\}$  is a support hyperplane for  $A$ . Similarly, defining the functional for  $x \in \mathbb{R}^2$ , by the formula  $f_2(x) = x_2$  we note it supports  $A$  at  $(0, 2)$  with support hyperplane  $\{(\lambda, 2) : \lambda \in \mathbb{R}\}$ .

**Theorem.** *A normed space  $X$  is strictly convex if and only if each support hyperplane for  $B_X$  supports  $B_X$  at only one point.*

As noted earlier, the unit sphere plays an important role for the space. In particular its shape. If there are no corners or sharp bends in it we will denote it as smooth. We now make this definition precise. Let  $x_0$  be an element in the unit sphere of  $X$ , a normed space. We say  $x_0$  is a **point of smoothness** of  $B_X$  if there is exactly one support hyperplane

for  $B_X$  that supports  $B_X$  at  $x_0$ . The space  $X$  is **smooth** if each point of  $S_X$  is a point of smoothness of  $B_X$ . Hence, assuming smoothness makes norm attaining functionals in the sphere unique. This result will be used in the next section [1], [5], [20].

**Corollary.** *Suppose that  $X$  is a smooth normed space. Then, for each  $x \in S_X$ , there is a unique  $x^*$  in  $S_X^*$  such that  $x^*x = 1$ .*

Analogues to the one for strictly convex spaces, we have an isometrically isomorphic result.

**Proposition.** *Every normed space that is isometrically isomorphic to a smooth normed space is itself smooth.*

We present three results that relate strict convexity with smoothness [1], [5], [20].

**Proposition.** *A normed space is smooth if its dual space is strictly convex.*

**Proposition.** *A normed space is strictly convex if its dual space is smooth.*

**Proposition.** *A reflexive normed space is strictly convex if and only if its dual space is smooth, and is smooth if and only if its dual space is strictly convex.*

Smoothness is also related to the Gâteaux derivative. Similarly to the derivative being connected to the smoothness of a real valued function in the plane, the Gâteaux derivative is another way of identifying the smoothness of the space. We define this derivative in terms of the right and left hand ones, if they coincide we then let this be the derivative. Let  $X$  be a normed space,  $x_0 \in S_X$  and  $y_0 \in X$ . Define

$$G_-(x_0, y_0) = \lim_{t \rightarrow 0^-} \frac{\|x_0 + ty_0\| - \|x_0\|}{t}$$

and

$$G_+(x_0, y_0) = \lim_{t \rightarrow 0^+} \frac{\|x_0 + ty_0\| - \|x_0\|}{t}.$$

Then  $G_-(x_0, y_0)$  and  $G_+(x_0, y_0)$  are, respectively, the **left hand** and **right hand Gâteaux derivative** of the norm at  $x_0$  in the direction  $y_0$ . The norm is **Gâteaux differentiable** at

$x_0$  in the direction  $y_0$  if  $G_-(x_0, y_0) = G_+(x_0, y_0)$ . It is denoted by  $G(x_0, y_0)$  and is referred to the Gâteaux derivative of the norm at  $x_0$  in the direction  $y_0$ . If the norm is Gâteaux differentiable at  $x_0$  in every direction  $y$ , then the norm is Gâteaux differentiable at  $x_0$ . If the norm is Gâteaux differentiable at every point in  $S_X$ , then the norm is said to be Gâteaux differentiable. We conclude this section with a result proved by Banach, which will be used in the next section, followed by a corollary stating the connection with smoothness [1], [5], [20].

**Theorem.** *Suppose that  $X$  is a normed space and that  $x_0 \in S_X$ . Then  $x_0$  is a point of smoothness of  $B_X$  if and only if the norm of  $X$  is Gâteaux differentiable at  $x_0$ . Furthermore, if  $x_0$  is a point of smoothness of  $B_X$  and  $x_0^*$  is the unique member of  $S_{X^*}$  that supports  $B_X$  at  $x_0$ , then the Gâteaux derivative of the norm of  $X$  at  $x_0$  in each direction  $y$  is given by the formula  $G(x_0, y) = R_{x_0^*}y$ .*

**Corollary.** *A normed space is smooth if and only if its norm is Gâteaux differentiable.*

We conclude by giving some examples of smooth and non-smooth spaces. For  $\mu$  a positive measure on a  $\sigma$ -algebra  $\Lambda$  of subsets of a set  $\Omega$  and for  $1 < p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , the dual space of  $L_p(\Omega, \Lambda, \mu)$  is isometrically isomorphic to  $L_q(\Omega, \Lambda, \mu)$ . Thus,  $L_p(\Omega, \Lambda, \mu)$  is smooth since  $L_q(\Omega, \Lambda, \mu)$  is strictly convex and hence  $(L_p(\Omega, \Lambda, \mu))^*$  is strictly convex as well [1]. Assuming the same measure space, and that  $\Omega$  is composed of the disjoint measurable subsets  $X_1$  and  $X_2$  we define the functionals  $f_1$  and  $f_2$  for  $g \in L_1(\Omega, \Lambda, \mu)$  by

$$f_1(g) = \int_{\Omega} I_{X_1}(x)g(x) + I_{X_2}(x)g(x) dx$$

and

$$f_2(g) = \int_{\Omega} I_{X_1}(x)g(x) - I_{X_2}(x)g(x) dx$$

where  $I_X(x)$  is the indicator function, that is

$$I_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X . \end{cases}$$

The condition

$$\|g\|_{L_1} = \int_{\Omega} |g| dx = 1$$

implies, where the supremum is taken over all  $g \in L_1(\Omega, \Lambda, \mu)$  for which  $\|g\|_{L_1} = 1$ ,

$$\|f_1\| = \sup\left\{\left|\int_{\Omega} g(x)dx\right|\right\} = 1.$$

Similarly, we note

$$\|f_2\| = 1,$$

that is  $f_1$  and  $f_2$  are unit dual vectors in  $(L_1(\Omega, \Lambda, \mu))^*$ . Letting  $g = (\mu(X_1))^{-1}I_{X_1}(x)$  we observe

$$\|g\|_{L_1} = 1$$

and

$$\langle g, f_1 \rangle = \langle g, f_2 \rangle = 1.$$

Hence,  $L_1(\Omega, \Lambda, \mu)$  is not smooth since  $f_1 \neq f_2$ . Similarly, assuming the same measure space, and that  $\Omega$  is composed of the disjoint measurable subsets  $X_1$  and  $X_2$  we define the functionals  $g_1$  and  $g_2$  for  $f \in L_{\infty}(\Omega, \Lambda, \mu)$  by

$$g_1(f) = \int_{\Omega} (\mu(X_1))^{-1}I_{X_1}(x)f(x) dx$$

and

$$g_2(f) = \int_{\Omega} (\mu(X_2))^{-1}I_{X_2}(x)f(x) dx.$$

Taking the supremum over all  $f \in L_{\infty}(\Omega, \Lambda, \mu)$  for which

$$\|f\|_{L_{\infty}} = \inf\{\alpha : \mu\{x : |f(x)| > \alpha\} = 0\} = 1,$$

it follows

$$\|g_1\| = \sup\{|\langle f, g_1 \rangle|\} = 1$$

and

$$\|g_2\| = \sup\{|\langle f, g_2 \rangle|\} = 1.$$

Furthermore, letting  $f(x) = I_\Omega(x)$  we note

$$\|f\|_{L_\infty} = \inf\{\alpha : \mu\{x : |I_\Omega(x)| > \alpha\} = 0\} = 1$$

and

$$\langle f, g_1 \rangle = \langle f, g_2 \rangle = 1.$$

By the same argument used for  $L_1(\Omega, \Lambda, \mu)$ , we conclude  $L_\infty(\Omega, \Lambda, \mu)$  is also not smooth.

## Singular Values

Singular values are well known in Hilbert Spaces. For a compact operator  $T : H \rightarrow H$ ,  $H$  a Hilbert space,  $T^*T$  and  $TT^*$  are both compact and self adjoint. By the Spectral Theorem (see, [23],[24]), there exists an orthonormal sequence  $\{x_n\}$  in  $H$  and a sequence of scalars  $\{\lambda_n\}$  such that for  $h \in H$ ,

$$T^*Th = \sum_{n=1}^{\infty} \lambda_n \langle h, x_n \rangle x_n.$$

The **singular values** of  $T$  are  $\lambda_n^{1/2}$ . In particular, if  $T$  is compact and self adjoint, the singular values are the absolute values of the eigenvalues. This section is based on [6], in it we investigate the existence of singular values in more general Banach spaces. We then use the results found to introduce the  $p$ -Laplacian problem [5].

### 2.1 Existence of Extremals

Our first result proves the existence of an element which can be used to obtain the operator norm. We then demonstrate examples of reflexive spaces and compact operators.

**Theorem 1.** *Let  $B_1, B_2$  be Banach spaces, with  $B_1$  reflexive, and suppose  $T : B_1 \rightarrow B_2$  is a compact operator. Then there exists a non-zero element  $f \in B_1$  such that*

$$\| T \| = \frac{\| Tf \|_{B_2}}{\| f \|_{B_1}}.$$

*Proof.* By the definition of the operator norm, we can find a bounded sequence  $\{f_n\}$  in the interior of the unitary ball of  $B_1$  such that

$$\| T \| - \frac{1}{n} \leq \frac{\| Tf_n \|}{\| f_n \|}.$$



Since  $B_1$  is reflexive, we can find a subsequence  $\{g_n\}$  of  $\{f_n\}$  which converges weakly to a  $g \in B_1$ , that is

$$g_n \rightharpoonup g.$$

By the compactness of  $T$ ,

$$T(g_n) \rightarrow T(g).$$

Since  $\{g_n\}$  is a subsequence of  $\{f_n\}$

$$\|T\| \leq \|Tg_n\| + \frac{1}{n}.$$

Taking the limit as  $n \rightarrow \infty$  and using the continuity of the norm we obtain

$$\|T\| \leq \lim_{n \rightarrow \infty} \|Tg_n\| = \|\lim_{n \rightarrow \infty} Tg_n\| = \|Tg\| \leq \frac{\|Tg\|}{\|g\|}$$

but by definition

$$\frac{\|Tg\|}{\|g\|} \leq \|T\|.$$

Thus,

$$\|T\| = \frac{\|Tg\|}{\|g\|}.$$

□

We note that by this result, for real  $\epsilon$ ,

$$\frac{\|T(f + \epsilon\phi)\|}{\|f + \epsilon\phi\|} \rightarrow \|T\|$$

as  $\epsilon \rightarrow 0$  and  $\phi \in B_1$ . In particular, if the derivative exists,

$$\left. \frac{d}{d\epsilon} \frac{\|T(f + \epsilon\phi)\|}{\|f + \epsilon\phi\|} \right|_{\epsilon=0} = 0$$

since this is a point of maximum value.

We now present examples of reflexive Banach spaces. Let  $(\Omega, \Lambda, \mu)$  be a positive measure space. For  $1 < p < \infty$  the functional space  $L_p(\Omega, \Lambda, \mu)$  of measurable functions with finite valued integral in  $\Omega$ , i.e.  $f \in L_p(\Omega, \Lambda, \mu)$  if

$$\|f\|_p = \left( \int_{\Omega} |f|^p d\mu \right)^{1/p} < \infty,$$

is reflexive (see, [1]). In order for the above to be a norm, equivalence is treated almost everywhere. By these we mean two functions are equal if they differ in a set of measure zero. The reflexivity of these spaces follows from the pairing of elements of the dual of  $L_p$  with  $L_{p'}$ ,  $p' = \frac{p}{p-1}$ . Furthermore, since finite dimensional normed spaces have the same dimension dual, it follows that they are also reflexive.

We now consider two examples of compact operators [23], [24]. Let  $M(x, y)$  be a complex-valued continuous function defined for  $-\infty < a \leq x, y \leq b < \infty$ . The operator  $M : C[a, b] \rightarrow C[a, b]$  defined by

$$(Mf)(x) = \int_a^b M(x, y)f(y)dy$$

is compact. Similarly, for a measurable complex valued function  $M(x, y)$  on the measure space  $(\Omega, \Lambda, \mu)$  for which

$$\int_{\Omega} |M(x, y)|^2 d\mu(x, y) < \infty$$

we define the operator  $M : L^2(\Omega) \rightarrow L^2(\Omega)$

$$(Mf)(x) = \int_{\Omega} M(x, y)f(y) d\mu(y).$$

$M$  is compact and is defined by the kernel  $M(x, y)$ .

## 2.2 Euler's Equation

In this subsection we show the existence of the derivative mentioned before. By letting the dual of the working spaces be strictly convex we are able to do so. Furthermore, we show the extremal from the previous result satisfies an Euler equation. For the theorem, we denote by  $L_i(g)$  the unique dual vector to  $g \in B_i$ , that is

$$\langle g, L_i(g) \rangle = \|g\|.$$

**Theorem 2.** *If  $T \neq 0$  and  $f$  is an extremal according to Theorem 1, with  $B_1^*$ ,  $B_2^*$  being strictly convex, then  $f$  satisfies the Euler equation*

$$T^*L_2(Tf) = \mu L_1(f)$$

where  $\mu = \frac{\|Tf\|_{B_2}}{\|f\|_{B_1}}$ .

Conversely, if  $f \neq 0$  satisfies the Euler equation above for some  $\mu$ , then  $\mu > 0$  and

$$\|Tf\| = \mu \|f\|.$$

*Proof.* Consider for  $\epsilon \in \mathbb{R}$

$$H(\epsilon) = \frac{\|T(f + \epsilon\phi)\|}{\|f + \epsilon\phi\|}.$$

Since

$$H'(\epsilon) = \lim_{t \rightarrow 0} \frac{H(\epsilon + t) - H(\epsilon)}{t},$$

we have

$$\begin{aligned} H'(0) &= \lim_{t \rightarrow 0} \frac{H(t) - H(0)}{t} = \lim_{t \rightarrow 0} \frac{\frac{\|T(f+t\phi)\|}{\|f+t\phi\|} - \frac{\|Tf\|}{\|f\|}}{t} \\ &= \lim_{t \rightarrow 0} \frac{\|T(f+t\phi)\| \|f\| - \|f+t\phi\| \|Tf\|}{t(\|f+t\phi\| \|f\|)}. \end{aligned}$$

Adding and subtracting  $\|f\| \|Tf\|$  we obtain the limit as  $t \rightarrow 0$  of

$$\begin{aligned} &\frac{\|T(f+t\phi)\| \|f\| + \|Tf\| \|f\| - \|Tf\| \|f\| - \|f+t\phi\| \|Tf\|}{t(\|f+t\phi\| \|f\|)} \\ &= \frac{\|f\| (\|T(f+t\phi)\| - \|Tf\|) - \|Tf\| (\|f+t\phi\| - \|f\|)}{t(\|f+t\phi\| \|f\|)}. \end{aligned}$$

Thus, the limit becomes

$$\begin{aligned} &= \lim_{t \rightarrow 0} \left[ \left( \frac{\|f\|}{\|f+t\phi\| \|f\|} \right) \left( \frac{\|Tf+tT\phi\| - \|Tf\|}{t} \right) \right. \\ &\quad \left. - \left( \frac{\|Tf\|}{\|f+t\phi\| \|f\|} \right) \left( \frac{\|f+t\phi\| - \|f\|}{t} \right) \right]. \end{aligned}$$

Since  $B_1^*$  and  $B_2^*$  are strictly convex,  $B_1$  and  $B_2$  are smooth and hence their norms are Gateaux differentiable. Letting  $Tx = \frac{Tf}{\|Tf\|}$  and  $Ty = \frac{T\phi}{\|T\phi\|}$ , we compute the limit of the first half of the equation above to be

$$\frac{\|f\|}{\|f\| \|f\|} \|Tf\| G(Tx, Ty) = \frac{\|Tf\|}{\|f\|} G(Tx, Ty).$$

Denoting by  $L_2(Tf)$  the unique dual vector to  $Tx$ , i.e.  $\|L_2(Tf)\| = 1$  and  $\langle Tx, L_2(Tf) \rangle = 1$ , we find

$$\frac{\|Tf\|}{\|f\|} G(Tx, Ty) = \frac{\|Tf\|}{\|f\|} \operatorname{Re} \langle Ty, L_2(Tf) \rangle = \frac{1}{\|f\|} \operatorname{Re} \langle T\phi, L_2(Tf) \rangle.$$

Similarly, letting  $x = \frac{f}{\|f\|}$  and  $y = \frac{\phi}{\|\phi\|}$ , we compute the limit of the second half of the equation to be

$$\frac{\|Tf\|}{\|f\| \|f\|} \|f\| G(x, y) = \frac{\|Tf\|}{\|f\|} G(x, y).$$

Denoting by  $L_1(f)$  the unique dual vector to  $x$ , i.e.  $\|L_1(f)\| = 1$  and  $\langle x, L_1(f) \rangle = 1$ , we find

$$\frac{\|Tf\|}{\|f\|} G(x, y) = \frac{\|Tf\|}{\|f\|} \operatorname{Re} \langle y, L_1(f) \rangle = \frac{\|Tf\|}{\|f\|^2} \operatorname{Re} \langle \phi, L_1(f) \rangle.$$

Note the unique existence of  $L_1(f)$  and  $L_2(Tf)$  are guaranteed by the smoothness of the spaces. We have now shown the derivative of  $H(\epsilon)$  exists at  $\epsilon = 0$ . Since  $f$  is an extremal, the derivative must be zero. We conclude

$$\frac{1}{\|f\|} \operatorname{Re} \langle T\phi, L_2(Tf) \rangle = \frac{\|Tf\|}{\|f\|^2} \operatorname{Re} \langle \phi, L_1(f) \rangle,$$

which implies

$$\operatorname{Re} \langle \phi, T^* L_2(Tf) \rangle = \frac{\|Tf\|}{\|f\|} \operatorname{Re} \langle \phi, L_1(f) \rangle$$

since the adjoint satisfies  $\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle$  for  $x \in B_1$  and  $y^* \in B_2^*$ . We note that for a complex linear functional  $g$ , the following holds [25]

$$g(x) = \operatorname{Re}(g(x)) - i \operatorname{Re}(g(ix)).$$

This implies

$$\begin{aligned} \langle \phi, T^* L_2(Tf) \rangle &= \operatorname{Re} \langle \phi, T^* L_2(Tf) \rangle - i \operatorname{Re} \langle i\phi, T^* L_2(Tf) \rangle \\ &= \frac{\|Tf\|}{\|f\|} \left( \operatorname{Re} \langle \phi, L_1(f) \rangle - i \operatorname{Re} \langle i\phi, L_1(f) \rangle \right) = \frac{\|Tf\|}{\|f\|} \langle \phi, L_1(f) \rangle, \end{aligned}$$

that is

$$T^*L_2(Tf) = \mu L_1(f)$$

with

$$\mu = \| Tf \| / \| f \| .$$

Conversely, assuming the Euler equation and  $x$  and  $Tx$  as before, we first note

$$\|Tf\| = \langle Tf, L_2(Tf) \rangle = \langle f, T^*L_2(Tf) \rangle$$

since  $\langle Tx, L_2(Tf) \rangle = 1$ . Similarly,

$$\|f\| = \langle f, L_1(f) \rangle$$

since  $\langle x, L_1(f) \rangle = 1$ . Using the Euler equation we obtain

$$\langle f, T^*L_2(Tf) \rangle = \mu \langle f, L_1(f) \rangle$$

which implies

$$\|Tf\| = \mu \|f\|$$

and  $\mu \geq 0$  since

$$\mu = \frac{\|Tf\|}{\|f\|}.$$

□

Using these results, we extend the terminology and refer to the  $\mu$  above as an eigenvalue. This and any other positive eigenvalues are called **singular values** of  $T$  [6]. This agrees with standard terminology when  $B_1$  and  $B_2$  are Hilbert spaces. Furthermore, the result of theorem 2 can be interpreted as the Euler equation which maximizes  $\|Tf\|$  subject to the constraint  $\|f\| = 1$ . We now demonstrate how the last two theorems can be used repeatedly [5]. Using the notation above and adjusting it to  $f = f_1$ ,  $\mu = \mu_1$ ,  $B_1 = X_1$ , and  $B_2 = Y_1$ , we define the subspaces

$$X_2 = \ker(L_1(f_1))$$

and

$$Y_2 = \ker(L_2(Tf_1)).$$

Since

$$\langle Tx, L_2(Tf_1) \rangle = \mu_1 \langle x, L_1(f_1) \rangle$$

for all  $x \in X_1$ , we note  $T$  maps  $X_2$  to  $Y_2$ . Furthermore, since  $X_2$  and  $Y_2$  are closed subspaces of reflexive spaces, they are reflexive. In fact,  $X_2^*$  and  $Y_2^*$  are strictly convex (see, [5]). We also note that the restriction,  $T_2$ , of  $T$  to  $X_2$  is a compact linear map. If it is a non-zero operator we can apply theorem 1 and theorem 2 to conclude there exists  $f_2 \in X_2/\{0\}$  such that

$$\langle T_2x, L_2(Tf_2) \rangle = \mu_2 \langle x, L_1(f_2) \rangle$$

for  $x \in X_2$  and where  $L_1(f_2)$  and  $L_2(Tf_2)$  denote the unique dual vectors of their respective subspace. We note  $\mu_2 \leq \mu_1$  and that this method can be repeated until the restriction of  $T$  to  $X_{n+1}$  is the zero operator. At this point, the range of  $T$  is the linear space spanned by  $Tf_1, \dots, Tf_n$ . In this case,  $T$  is shown to be of finite rank, which certainly is the case for any finite dimensional normed space. If the space is infinite dimensional  $T$  may not be of finite rank. Yet, the sequence of eigenvalues  $\{\mu_n\}$  tend to zero with the largest of them being  $\mu_1 = \|T\|$  (see, [5]).

### 2.3 Singular Values of the Adjoint

**Theorem.** *If  $B_1$ ,  $B_2$  and their duals are all strictly convex, then the singular values of  $T$  coincide with those of  $T^*$ .*

*Proof.* Assume  $f \in B_1/\{0\}$  satisfies the Euler equation for some  $\mu > 0$ . If we define by  $L'_1$  the map from  $B_1^*$  to its unique dual vector, then

$$L'_1(L_1(f)) = \frac{f}{\|f\|}$$

since

$$\left\langle \frac{f}{\|f\|}, L_1(f) \right\rangle = 1.$$

Furthermore,

$$L'_1(\mu L_1(f)) = \frac{f}{\|f\|}$$

since

$$\|\mu L_1(f)\| = \left\langle \frac{f}{\|f\|}, \mu L_1(f) \right\rangle = \mu \left\langle \frac{f}{\|f\|}, L_1(f) \right\rangle .$$

Defining  $L'_2$  in a similar way for  $B_2^*$  it follows

$$L'_2(L_2(Tf)) = \frac{Tf}{\|Tf\|}.$$

By the Euler equation

$$\begin{aligned} TL'_1(T^* L_2(Tf)) &= TL'_1(\mu L_1(f)) \\ &= \frac{Tf}{\|f\|} = \frac{\|Tf\|}{\|f\|} L'_2(L_2(Tf)). \end{aligned}$$

Letting  $g = L_2(Tf) \neq 0$ , the equation above becomes

$$TL'_1(T^* g) = \lambda L'_2(g)$$

where  $\lambda = \|Tf\|/\|f\|$ . Thus any singular value for  $T$  is a singular value for  $T^*$ . The converse also holds since  $T^{**} = T$ .  $\square$

## 2.4 The $p$ -Laplacian

The above results allow us to show the existence of solutions to the one dimensional  $p, q$ -Laplacian Dirichlet eigenvalue problem. To do so, we consider the interval  $I = (a, b)$  in  $\mathbb{R}$ . For  $1 < p < \infty$ , we let  $B_1 = W_0^{1,p}(I)$  by which we mean the completion of  $C_0^\infty(I)$ , the space of all infinitely differentiable functions with compact support in  $I$ , with respect to the norm

$$\|f\|_{W_0^{1,p}(I)} := \|f'\|_{L_p(I)}$$

where  $f'$  denotes the distributional derivative of  $f$ .  $B_1$  is a closed subspace of the Sobolev space  $W^{1,p}(I)$ . By Friedrichs inequality

$$\|u\|_{L_p} \leq (b - a) \|u'\|_{L_p}, \quad u \in W_0^{1,p}(I)$$

they have equivalent norms, hence  $B_1$  is also a Banach Space [5]. For  $1 < q < \infty$  we let  $B_2$  be the Banach space  $L_q(I)$ . The natural embedding  $id : B_1 \rightarrow B_2$  is compact [25]. Furthermore, both  $B_1$  and  $B_2$  are reflexive and strictly convex [5]. We compute the Gâteaux derivative of the norm of  $B_2$  implying  $B_1^*$  and  $B_2^*$  are strictly convex. We first note that for  $f, g \in B_2$

$$\lim_{t \rightarrow 0} \frac{\|f + tg\| - \|f\|}{\|f + tg\|^q - \|f\|^q} = \lim_{t \rightarrow 0} \frac{(\|f + tg\|)'}{q\|f + tg\|^{q-1}(\|f + tg\|)'} = (q\|f\|^{q-1})^{-1}.$$

From this we conclude

$$G(f, g) = \lim_{t \rightarrow 0} \frac{\|f + tg\|^q - \|f\|^q}{t} (q^{-1}\|f\|^{1-q}).$$

To compute the limit on the right we define the functional

$$J(x) = \int_0^x qs^{q-1} ds.$$

Denoting by  $j(x) = |x|$ , we note pointwise in  $I$

$$\left| \frac{J(j(f(x) + tg(x)) - J(j(f(x)))}{t} \right| = \left| \frac{|f(x) + tg(x)|^q - |f(x)|^q}{t} \right|.$$

Furthermore, taking the limit as  $t$  tends to 0, the left hand side becomes the Gâteaux derivative of the composed functional  $J(j(x))$  at  $f(x)$  in the direction  $g(x)$ . By the chain rule of the Gâteaux derivative

$$\lim_{t \rightarrow 0} \frac{J(j(f(x) + tg(x)) - J(j(f(x)))}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \int_{|f(x)|}^{|f(x)+tg(x)|} qs^{q-1} ds = q|f(x)|^{q-2} f(x)g(x)$$

pointwise on  $I$  since for  $f(x) \neq 0$

$$\lim_{t \rightarrow 0} \frac{|f(x) + tg(x)| - |f(x)|}{t} = \frac{f(x)g(x)}{|f(x)|}$$

by the usual chain rule. Denoting by  $r(\lambda) = |f(x) + \lambda tg(x)|^q$ , we have pointwise in  $I$ , for some  $\lambda_1 \in (0, 1)$

$$\left| \frac{|f(x) + tg(x)|^q - |f(x)|^q}{t} \right| \leq q|g(x)||f(x) + \lambda_1 tg(x)|^{q-1}$$



by the mean value theorem. By Hölder's inequality the right hand side is integrable, hence by the Lebesgue's dominated convergence theorem [20]

$$\lim_{t \rightarrow 0} \frac{\|f + tg\|^q - \|f\|^q}{t} = \lim_{t \rightarrow 0} \int_I \frac{|f(x) + tg(x)|^q - |f(x)|^q}{t} dx = \int_I q|f(x)|^{q-2} f(x)g(x) dx.$$

This implies

$$G(f, g) = \|f\|^{1-q} \int_I |f(x)|^{q-2} f(x)g(x) dx.$$

On the other hand, for  $h \in B_1$

$$\begin{aligned} -\|h\|_{W_0^{1,p}(I)}^{1-p} \int_I (|h'(x)|^{p-2} h'(x))' h(x) dx &= \|h\|_{W_0^{1,p}(I)}^{1-p} \int_I |h'(x)|^{p-2} h'(x) h'(x) dx \\ &= \|h\|_{W_0^{1,p}(I)}^{1-p} \int_I |h'(x)|^p dx = \|h\|_{W_0^{1,p}(I)}. \end{aligned}$$

Hence, the Gâteaux derivative with respect to the norm of  $B_1$  is

$$G(f, g) = -\|f\|^{1-p} \int_I (|f'(x)|^{p-2} f'(x))' g(x) dx.$$

Applying theorem 1, we can now find a function  $f_1 \in S_{B_1}$ , with  $\|f_1\|_{L_q(I)} = \|id\| = \lambda_1$ , such that for all  $g \in B_1$

$$\langle g, L_2(f) \rangle = \lambda_1 \langle g, L_1(f) \rangle.$$

Since the dual vectors coincide with their corresponding Gâteaux derivative, we obtain the formulation

$$\int_I |f_1(x)|^{q-2} f_1(x)g(x) dx = -\lambda_1^q \int_I (|f_1'(x)|^{p-2} f_1'(x))' g(x) dx.$$

Hence,  $f_1$  is a weak solution to the Dirichlet  $p, q$ -Laplacian eigenvalue problem

$$-(|u'|^{p-2} u')' = \lambda |u|^{q-2} u, \quad u(a) = u(b) = 0.$$

For higher dimensions, this method can be still used with the corresponding adaptations. In particular, for  $I \subset \mathbb{R}^n$ ,  $q = p$ , and giving  $B_1$  the equivalent norm

$$\|u\|^p = \int_I |\nabla u|^p dx,$$

we can show the existence of weak solutions to the  $p$ -Laplacian eigenvalue problem [5]

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u, \quad u = 0 \text{ on } \partial I.$$

## Generalized Trigonometric Functions

The generalized trigonometric functions arise as eigenfunctions of the  $p$ -Laplacian problem [5]. In this section we motivate their definition in a more elementary manner. We consider the integral

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt$$

defined for  $x \in [0, 1]$ . Using the substitution  $t = \sin(\theta)$ , we solve the integral to obtain

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt = \int_0^{\arcsin(x)} d\theta = \arcsin(x).$$

In particular, when  $x = 1$  we obtain an analytical definition of  $\pi$

$$\int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \arcsin(1) = \frac{\pi}{2}.$$

We note that the inverse function of  $x \rightarrow \int_0^x \frac{1}{\sqrt{1-t^2}} dt$  could be used to define  $\sin(\theta)$ . By varying the exponent of  $t$  and the  $n^{\text{th}}$  root taken we obtain the generalized trigonometric functions. We study this generalization and their properties in this section which is based on [7],[8] and chapter two of [5].

### 3.1 Generalization and Basic Properties

Let  $1 < p < \infty$  and define a differentiable function  $F_p : [0, 1] \rightarrow \mathbb{R}$  by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt$$

Taking its derivative one can verify  $F_p$  is strictly increasing. By analogy with  $p = 2$ , we denote its inverse by  $\sin_p$ . Note  $F_2 = \arcsin$  and  $F_2^{-1} = \sin$ .  $F_p$  is defined on the interval  $[0, \pi_p/2]$ , where

$$\pi_p = 2 \int_0^1 (1-t^p)^{-1/p} dt.$$

Thus  $\sin_p$  is strictly increasing on  $[0, \pi_p/2]$ ,  $\sin_p(0) = 0$  and  $\sin_p(\pi_p/2) = 1$ . To extend  $\sin_p$  to  $[0, \pi_p]$  we define

$$\sin_p(x) = \sin_p(\pi_p - x), \quad x \in [\pi_p/2, \pi_p];$$

extension to  $[-\pi_p, \pi_p]$  is made by oddness;  $\sin_p$  extends to  $\mathbb{R}$  by  $2\pi_p$ -periodicity. This extension is continuously differentiable on  $\mathbb{R}$ .

A function  $\cos_p : \mathbb{R} \rightarrow \mathbb{R}$  is defined as expected by

$$\cos_p(x) = \frac{d}{dx} \sin_p(x), \quad x \in \mathbb{R}.$$

$\cos_p$  is even,  $2\pi_p$ -periodic and odd about  $\pi_p/2$ .

**Theorem 3.** *If  $x \in [0, \pi_p/2]$ , then  $\cos_p(x) = (1 - \sin_p^p(x))^{1/p}$  and*

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1.$$

*Proof.* Letting  $y = \sin_p(x)$ , we make use of the following identity  $f'(x) = \frac{1}{\frac{d}{dx} f^{-1}[f(x)]}$  to obtain

$$\cos_p(x) = \frac{1}{\frac{d}{dx} \int_0^y (1 - t^p)^{-1/p} dt} = \frac{1}{(1 - y^p)^{-1/p}} = (1 - y^p)^{1/p}.$$

Thus  $\cos_p$  is strictly decreasing on  $[0, \pi_p/2]$ ,  $\cos_p(0) = 1$  and  $\cos_p(\pi_p/2) = 0$ . Now raising to the  $p$  both  $\sin_p$  and  $\cos_p$ , we obtain the familiar identity

$$\sin_p^p(x) + \cos_p^p(x) = y^p + (1 - y^p)^{p/p} = 1 + y^p - y^p = 1.$$

□

This holds for all  $x \in \mathbb{R}$  by symmetry and periodicity. The extended  $\sin_p$  function belongs to  $C^1(\mathbb{R})$ , yet it is far from being real analytic on  $\mathbb{R}$ . Observe that letting  $y = \sin_p(x)$  and taking its second derivative making use of the previous result

$$\frac{d}{dx} y' = \frac{1}{p} (1 - y^p)^{\frac{1}{p}-1} (-py^{p-1}) y'$$

$$= -(1 - y^p)^{\frac{1}{p}-1} (y^{p-1}) (1 - y^p)^{1/p} = -\frac{y^{p-1} (1 - y^p)^{\frac{2}{p}}}{1 - y^p}$$

thus, it is not continuous at  $\pi_p/2$  if  $2 < p < \infty$ , yet it maintains real analytic on  $[0, \pi_p/2)$ .

**Theorem 4.**  $\pi_p = \frac{2\pi}{p \sin(\pi/p)}$  and decreases as  $p$  increases. Furthermore,  $p \pi_p = p' \pi_{p'}$ ,  $p$  and  $p'$  being Hölder conjugates.

*Proof.* First, to calculate  $\pi_p$  we make the change of variable  $t = s^{1/p}$  leaving us with  $dt = \frac{1}{p} s^{\frac{1}{p}-1} ds$ . Substituting this into the original formula

$$\pi_p = \frac{2}{p} \int_0^1 (1-s)^{-\frac{1}{p}} s^{\frac{1}{p}-1} ds = 2p^{-1} B(1-1/p, 1/p) = 2p^{-1} \Gamma(1-1/p) \Gamma(1/p),$$

where  $B$  is the beta function. Using the formula  $\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$

$$p \pi_p / 2 = \Gamma(1-1/p) \Gamma(1/p) = \frac{\pi}{\sin(\pi/p)} \implies \pi_p = \frac{2\pi}{p \sin(\pi/p)}$$

Note that  $\pi_2 = \pi$  as expected. Finally, using the above result we obtain

$$p \pi_p = 2\Gamma(1-1/p) \Gamma(1/p) = 2\Gamma(1/p') \Gamma(1-1/p') = p' \pi_{p'}.$$

□

**Corollary 1.**

$$\lim_{p \rightarrow 1} \pi_p = \infty, \quad \lim_{p \rightarrow \infty} \pi_p = 2, \quad \lim_{p \rightarrow 1} (p-1)\pi_p = \lim_{p \rightarrow \infty} \pi_{p'} = 2.$$

*Proof.* The first limit follows from direct substitution. For the second one, note that

$$\cos(x) < \frac{\sin(x)}{x} < 1.$$

As  $x \rightarrow 0$  the inside value tends to 1 by the squeeze theorem. Rewriting the limit with  $x = \frac{\pi}{p}$ ,

$$\lim_{p \rightarrow \infty} \pi_p = 2 \lim_{p \rightarrow \infty} \left( \frac{\sin(\pi/p)}{\pi/p} \right)^{-1} = 2 \lim_{x \rightarrow 0} \left( \frac{\sin(x)}{x} \right)^{-1} = 2.$$

For the last result, since  $p \rightarrow 1$  we have  $p' \rightarrow \infty$ . Observing that

$$(p-1)\pi_p = \frac{p-1}{p} \frac{2\pi}{\sin(\pi/p)} = \frac{1}{p'} \frac{2\pi}{\sin(\pi/p)} = \frac{p \pi_p}{p'} = \pi_{p'}$$

we conclude

$$\lim_{p \rightarrow 1} (p-1)\pi_p = \lim_{p' \rightarrow \infty} \pi_{p'} = 2$$

□

The analogue of the tangent function is

$$\tan_p(x) = \frac{\sin_p(x)}{\cos_p(x)},$$

defined for values of  $x$  at which  $\cos_p(x) \neq 0$ . Hence  $\tan_p(x)$  is defined for all  $x \in \mathbb{R}$  except for the points  $(k + 1/2)\pi_p$  ( $k \in \mathbb{Z}$ );  $\tan_p$  is odd,  $\pi_p$ -periodic, and  $\tan_p 0 = 0$ .

**Corollary 2.** *On  $(-\pi_p/2, \pi_p/2)$ ,  $\tan_p$  has its derivative equal to  $1 + |\tan_p(x)|^p$ . Letting  $A$  be the inverse on this interval,  $A'(t) = 1/(1 + |t|^p)$ ,  $t \in \mathbb{R}$ .*

*Proof.* To obtain the derivative, we use our previous result connecting  $\sin_p$  and  $\cos_p$ . Letting  $y = \sin_p(x)$

$$\begin{aligned} \frac{d}{dx} \tan_p(x) &= \frac{d}{dx} \frac{y}{(1 - y^p)^{1/p}} = \frac{(1 - y^p)^{1/p} y' + y(1 - y^p)^{2/p-1} y^{p-1}}{(1 - y^p)^{2/p}} \\ &= \frac{y'}{(1 - y^p)^{1/p}} + \frac{y^p}{1 - y^p} = 1 + \frac{|\sin_p(x)|^p}{|\cos_p(x)|^p} = 1 + |\tan_p(x)|^p. \end{aligned}$$

To calculate the derivative of the inverse, we use  $f'(x) = \frac{1}{\frac{d}{dx} f^{-1}[f(x)]}$  to obtain

$$A'(t) = \frac{1}{\frac{d}{dx} \tan_p(A(t))} = \frac{1}{1 + |\tan_p(A(t))|^p} = \frac{1}{1 + |t|^p}.$$

□

**Corollary 3.** *For all  $x \in [0, \pi_p/2]$*

$$\frac{d}{dx} \cos_p^{p-1}(x) = -(p-1) \sin_p^{p-1}(x), \quad \frac{d}{dx} \sin_p^{p-1}(x) = (p-1) \sin_p^{p-2}(x) \cos_p(x).$$

*Proof.* Letting  $y = \sin(x)$  we obtain

$$\begin{aligned} \cos_p^{p-1}(x) &= (1 - y^p)^{\frac{(p-1)}{p}} = (1 - y^p)^{1-1/p} \\ &= (1 - 1/p)(1 - y^p)^{-1/p} (-py^{p-1})(1 - y^p)^{1/p} = (1 - p)(y^{p-1}). \end{aligned}$$

The second derivative follows directly

$$\frac{d}{dx} (y^{p-1}) = (p-1)y^{p-2}y' = (p-1)y^{p-2}(1 - y^p)^{1/p}.$$

□

**Theorem 5.** For all  $y \in [0, 1]$ ,

$$\begin{aligned}\cos_p^{-1}(y) &= \sin_p^{-1}(1 - y^p)^{1/p}, & \sin_p^{-1}(y) &= \cos_p^{-1}(1 - y^p)^{1/p} \\ \frac{2}{\pi_p} \sin_p^{-1}(y) + \frac{2}{\pi_{p'}} \sin_{p'}^{-1}(1 - y^p)^{1/p'} &= 1, & \cos_p^p(\pi_p y/2) &= \sin_{p'}^{p'}(\pi_{p'}(1 - y)/2).\end{aligned}$$

*Proof.* Let  $\cos_p(x) = y$ , then  $x = \cos_p^{-1}(y)$ . Inverting our previous result

$$\cos_p(x) = (1 - \sin_p^p(x))^{1/p} \text{ implies } x = \sin_p^{-1}(1 - \cos_p^p(x))^{1/p}.$$

For the second equation let  $y = \sin_p(x)$ . For the third equality, we apply the change of variables  $s = (1 - t^{p'})^{1/p}$ ,  $dt = p(1 - t^{p'})^{1-1/p}(-1/p')t^{1-p'} ds$ , on the integral

$$\begin{aligned}\sin_{p'}^{-1}(1 - y^p)^{1/p'} &= \int_0^{(1-y^p)^{1/p'}} (1 - t^{p'})^{-1/p'} dt = -\frac{p}{p'} \int_1^y (1 - s^p)^{-1/p} ds \\ &= \frac{p}{p'} \left[ \int_0^1 (1 - s^p)^{-1/p} - \int_0^y (1 - s^p)^{-1/p} \right] ds = \frac{\pi_{p'}}{\pi_p} \left( \frac{\pi_p}{2} - \sin_p^{-1}(y) \right) \\ &= \frac{\pi_{p'}}{2} - \frac{\pi_{p'}}{\pi_p} \sin_p^{-1}(y).\end{aligned}$$

Multiplying both sides by  $2/\pi_{p'}$  we obtain the desired result

$$\frac{2}{\pi_{p'}} \sin_{p'}^{-1}(1 - y^p)^{1/p'} = 1 - \frac{2}{\pi_p} \sin_p^{-1}(y).$$

Using this last result and previous equations we obtain the fourth equation

$$\cos_p^p(\pi_p y/2) = 1 - \sin_p^p(\pi_p y/2) := 1 - x^p \text{ implies } x = \sin_p(\pi_p y/2)$$

and

$$y = \frac{2}{\pi_p} \sin_p^{-1}(x) = 1 - \frac{2}{\pi_{p'}} \sin_{p'}^{-1}(1 - x^p)^{1/p'}$$

hence

$$(y - 1)(-\pi_{p'}/2) = \sin_{p'}^{-1}(1 - x^p)^{1/p'}.$$

Taking  $\sin_{p'}^{p'}$  of both sides concludes the proof,

$$1 - x^p = \sin_{p'}^{p'}(\pi_{p'}(1 - y)/2).$$

□

### 3.2 A Further Generalization

We now move to a more refined extension of the trigonometric functions. Letting  $p, q \in (1, \infty)$ , we define

$$\pi_{p,q} = 2 \int_0^1 (1 - t^q)^{-1/p} dt.$$

This coincides with  $\pi_p$  when  $p = q$ . Using the substitution  $s = t^q$  we obtain

$$\pi_{p,q} = 2q^{-1} \int_0^1 (1 - s)^{-1/p} s^{1/q-1} ds = 2q^{-1} B((p-1)/p, 1/q).$$

One can see that  $\pi_{p,q}$  decreases as either  $p$  or  $q$  increases, with the other held constant, and

$$\lim_{p \rightarrow \infty} \pi_{p,q} = 2 \quad (1 < q < \infty), \quad \lim_{q \rightarrow \infty} \pi_{p,q} = 2 \quad (1 < p < \infty).$$

Keeping the analogy with the case  $p = q$  we define  $\sin_{p,q}$  on the interval  $[0, \pi_{p,q}/2]$  to be the inverse of the strictly increasing function  $F_{p,q} : [0, 1] \rightarrow [0, \pi_{p,q}/2]$  given by

$$F_{p,q}(x) = \int_0^x (1 - t^q)^{-1/p} dt.$$

This extends to all of the real line by the process of symmetry and  $2\pi_{p,q}$ -periodicity as for the case  $p = q$ . The function  $\cos_{p,q}$  is defined to be the derivative of  $\sin_{p,q}$ , and it follows that for all  $x \in \mathbb{R}$ ,

$$|\sin_{p,q}(x)|^q + |\cos_{p,q}(x)|^p = 1.$$

The  $p, q$  generalization has the following properties:

**Theorem 6.** For  $x \in [0, \pi_{p,q}/2)$ ,

$$\frac{d}{dx} \cos_{p,q}(x) = -\frac{q}{p} (\cos_{p,q}(x))^{2-p} (\sin_{p,q}(x))^{q-1}, \quad \frac{d}{dx} \tan_{p,q}(x) = 1 + \frac{q (\sin_{p,q}(x))^q}{p (\cos_{p,q}(x))^p}$$

$$\frac{d}{dx} (\cos_{p,q}(x))^{p-1} = -\frac{q(p-1)}{p} (\sin_{p,q}(x))^{q-1}, \quad \frac{d}{dx} (\sin_{p,q}(x))^{p-1} = (p-1) (\sin_{p,q}(x))^{p-2} (\cos_{p,q}(x))$$

**Theorem 7.** For  $y \in [0, 1]$ ,

$$\cos_{p,q}^{-1}(y) = \sin_{p,q}^{-1}((1 - y^p)^{1/q}), \quad \sin_{p,q}^{-1}(y) = \cos_{p,q}^{-1}((1 - y^q)^{1/p}),$$

$$\begin{aligned} \frac{2}{\pi_{p,q}} \sin_{p,q}^{-1}(y^{1/q}) + \frac{2}{\pi_{q',p'}} \sin_{q',p'}^{-1}((1-y)^{1/p'}) &= 1, \\ (\cos_{p,q}(\pi_{p,q}y/2))^p &= (\sin_{q',p'}(\pi_{q',p'}(1-y)/2))^{p'}. \end{aligned}$$

In addition,

**Theorem 8.** For  $p, q \in (2, \infty)$  and  $\theta \in (0, \pi_{p,q}/2]$ ,

$$\frac{2}{\pi_{p,q}} \leq \frac{\sin_{p,q}(\theta)}{\theta} \leq 1.$$

Using the results above it can be shown that

$$\pi_{p,q} = \begin{cases} 2p', & \text{if } 1 \leq p \leq \infty, q = 1, \\ 2, & \text{if } 1 \leq p \leq \infty, q = \infty, \\ \infty, & \text{if } p = 1, 1 \leq q \leq \infty, \\ 2, & \text{if } p = \infty, 1 \leq q \leq \infty. \end{cases}$$

Corresponding values of  $\sin_p(x)$  and  $\cos_p(x)$  are

$$\sin_{p,q}(x) = \begin{cases} 1 - (1 - x/p')^{p'}, & \text{if } 1 \leq p \leq \infty, q = 1, \\ x, & \text{if } 1 \leq p \leq \infty, q = \infty, \\ x, & \text{if } p = \infty, 1 \leq q \leq \infty, \end{cases}$$

and

$$\cos_{p,q}(x) = \begin{cases} (1 - x/p')^{1/(p-1)}, & \text{if } 1 \leq p \leq \infty, q = 1, \\ 1, & \text{if } 1 \leq p \leq \infty, q = \infty, \\ 1, & \text{if } p = \infty, 1 \leq q \leq \infty. \end{cases}$$

We conclude this section with two results involving the Hölder conjugate pair of  $p, q$ , i.e.

$p' = \frac{p}{p-1}$  and  $q' = \frac{q}{q-1}$ . The first generalizes the symmetry found before:



**Lemma 1.** Let  $p, q \in (1, \infty)$ . Then

$$\pi_{p,q} = \frac{p'}{q} \pi_{q',p'}.$$

*Proof.* Making the change of variables  $y = (1 - t^q)^{1/p'}$  we obtain  $t = (1 - y^{p'})^{1/q}$ . Hence  $dt = (1/q)(1 - y^{p'})^{1/q-1}(-p'y^{p'-1})dy$  and

$$\begin{aligned} \frac{\pi_{p,q}}{2} &= \int_0^1 (1 - t^q)^{-1/p} dt = \int_1^0 (1 - (1 - y^{p'}))^{-1/p} (-p/q)(1 - y^{p'})^{1/q-1} (y^{p'-1}) dy \\ &= \frac{p}{q} \int_0^1 y^{-p'/p} (1 - y^{p'})^{1/q-1} (y^{p'-1}) dy = \frac{p}{q} \int_0^1 y^{-p'/p+p'-1} (1 - y^{p'})^{1/q-1} dy \\ &= \frac{p}{q} \int_0^1 y^{-p'(1/p-1)-1} (1 - y^{p'})^{1/q-1} dy = \frac{p}{q} \int_0^1 y^{-p'(-1/p')-1} (1 - y^{p'})^{1/q-1} dy \\ &= \frac{p}{q} \int_0^1 (1 - y^{p'})^{-1/q'} dy = \frac{p}{q} \frac{\pi_{p',q'}}{2}. \end{aligned}$$

□

**Lemma 2.** Let  $1 < p, q < \infty$ . Then the following inequalities hold.

$$\pi_{p,q} \leq \pi_{q',q} \quad \text{if } p' \leq q.$$

$$\pi_{p,q} \leq \frac{p'}{q} \pi_{p,p'} \quad \text{if } p' > q.$$

*Proof.* The condition  $p' \leq q$  implies

$$\frac{1}{q} \leq \frac{1}{p'} = 1 - \frac{1}{p},$$

hence

$$\frac{1}{p} \leq 1 - \frac{1}{q} = \frac{1}{q'}.$$

Thus  $q' \leq p$  and the first inequality follows from the decreasing property of  $\pi_{p,q}$  for fixed  $q$ . Similarly, the condition  $p' > q$  implies  $q' > p$ . Applying the first result we conclude

$$\pi_{p,q} = \frac{p'}{q} \pi_{q',p'} < \frac{p'}{q} \pi_{p,p'}.$$

□

## Basis Properties of the Generalized Trigonometric Functions

We continue investigating the properties of the generalized trigonometric functions in this chapter by detailing their recently shown basis properties [12]. In connection with the basis properties of the Fourier series, we show the system  $\{\prod_{i=1}^d \sin_{p,q}(n_i \pi_{p,q} x_i)\}$  is a basis for  $L^r((0, 1)^d)$  where  $r \in (1, \infty)$  and  $d = 1, 2, 3$ .

### 4.1 Strategy for Obtaining the Basis Result

In this section we go over the results used in the main proof. A sequence  $\{x_i\}$  of a Banach space  $X$  with norm  $\|\cdot\|$  is a basis for the space if for any  $x \in X$  there exists a unique sequence of scalars  $\{s_i\}$  such that  $x = \sum_{i=1}^{\infty} s_i x_i$ , that is

$$\lim_{n \rightarrow \infty} \left\| x - \sum_{i=1}^n s_i x_i \right\| = 0.$$

With a basis  $\{x_i\}$  we can define a sequence of linear functionals by  $f_i(x) = s_i$ , where  $x = \sum_{i=1}^{\infty} s_i x_i$ . These functionals are referred to as the associated sequence of coefficient functionals. In general topological linear spaces, a Schauder basis is a basis for which its associated sequence of coefficient functionals are continuous in the space. In Banach spaces these two notions are equivalent, indeed ([10] theorem 3.1, p. 20):

**Theorem 9.** *Let  $\{x_n\}$  be a basis of a Banach space  $X$ . Then the coefficient functionals  $f_n$  associated to the basis  $\{x_n\}$  are continuous on  $X$ , i.e.  $f_n \in X^*$ .*

Thus, the aim of this section is in finding a new Schauder basis. Vital to this pursuit is the following theorem ([10] example 11.1, p. 342-345):

**Theorem 10.** *For  $r \in (0, \infty)$  and  $f \in L^r((0, 1)^n)$  the sine Fourier partial sums  $S_1$  converge in  $L^r$ -norm to  $f$  in the Pringsheim sense, i.e.:*

$$\|f - S_l\| \rightarrow 0 \quad \text{as} \quad \min\{l_1, l_2, \dots, l_n\} \rightarrow \infty,$$

where  $\mathbf{l} := (l_1, l_2, \dots, l_n) \in \mathbb{N}^n$ ,

$$S_{\mathbf{l}} := \sum_{k_i \leq l_i} \hat{f}(k_1, k_2, \dots, k_n) \prod_{j=1}^n \sin(\pi k_j x_j).$$

and

$$\hat{f}(k_1, k_2, \dots, k_n) := 2^n \int_{(0,1)^n} f(x_1, x_2, \dots, x_n) \prod_{j=1}^n \sin(\pi k_j x_j) dx_1 \dots dx_n.$$

This result follows from the density of the Fourier series in  $L^r((-1, 1)^n)$ , that is, denoting by  $\mathbf{k} := (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$ , we have for  $f \in L^r((-1, 1)^n)$

$$\|f - \sum_{k_i \leq l_i} \hat{f}(k_1, k_2, \dots, k_n) e^{2\pi i \mathbf{k} \cdot \mathbf{x}}\| \rightarrow 0 \quad \text{as } \min\{l_1, l_2, \dots, l_n\} \rightarrow \infty.$$

We recall the Fourier coefficients of  $f$  are given by

$$\hat{f}(k_1, k_2, \dots, k_n) = \int_{(0,1)^n} f(x_1, x_2, \dots, x_n) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} dx_1 \dots dx_n.$$

Using this well known result, and the fact that any  $f \in L^r((0, 1)^n)$  can be uniquely extended to  $(-1, 1)^n$  as an odd function we obtain theorem 10.

Starting with this basis, the strategy will be to construct a linear bi-continuous operator  $T : L^r((0, 1)^n) \rightarrow L^r((0, 1)^n)$  having the property

$$T\left(\prod_{j=1}^n \sin(n_j \pi x_j)\right) = \prod_{j=1}^n \sin_{p,q}(n_j \pi_{p,q} x_j).$$

This will imply the result thanks to the following theorem ([9] p.75),

**Theorem 11.** *Let  $\{x_n\}$  be a basis of a Banach space  $X$ . If there exists a bounded operator  $T : X \rightarrow X$  that is invertible,  $T^{-1}$  exists and it is bounded, such that*

$$T(x_n) = y_n$$

for all  $n$ , then  $\{y_n\}$  too is a basis for  $X$ .

Hence, the problem reduces to constructing an operator with the desired properties. We will see such mapping will be composed of a series of isometries. The required boundedness

of the mapping will impose restrictions on the range of values of  $p, q$  for which the result holds as well as on the maximum number of dimensions. We conclude this subsection by stating the result that will be used to show the invertibility of the operator ([11] p.196):

**Theorem 12.** *Let  $H$  and  $A$  be operators from  $X$  to  $Y$ . Let  $H^{-1}$  exist and belong to  $B(X, Y)$ . Let  $A$  be bounded satisfying the inequality*

$$\|A\| \|H^{-1}\| < 1.$$

*Then  $S = H + A$  is invertible with  $S^{-1} \in B(X, Y)$ .*

## 4.2 Sufficient Conditions for Basis Properties

We will use the notation  $\mathbf{xk} = (x_1 k_1, x_2 k_2, \dots, x_n k_n)$  in this section. For  $\mathbf{x} \in (0, 1)^n$ , the function

$$\tau_{\mathbf{l}, q}(\mathbf{x}) := \prod_{j=1}^n \sin_{p, q}(l_j \pi_{p, q} x_j)$$

can be expressed as

$$\sum_{\mathbf{k} \in \mathbb{N}^n} \hat{\tau}_{\mathbf{l}, q}(k_1, k_2, \dots, k_n) \prod_{j=1}^n \sin(\pi k_j x_j)$$

by theorem 10. We note

$$\begin{aligned} \hat{\tau}_{\mathbf{l}, q}(k_1, k_2, \dots, k_n) &= 2^n \int_{(0,1)^n} \prod_{j=1}^n \sin_{p, q}(l_j \pi_{p, q} x_j) \prod_{j=1}^n \sin(\pi k_j x_j) dx_1 \dots dx_n \\ &= 2^n \int_{(0,1)^n} \prod_{j=1}^n \sin_{p, q}(l_j \pi_{p, q} x_j) \sin(\pi k_j x_j) dx_1 \dots dx_n. \end{aligned}$$

By symmetry around half of the interval when  $l_i = 1$ ,

$$\begin{aligned} &\int_{(0,1)} \sin_{p, q}(\pi_{p, q} x_i) \sin(\pi k_i x_i) dx_i = \\ &\int_{(0,1/2)} \sin_{p, q}(\pi_{p, q} x_i) \sin(\pi k_i x_i) dx_i - \int_{(0,1/2)} \sin_{p, q}(\pi_{p, q} x_i) \sin(\pi k_i x_i) dx_i = 0 \end{aligned}$$

if  $k_i$  is even. Hence,  $\hat{\tau}_{\mathbf{1}p,q}(k_1, k_2, \dots, k_n) = 0$  when any component is even. Furthermore, since  $\tau_{\mathbf{1}p,q}(\mathbf{x}) = \tau_{\mathbf{1}p,q}(\mathbf{l}\mathbf{x})$

$$\begin{aligned}\hat{\tau}_{\mathbf{1}p,q}(\mathbf{k}) &= 2^n \int_{(0,1)^n} \tau_{\mathbf{1}p,q}(\mathbf{l}\mathbf{x}) \prod_{j=1}^n \sin(\pi k_j x_j) dx_1 \dots dx_n \\ &= 2^n \sum_{\mathbf{r} \in \mathbb{N}^n} \hat{\tau}_{\mathbf{1}p,q}(\mathbf{r}) \int_{(0,1)^n} \prod_{j=1}^n \sin(\pi r_j l_j x_j) \sin(\pi k_j x_j) dx_1 \dots dx_n \\ &= 2^n \sum_{r_j \text{ odd}} \hat{\tau}_{\mathbf{1}p,q}(\mathbf{r}) \int_{(0,1)^n} \prod_{j=1}^n \sin(\pi r_j l_j x_j) \sin(\pi k_j x_j) dx_1 \dots dx_n.\end{aligned}$$

Noting

$$\int_{(0,1)} \sin(\pi r_j l_j x_j) \sin(\pi k_j x_j) dx_j = 1/2$$

if  $r_j l_j = k_j$  and 0 otherwise we obtain the following lemma

**Lemma 3.**

$$\hat{\tau}_{\mathbf{1}p,q}(\mathbf{k}) = \hat{\tau}_{\mathbf{1}p,q}(\mathbf{r})$$

if  $r_j l_j = k_j$  for odd  $r_j$  and 0 otherwise.

The following estimates will be needed later [7]:

**Lemma 4.** *Let  $1 < p, q < \infty$  and  $m_j$  odd, then*

$$|\hat{f}_{\mathbf{1}p,q}(m_j)| = \left| 2 \int_{(0,1)} \sin_{p,q}(\pi_{p,q} x_j) \sin(\pi m_j x_j) dx_j \right| \leq \frac{4\pi_{p,q}}{(\pi m_j)^2}.$$

*Proof.* We proceed by exploiting the symmetry of these functions and by repeated application of integration by parts. Indeed,

$$\begin{aligned}\hat{f}_{\mathbf{1}p,q}(m_j) &= 2(2) \int_{(0,1/2)} \sin_{p,q}(\pi_{p,q} x_j) \sin(\pi m_j x_j) dx_j \\ &= 4 \left[ - \frac{\cos(m_j \pi x_j) \sin_{p,q}(\pi_{p,q} x_j)}{m_j \pi} \Big|_0^{1/2} + \frac{\pi_{p,q}}{m_j \pi} \int_{(0,1/2)} \cos_{p,q}(\pi_{p,q} x_j) \cos(\pi m_j x_j) dx_j \right] \\ &= \frac{4\pi_{p,q}}{m_j \pi} \left[ \frac{\sin(m_j \pi x_j) \cos_{p,q}(\pi_{p,q} x_j)}{m_j \pi} \Big|_0^{1/2} - \frac{1}{m_j \pi} \int_{(0,1/2)} \frac{d}{dx_j} \cos_{p,q}(\pi_{p,q} x_j) \sin(\pi m_j x_j) dx_j \right]\end{aligned}$$

$$= -\frac{4\pi_{p,q}}{(m_j\pi)^2} \int_{(0,1/2)} \frac{d}{dx_j} \cos_{p,q}(\pi_{p,q}x_j) \sin(\pi m_j x_j) dx_j.$$

Using the substitution  $s = \cos_{p,q}(\pi_{p,q}x_j)$  we obtain  $x_j = \frac{\cos_{p,q}^{-1}(s)}{\pi_{p,q}}$ , hence

$$\hat{f}_{1p,q}(m_j) = \frac{4\pi_{p,q}}{(m_j\pi)^2} \int_{(0,1)} \sin\left(\pi m_j \frac{\cos_{p,q}^{-1}(s)}{\pi_{p,q}}\right) ds,$$

the result follows.  $\square$

By a previous result we obtain the following lemma [7]:

**Lemma 5.** For  $1 < p, q < \infty$  and  $m_j = 1$

$$\hat{f}_{1p,q}(m_j) \geq \frac{8}{\pi^2}.$$

*Proof.* By theorem 8 from section 3, for  $x_j \in (0, 1/2)$ ,  $\sin_{p,q}(\pi_{p,q}x_j) \geq 2x_j$ . Hence,

$$\begin{aligned} \hat{f}_{1p,q}(m_j) &= 4 \int_{(0,1/2)} \sin_{p,q}(\pi_{p,q}x_j) \sin(\pi x_j) dx_j \geq 4 \int_{(0,1/2)} 2x_j \sin(\pi x_j) dx_j \\ &= 8 \left[ -\frac{x_j \cos(\pi x_j)}{\pi} \Big|_0^{1/2} + \frac{1}{\pi} \int_{(0,1/2)} \cos(\pi x_j) dx_j \right] \\ &= 8 \left[ \frac{1}{\pi^2} (\sin(\pi x_j)) \Big|_0^{1/2} \right] = \frac{8}{\pi^2}. \end{aligned}$$

$\square$

We define the extension of a function  $f : [0, 1]^n \rightarrow \mathbb{R}$  to the function  $\tilde{f} : [0, \infty)^n \rightarrow \mathbb{R}$  given by the following rules:

$$\tilde{f}(\mathbf{x}) = -\tilde{f}(2\mathbf{k} - \mathbf{x}) \text{ for } \mathbf{x} \in \prod_{j=1}^n [k_j, k_j + 1) \in \mathbb{N}^n,$$

$$\text{and } \tilde{f} \equiv f \text{ on } [0, 1)^n.$$

We remark that given  $r \in (1, \infty)$ , for each  $\mathbf{k} \in \mathbb{N}^n$ , the map

$$M_{\mathbf{k}} : L^r((0, 1)^n) \longrightarrow L^r((0, 1)^n),$$

which is defined as  $M_{\mathbf{k}}(g)(\mathbf{x}) := \tilde{g}(\mathbf{x}\mathbf{k})$ , is well defined and linear. In fact it is an isometry as we now demonstrate.

$$\begin{aligned} \|M_{\mathbf{k}}(g)\|^r &= \int_{(0,1)^n} |M_{\mathbf{k}}g(\mathbf{x})|^r d\mathbf{x} = \int_{(0,1)^n} |\tilde{g}(\mathbf{x}\mathbf{k})|^r d\mathbf{x} \\ &= \left(\prod_{j=1}^n k_j\right)^{-1} \int_{\prod_{i=1}^n (0, k_i)} |\tilde{g}(\mathbf{t})|^r d\mathbf{t} = \prod_{j=1}^n \sum_{l_j=1}^{k_j} k_j^{-1} \int_{\prod_{i=1}^n (l_i-1, l_i)} |\tilde{g}(\mathbf{t})|^r d\mathbf{t} \\ &= \prod_{j=1}^n \sum_{l_j=1}^{k_j} k_j^{-1} \int_{(0,1)^n} |g(\mathbf{t})|^r d\mathbf{t} = \int_{(0,1)^n} |g(\mathbf{t})|^r d\mathbf{t} = \|g\|^r. \end{aligned}$$

Hence,  $\|M_{\mathbf{k}}\| = 1$  as stated. Denoting  $\hat{\tau}_{\mathbf{1}p,q}(\mathbf{r}) := \tau_{p,q}(\mathbf{r})$ , we define the linear operator  $T : L^r((0,1)^n) \rightarrow L^r((0,1)^n)$  by

$$T(g) := \sum_{\mathbf{k} \in \mathbb{N}^n} \tau_{p,q}(\mathbf{k}) M_{\mathbf{k}}(g).$$

This operator is well defined and bounded since by lemma 4

$$\begin{aligned} \|T\| &= \sum_{\mathbf{k} \in \mathbb{N}^n} |\tau_{p,q}(\mathbf{k})| = \sum_{k_j \text{ odd}} \prod_{j=1}^n |\hat{f}_{1p,q}(k_j)| \\ &\leq \left(\frac{4\pi_{p,q}}{\pi^2}\right)^n \left(\sum_{l \text{ odd}} l^{-2}\right)^n < \infty. \end{aligned}$$

At this point, we verify the condition needed to apply theorem 11, namely

$$T\left(\prod_{j=1}^n \sin(n_j \pi x_j)\right) = \tau_{\mathbf{n}p,q}(\mathbf{x}).$$

Since

$$M_{\mathbf{k}}\left(\prod_{j=1}^n \sin(n_j \pi x_j)\right) = M_{\mathbf{1}}\left(\prod_{j=1}^n \sin(k_j n_j \pi x_j)\right) = \prod_{j=1}^n \sin(k_j n_j \pi x_j),$$

we compute

$$\begin{aligned} T\left(\prod_{j=1}^n \sin(n_j \pi x_j)\right) &= \sum_{\mathbf{k} \in \mathbb{N}^n} \tau_{p,q}(\mathbf{k}) \prod_{j=1}^n \sin(k_j n_j \pi x_j) \\ &= \sum_{k_j \text{ odd}} \tau_{p,q}(\mathbf{k}) \prod_{j=1}^n \sin(k_j n_j \pi x_j) = \sum_{k_j \text{ odd}} \hat{\tau}_{\mathbf{n}p,q}(\mathbf{k}\mathbf{n}) \prod_{j=1}^n \sin(k_j n_j \pi x_j) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\mathbf{k} \in \mathbb{N}^n} \hat{\tau}_{\mathbf{n}p,q}(\mathbf{k}\mathbf{n}) \prod_{j=1}^n \sin(k_j n_j \pi x_j) = \sum_{\mathbf{l} \in \mathbb{N}^n} \hat{\tau}_{\mathbf{n}p,q}(\mathbf{l}) \prod_{j=1}^n \sin(l_j \pi x_j) \\
&= \prod_{j=1}^n \sin_{p,q}(n_j \pi x_j) = \tau_{\mathbf{n}p,q}(\mathbf{x}).
\end{aligned}$$

Using theorem 12 with  $H = \tau_{p,q}(\mathbf{1}) \cdot id$  and  $A = T - \tau_{p,q}(\mathbf{1}) \cdot id$ , we observe  $T = A + H$  is invertible if

$$\|T - \tau_{p,q}(\mathbf{1}) \cdot id\| < |\tau_{p,q}(\mathbf{1})|,$$

since  $\|H^{-1}\| = |\tau_{p,q}(\mathbf{1})|^{-1}$ . Satisfying this inequality will imply the basis properties of the  $\sin_{p,q}(x)$  system. It will also provide sufficient restrictions on  $p, q$  and  $n$ . Let us bound from above the left hand side:

$$\begin{aligned}
&\|T - \tau_{p,q}(\mathbf{1}) \cdot id\| \leq \sum_{\mathbf{k} \in \{\mathbb{N}^{n-1}\}} |\tau_{p,q}(\mathbf{k})| \\
&= \sum_{s=1}^{n-1} \left[ \binom{n}{s} [\hat{f}_{1p,q}(1)]^s \sum_{\substack{k_i > 1 \\ i > s}} \prod_{j=s+1}^n \hat{f}_{1p,q}(k_j) \right] + \sum_{k_i > 1} \prod_{j=1}^n \hat{f}_{1p,q}(k_j) \\
&= \sum_{s=1}^{n-1} \binom{n}{s} [\hat{f}_{1p,q}(1)]^s \sum_{j>1} \left( \hat{f}_{1p,q}(j) \right)^{n-s} + \sum_{j>1} \left( \hat{f}_{1p,q}(j) \right)^n \\
&\leq \sum_{s=1}^{n-1} \binom{n}{s} \left[ \frac{4\pi_{p,q}}{\pi^2} \right]^s \left( \frac{4\pi_{p,q}}{\pi^2} \left( \sum_{j \text{ odd}} j^{-2} - 1 \right) \right)^{n-s} + \left( \frac{4\pi_{p,q}}{\pi^2} \left( \sum_{j \text{ odd}} j^{-2} - 1 \right) \right)^n \\
&= \left( \frac{4\pi_{p,q}}{\pi^2} \right)^n \left[ \sum_{s=1}^{n-1} \binom{n}{s} \left( \frac{\pi^2}{8} - 1 \right)^{n-s} + \left( \frac{\pi^2}{8} - 1 \right)^n \right] \\
&= \left( \frac{4\pi_{p,q}}{\pi^2} \right)^n \left[ \sum_{s=0}^n \binom{n}{s} \left( \frac{\pi^2}{8} - 1 \right)^{n-s} - 1 \right] \\
&= \left( \frac{4\pi_{p,q}}{\pi^2} \right)^n \left[ \left( 1 + \frac{\pi^2}{8} - 1 \right)^n - 1 \right] \\
&= \left( \frac{4\pi_{p,q}}{\pi^2} \right)^n \left[ \left( \frac{\pi^2}{8} \right)^n - 1 \right].
\end{aligned}$$

For the right hand side, we apply lemma 3 to conclude

$$\tau_{p,q}(\mathbf{1}) = (\hat{f}_{1p,q}(1))^n \geq \left( \frac{8}{\pi^2} \right)^n.$$



We state these findings as the following lemma:

**Lemma 6.** *The operator  $T : L^r((0, 1)^n) \rightarrow L^r((0, 1)^n)$  defined above is a homeomorphism if  $\pi_{p,q} \leq \frac{16}{(\pi^{2n} - 8^n)^{1/n}}$ . Furthermore,  $T(\prod_{j=1}^n \sin(k_j \pi x_j)) = \tau_{\mathbf{k},q}(\mathbf{x})$  for all  $\mathbf{k} \in \mathbb{N}^n$ .*

*Proof.* By the previous remarks on theorem 12 it suffices to show

$$\|T - \tau_{p,q}(\mathbf{1}) \cdot id\| < |\tau_{p,q}(\mathbf{1})|.$$

By the bounds found, it is sufficient for

$$\left(\frac{4\pi_{p,q}}{\pi^2}\right)^n \left[\left(\frac{\pi^2}{8}\right)^n - 1\right] \leq \left(\frac{8}{\pi^2}\right)^n$$

This is equivalent to

$$(4\pi_{p,q})^n \leq \frac{64^n}{\pi^{2n} - 8^n},$$

The result follows. □

In fact, for  $p, q \in (1, 2)$  we can improve this bound. To do so, we need the following result:

**Lemma 7.** *If  $1 < p < p' < \infty$ ,  $1 < q < q' < \infty$  the function*

$$w(x) = \frac{\sin_{p,q}^{-1}(x)}{\sin_{p',q'}^{-1}(x)}$$

*is strictly increasing on  $(0, 1)$ .*

*Proof.* Letting  $u(x) = \sin_{p,q}^{-1}(x)$  and  $v(x) = \sin_{p',q'}^{-1}(x)$ , we observe by the chain rule

$$w'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)},$$

hence  $w'(x)$  is positive if (negative)

$$u'(x)v(x) > u(x)v'(x) \quad (<).$$

Dividing by  $u'(x)$ , note this is a positive quantity in all of  $(0, 1)$ , we obtain the equivalent condition: positive if (negative)

$$v(x) > \frac{u(x)v'(x)}{u'(x)} \quad (<).$$

We define  $s(x) = v(x) - \frac{u(x)v'(x)}{u'(x)}$ . It is sufficient to show  $s(x)$  is positive. By direct computations, we note that  $u'(x) = (1 - x^q)^{-1/p}$  and

$$u''(x) = -\frac{1}{p}(1 - x^q)^{-1/p-1}(-qx^{q-1}) = \frac{q}{p}(1 - x^q)^{-1}(x^{q-1})(1 - x^q)^{-1/p}.$$

Denoting  $\lambda_u := \frac{q}{p}(1 - x^q)^{-1}(x^{q-1})$ , we see  $u''(x) = \lambda_u u'(x)$ . An analogous definition for  $v(x)$ , yields  $v''(x) = \lambda_v v'(x)$  where  $v'(x) = (1 - x^{q'})^{-1/p'}$ . Hence,

$$\begin{aligned} s'(x) &= v'(x) - \frac{(u'(x)v'(x) + u(x)v''(x))u'(x) - u(x)v'(x)u''(x)}{[u'(x)]^2} \\ &= \frac{-u(x)v''(x)u'(x) + u(x)v'(x)u''(x)}{[u'(x)]^2} \\ &= \frac{u(x)}{[u'(x)]^2} [v'(x)u''(x) - v''(x)u'(x)] \\ &= \frac{u(x)u'(x)v'(x)}{[u'(x)]^2} [\lambda_u - \lambda_v] \\ &= \frac{u(x)v'(x)}{u'(x)x} \left[ \frac{q}{p} \frac{x^q}{1 - x^q} - \frac{q'}{p'} \frac{x^{q'}}{1 - x^{q'}} \right]. \end{aligned}$$

We show  $s(x)$  is positive by demonstrating it is increasing on  $(0, 1)$ . To do so, we introduce the function

$$h(x) = \frac{x^{q'}(1 - x^q)}{x^q(1 - x^{q'})} = \frac{x^{q'} - x^{q+q'}}{x^q - x^{q+q'}}.$$

Computing its derivative we find

$$\begin{aligned} h'(x) &= \frac{(q'x^{q'-1} - (q + q')x^{q+q'-1})(x^q - x^{q+q'}) - (x^{q'} - x^{q+q'})(qx^{q-1} - (q + q')x^{q+q'-1})}{[x^q(1 - x^{q'})]^2} \\ &= \frac{1}{[x^q(1 - x^{q'})]^2} \left[ (q'x^{q+q'-1} - (q + q')x^{2q+q'-1} - q'x^{q+2q'-1} + (q + q')x^{2q+2q'-1}) \right. \\ &\quad \left. - (qx^{q+q'-1} - (q + q')x^{q+2q'-1} - qx^{2q+q'-1} + (q + q')x^{2q+2q'-1}) \right] \\ &= \frac{1}{[x^q(1 - x^{q'})]^2} \left[ q'x^{q+q'-1} - q'x^{2q+q'-1} - qx^{q+q'-1} + qx^{q+2q'-1} \right] \\ &= \frac{x^{q+q'-1}}{[x^q(1 - x^{q'})]^2} \left[ (q' - q) + (qx^{q'} - q'x^q) \right]. \end{aligned}$$

The first term above is positive on  $(0, 1)$ . For the second one we note

$$\begin{aligned} \left[ (q' - q) + (qx^{q'} - q'x^q) \right]' &= qq'x^{q'-1} - qq'x^{q-1} \\ &= qq'(x^{q'-1} - x^{q-1}) < 0 \end{aligned}$$

since  $q' > q$ . Hence this term is strictly decreasing on  $(0, 1)$ . Furthermore

$$\lim_{x \rightarrow 1} \left[ (q' - q) + (qx^{q'} - q'x^q) \right] = (q' - q) + (q - q') = 0.$$

Thus  $h'(x) > 0$  on  $(0, 1)$  and we conclude  $h(x)$  is strictly increasing. Computing the maximum value  $h(x)$  approaches, we find

$$\lim_{x \rightarrow 1} \frac{x^{q'} - x^{q+q'}}{x^q - x^{q+q'}} = \lim_{x \rightarrow 1} \frac{q'x^{q'-1} - (q + q')x^{q+q'-1}}{qx^{q-1} - (q + q')x^{q+q'-1}} = \frac{q}{q'}.$$

This implies the inequality

$$q'h(x) < q$$

on  $(0, 1)$ . Hence, on  $(0, 1)$

$$-\frac{q}{p} < \frac{q'}{p'}h(x) - \frac{q}{p} < q\left(\frac{1}{p'} - \frac{1}{p}\right) < 0.$$

This implies

$$\frac{q'}{p'}h(x) < \frac{q}{p},$$

or equivalently

$$\frac{q'}{p'} \frac{x^{q'}}{1 - x^{q'}} < \frac{q}{p} \frac{x^q}{1 - x^q}$$

on  $(0, 1)$ . This implies  $s'(x)$  is positive, hence  $s(x)$  is strictly increasing. Since

$$\lim_{x \rightarrow 0} s(x) = 0,$$

we conclude  $s(x) > 0$  on  $(0, 1)$ . This proves the result.  $\square$

**Corollary 4.** *If  $1 < p < p' < \infty$  and  $1 < q < q' < \infty$*

$$\frac{\sin_{p,q}^{-1}(x)}{\pi_{p,q}} < \frac{\sin_{p',q'}^{-1}(x)}{\pi_{p',q'}}$$

for  $x \in (0, 1)$ . If  $x \in (0, 1/2)$

$$\sin_{p',q'}(\pi_{p',q'}x) < \sin_{p,q}(\pi_{p,q}x),$$

and uniformly on  $(0, 1)$

$$1 < \frac{\frac{\sin_{p',q'}^{-1}(x)}{\pi_{p',q'}}}{\frac{\sin_{p,q}^{-1}(x)}{\pi_{p,q}}} < \frac{\pi_{p,q}}{\pi_{p',q'}}.$$

*Proof.* Since

$$\lim_{x \rightarrow 1} \frac{\sin_{p,q}^{-1}(x)}{\sin_{p',q'}^{-1}(x)} = \frac{\pi_{p,q}}{\pi_{p',q'}},$$

we can apply lemma 7 to conclude the first inequality. For the second inequality observe that if  $x \in (0, 1/2)$ , there exist  $y_1, y_2 \in (0, 1)$  such that

$$\sin_{p',q'}(\pi_{p',q'}x) = y_1$$

and

$$\sin_{p,q}(\pi_{p,q}x) = y_2.$$

Thus

$$x = \frac{\sin_{p,q}^{-1}(y_2)}{\pi_{p,q}} = \frac{\sin_{p',q'}^{-1}(y_1)}{\pi_{p',q'}},$$

and by the first inequality we conclude

$$\sin_{p',q'}^{-1}(y_1) < \sin_{p',q'}^{-1}(y_2).$$

By the strictly increasing property of this function we obtain  $y_1 < y_2$  as stated. For the last inequality, note that by lemma 7, the function

$$\frac{1}{w(x)} = \frac{\sin_{p',q'}^{-1}(x)}{\sin_{p,q}^{-1}(x)}$$

is strictly decreasing on  $(0, 1)$  with its image contained in the interval  $(\pi_{p',q'}/\pi_{p,q}, 1)$ . The result follows from this observation.  $\square$

By corollary 4, we obtain

$$\begin{aligned}\tau_{p,q}(\mathbf{1}) &= \left(2 \int_{(0,1)} \sin_{p,q}(\pi_{p,q}x) \sin(\pi x) dx\right)^n \\ &\geq \left(2 \int_{(0,1)} \sin(\pi x) \sin(\pi x) dx\right)^n = 1.\end{aligned}$$

Using this improvement yields the following corollary:

**Corollary 5.** *The system  $\{\sin_{p,q}(k_1\pi_{p,q}x_1) \sin_{p,q}(k_2\pi_{p,q}x_2) \dots \sin_{p,q}(k_n\pi_{p,q}x_n)\}_{(k_1,k_2,\dots,k_n) \in \mathbb{N}^n}$  is a basis in  $L^r((0,1)^n)$  if  $1 < p < 2, 1 < q < 2$  and*

$$\pi_{p,q} < \frac{2\pi^2}{(\pi^{2n} - 8^n)^{1/n}}$$

or if either  $p \geq 2$  or  $q \geq 2$ , and

$$\pi_{p,q} < \frac{16}{(\pi^{2n} - 8^n)^{1/n}}.$$

*Proof.* By lemma 6 followed by theorem 11 the result follows. The case  $p, q < 2$  follows similarly once the new upper bound is incorporated into lemma 6.  $\square$

Picking  $p_0, p_1$  such that

$$\pi_{p_0,p_0} = \frac{2\pi^2}{(\pi^4 - 64)^{1/2}},$$

and

$$\pi_{p_1,p_1} = \frac{16}{(\pi^4 - 64)^{1/2}}$$

we obtain the following particular result:

**Corollary 6.** *The system  $\{\sin_{p,p}(k_1\pi_{p,p}x_1) \sin_{p,p}(k_2\pi_{p,p}x_2)\}_{(k_1,k_2) \in \mathbb{N}^2}$  is a basis in  $L^r((0,1)^2)$  if  $p \in (p_0, 2) \cup (p_1, \infty)$ .*

*Proof.* This is an immediate consequence of corollary 5 and the fact  $\pi_{p,p}$  decreases as  $p$  increases.  $\square$

By the decreasing property of  $\pi_{p,q}$ , if  $p, q \in (1, 2)$

$$\pi_{p,q} > \pi.$$

Since

$$\pi > \frac{2\pi^2}{(\pi^{2n} - 8^n)^{1/n}}$$

for  $n > 2$ , an analogues result to corollary 2 for  $n = 3$  is limited to:

**Corollary 7.** *The system  $\{\sin_{p,p}(k_1\pi_{p,p}x_1) \sin_{p,p}(k_2\pi_{p,p}x_2) \sin_{p,p}(k_3\pi_{p,p}x_3)\}_{(k_1,k_2,k_3) \in \mathbb{N}^3}$  is a basis in  $L^r((0, 1)^3)$  if  $p \in (p_2, \infty)$ , where*

$$\pi_{p_2,p_2} = \frac{16}{(\pi^6 - 8^3)^{1/3}}.$$

We conclude this subsection by remarking that since  $2 \leq \pi_{p,q}$  for all  $p, q \in (1, \infty)$  and

$$\frac{16}{(\pi^{2n} - 8^n)^{1/n}} < 2$$

for  $n > 3$ , the applicability of corollary 5 is restrained to  $n = 1$ ,  $n = 2$ , and  $n = 3$ .

### 4.3 Bound Improvements and Concluding Remarks

We conclude this section improving the previous results with the help of corollary 4.

By corollary 4, for  $2 < p$

$$\frac{\pi_{p,p}}{\pi} \frac{\sin_{p,p}^{-1}(x)}{\pi_{p,p}} < \frac{\sin^{-1}(x)}{\pi}$$

on  $(0, 1)$ . Letting  $y = \sin_{p,p}(\pi_{p,p} x)$  we obtain

$$\frac{\pi_{p,p}}{\pi} x < \frac{\sin^{-1}(y)}{\pi}.$$

Since  $\sin(x)$  is strictly increasing,

$$\sin(\pi_{p,p} x) < y = \sin_{p,p}(\pi_{p,p} x)$$

on  $(0, 1/2)$ . Hence,

$$\hat{f}_{1p,p}(1) = 4 \int_{(0,1/2)} \sin_{p,p}(\pi_{p,p} x) \sin(\pi x) dx > 4 \int_{(0,1/2)} \sin(\pi_{p,p} x) \sin(\pi x) dx$$

$$\begin{aligned}
&= 2 \left( \frac{1}{\pi - \pi_{p,p}} \sin \left( \frac{\pi - \pi_{p,p}}{2} \right) - \frac{1}{\pi + \pi_{p,p}} \sin \left( \frac{\pi + \pi_{p,p}}{2} \right) \right) \\
&= 2 \cos \left( \frac{\pi_{p,p}}{2} \right) \left( \frac{1}{\pi - \pi_{p,p}} - \frac{1}{\pi + \pi_{p,p}} \right) \\
&= \frac{4\pi_{p,p} \cos \left( \frac{\pi_{p,p}}{2} \right)}{(\pi - \pi_{p,p})(\pi + \pi_{p,p})} = w(\pi_{p,p}).
\end{aligned}$$

We note  $w$  is increasing in  $(\pi/2, \pi)$  and

$$\lim_{p \rightarrow 2^+} w(\pi_{p,p}) = \lim_{p \rightarrow 2^+} \frac{4[\pi'_{p,p} \cos \left( \frac{\pi_{p,p}}{2} \right) - \pi_{p,p} \sin \left( \frac{\pi_{p,p}}{2} \right) \frac{\pi'_{p,p}}{2}]}{-2\pi_{p,p}\pi'_{p,p}} = 1.$$

Recalling  $\pi_{p,p}$  increases to  $\pi$  as  $p$  decreases to 2, it follows that if for any  $\delta > 0$  and  $\pi_{p^*,p^*} > \pi/2$

$$w(\pi_{p^*,p^*}) > 1 - \delta.$$

Thus

$$w(\pi_{p,p}) > 1 - \delta$$

for any  $p$  with  $2 \leq p < p^*$ . Additionally

$$w(\pi_{2.33,2.33}) = w \left( \frac{2\pi}{2.33 \sin \left( \frac{\pi}{2.33} \right)} \right) > \frac{93}{100} > \frac{8}{\pi^2}.$$

This implies

$$w(\pi_{p,p}) = \hat{f}_{1,p,p}(1) > \frac{93}{100}$$

for  $p \in [2, 2.33]$ . Noting the inequality

$$\left( \frac{4\pi_{p,p}}{\pi^2} \right)^2 \left( \left( \frac{\pi^2}{8} \right)^2 - 1 \right) < .93^2$$

holds for  $p < 2.33$ , we obtain the following improvement to corollary 6, where  $p_0$  is defined as before:

**Lemma 8.** *The system  $\{\sin_{p,p}(k_1\pi_{p,p}x_1) \sin_{p,p}(k_2\pi_{p,p}x_2)\}_{(k_1,k_2) \in \mathbb{N}^2}$  is a basis in  $L^r((0,1)^2)$  if  $p \in (p_0, \infty)$ .*

Using this result we can conclude the following theorem

**Theorem 13.** For  $r \in (1, \infty)$ , the system  $\{\sin_{p,q}(k_1\pi_{p,q}x_1) \sin_{p,q}(k_2\pi_{p,q}x_2)\}_{(k_1,k_2) \in \mathbb{N}^2}$  is a basis in  $L^r((0,1)^2)$  if  $(p,q) \in (p_0, \infty) \times (p_0, \infty)$ .

*Proof.* If  $2 \leq p, q < \infty$ , by the bound improvement in lemma 8, the following holds

$$\pi_{p,q} < \pi_{\min\{p,q\}, \min\{p,q\}} < \frac{16}{\sqrt{\pi^4 - 64}}.$$

Hence the basis property holds for  $(p, q)$ . Similarly, if  $p \in (p_0, 2)$  and  $q \in (p_0, 2)$ ,

$$\pi_{p,q} < \pi_{\min\{p,q\}, \min\{p,q\}} < \frac{2\pi^2}{\sqrt{\pi^4 - 64}},$$

and by corollary 6 the basis property holds as well. □

Repeating the same idea using corollary 7, we obtain our last result:

**Theorem 14.** For  $r \in (1, \infty)$ , the system  $\{\sin_{p,q}(k_1\pi_{p,q}x_1) \sin_{p,q}(k_2\pi_{p,q}x_2) \sin_{p,q}(k_3\pi_{p,q}x_3)\}_{(k_1,k_2,k_3) \in \mathbb{N}^3}$  is a basis in  $L^r((0,1)^3)$  if  $(p,q) \in (p_2, \infty) \times (p_2, \infty)$ .

In conclusion, we have continued showing properties of the generalized trigonometric functions. In particular, we found the  $\sin_{p,q}(x)$  functions can be used to obtain a Schauder basis for  $L^r((0,1)^n)$ ,  $r \in (1, \infty)$  and  $n = 1, 2, 3$ . With the two dimensional generalized Fourier system one could begin the development of non-orthogonal systems in the treatment of signal processing. The lesser degree of smoothness of these generalized functions could be an indispensable tool in studying image processing in the case of a discontinuous gradient. In principle, the smoothness of the generalized trigonometric functions could be adjusted by a variation of the parameters  $p$  and  $q$ .



## Open Problems and Future Direction

We conclude this work by referencing further basis results and by introducing the space of variable exponent and the corresponding  $p(x)$ -Laplacian problem. We state future directions of study and reference to related open problems.

### 5.1 Other Basis Results

There has been further findings on the basis properties of the generalized trigonometric functions. The functions  $S_{\frac{1}{p}}(x)$  and  $C_{\frac{1}{p}}(x)$  introduced by Lindqvist and Peetre are generalizations of the sine and cosine functions as well and are related to the ones covered in chapter 3 by the following formulas (see, [12],[13],[17]):

$$S_{\frac{1}{p}}(x) = \sin_{p,p'}(x) \quad \text{and} \quad C_{\frac{1}{p}}(x) = (\cos_{p,p'}(x))^{p-1}$$

for  $x \in \mathbb{R}$ ,  $p' = \frac{p}{p-1}$ . It can be shown that for  $x \in \mathbb{R}$

$$\frac{d}{dx} C_{\frac{1}{p}}(x) = -(S_{\frac{1}{p}}(x))^{p-1}, \quad \frac{d}{dx} S_{\frac{1}{p}}(x) = (C_{\frac{1}{p}}(x))^{p-1},$$

and

$$S_{p'}(x)^p + (C_{p'}(x))^p = 1.$$

Under appropriate conditions, the generalized functions  $S_{\frac{1}{p}}(x)$  and  $C_{\frac{1}{p}}(x)$  also form a Schauder basis for  $L^r((0,1)^d)$ ,  $r \in (1, \infty)$ ,  $d = 1, 2, 3$  [15],[17]. Furthermore, the generalized exponent function for  $x \in \mathbb{R}$

$$E_{\frac{1}{p}}(ix) = C_{\frac{1}{p}}(x) + iS_{\frac{1}{p}}(x),$$

can be shown to form a Schauder basis for  $L^r((-1,1)^d)$ ,  $r \in (1, \infty)$ ,  $d = 1, 2, 3$  (see, [16],[17]). As noted in the end of chapter 4, the basis properties of the generalized trigonometric functions is limited to three dimensions. It remains unsolved to show their basis properties in higher dimensions.

Equations analogues to the angle sum formulas could be another interesting problem to investigate. It was shown in [7] that for  $x \in [0, \pi_{4/3,4}/4)$ ,

$$\sin_{4/3,4}(2x) = \frac{2 \sin_{4/3,4}(x)(\cos_{4/3,4}(x))^{1/3}}{(1 + 4(\sin_{4/3,4}(x))^4(\cos_{4/3,4}(x))^{4/3})^{1/2}}.$$

This result was obtained using addition formulas for elliptic functions. It is yet to be shown general relationships for  $\sin_{p,q}(x + y)$  in terms of  $\sin_{p,q}(x)$  and  $\sin_{p,q}(y)$ .

## 5.2 Variable Exponent Spaces and the $p(x)$ -Laplacian

Along with the continuing study of the  $p$ -Laplacian problem and its variants, a natural future direction for investigation is the theory of variable exponent. We briefly define this generalized space and its analogue  $p(x)$ -Laplacian problem.

Let  $\Omega$  be a Lebesgue  $n$ -measurable subset of  $\mathbb{R}^n$ . We denote by  $\mathcal{M}(\Omega)$  the family of all extended complex measurable functions on  $\Omega$ . By  $P(\Omega)$  we denote the subset of  $\mathcal{M}(\Omega)$  consisting of all the functions  $p$  that map  $\Omega$  into  $(1, \infty)$  and satisfy

$$p_- := \operatorname{ess\,inf}_{x \in \Omega} p(x) > 1, \quad p_+ := \operatorname{ess\,sup}_{x \in \Omega} p(x) < \infty.$$

For  $f \in \mathcal{M}(\Omega)$  and  $p \in P(\Omega)$  we define the convex modular in the sense of Musielak ([18])

$$\rho_p(f) = \int_{\Omega} |f(x)|^{p(x)} dx$$

and the function space

$$L_{p(\cdot)}(\Omega) := \{f \in \mathcal{M}(\Omega) : \rho_p(f/\lambda) < \infty \text{ for some } \lambda > 0\}.$$

Endowed with the norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\},$$

the space  $L_{p(\cdot)}(\Omega)$  is a Banach space. This space generalizes the classical  $L_p(\Omega)$  space since the latter is obtained from the former when  $p(x)$  is a constant function. This new

generalized space poses significant new challenges such as the lack of translation invariance and the failure of Young's inequality. We reference [5],[18], and [19] for a treatment of these spaces and the results presented here. The  $p(x)$ -Laplacian eigenvalue problem now becomes

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{p(x)-2}u.$$

Aside from the mathematical intrinsic value of this generalized operator, there are many real world applications for these equations. The modeling of electrorheological fluids and image restoration are two important applications suitable in this mathematical framework (see, [20],[21]). Developing efficient algorithms for implementing variable exponent theory will be a future continuation of this work as well as the continued study of the classical theory.

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## Curriculum Vitae

I, Luis Raul Suarez Salas, was raised across the border in Cd. Juarez Mexico. I started school in the U.S. until the seventh grade. I've always had an interest in science, particularly during my last year of high school. I began school at UTEP in the fall of 2012 as a physics major. Thanks to an NSF Scholarship from UTEP and other merit-based awards, I was able to live on campus and focus on my studies. After taking an applied analysis course with my now mentor, I became interested in mathematics. Having completed two summer research opportunities, I was convinced of its ubiquitousness in science. Thanks to the MARC-U STAR program at UTEP, I began researching with Dr. Osvaldo Méndez from the mathematics department my junior year. He guided me through the learning of proof-based mathematics. I graduated with an honors physics degree from UTEP the summer of 2016. I also learned about a different branch of mathematics working with Dr. Dogan that summer thanks to the SURPASS program. I then enrolled in the master's program in mathematics that fall at UTEP. In it, I amplified my passion for teaching. I began teaching a geometry course for the Upward Bound Program of El Paso on Saturdays that year. I also began to teach a STEM robotics class last fall at the high school level. I hope to have a positive impact on the generations of future scientists. In particular, I hope to increase the number of underrepresented minorities in science. I completed my master's thesis project on May of 2018. I plan to begin a doctorate program in mathematics at Michigan State University in July of this year. I hope to have an impact on methods for implementing theoretical models, in particular the  $p(x)$ -Laplacian operator.

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