

1-1-2021

Can Ideas Behind Ancient Egyptian Fractions Speed up Modern Computers?

Olga Kosheleva

The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich

The University of Texas at El Paso, vladik@utep.edu

Follow this and additional works at: https://scholarworks.utep.edu/cs_techrep



Part of the [Computer Sciences Commons](#), and the [Mathematics Commons](#)

Comments:

Technical Report: UTEP-CS-21-12

Published in *Applied Mathematical Sciences* 2021, Vol. 15, No. 3, pp. 113-118.

Recommended Citation

Kosheleva, Olga and Kreinovich, Vladik, "Can Ideas Behind Ancient Egyptian Fractions Speed up Modern Computers?" (2021). *Departmental Technical Reports (CS)*. 1545.

https://scholarworks.utep.edu/cs_techrep/1545

This Article is brought to you for free and open access by the Computer Science at ScholarWorks@UTEP. It has been accepted for inclusion in Departmental Technical Reports (CS) by an authorized administrator of ScholarWorks@UTEP. For more information, please contact lweber@utep.edu.

Can Ideas Behind Ancient Egyptian Fractions Speed up Modern Computers?

Olga Kosheleva¹ and Vladik Kreinovich²

¹Department of Teacher Education

²Department of Computer Science

University of Texas at El Paso

500 W. University

El Paso, TX 79968, USA

olgak@utep.edu, vladik@utep.edu

Abstract

To divide two numbers a and b , modern computers use an algorithm which is more efficient than what we humans normally do: they compute $a \cdot (1/b)$, where for all sufficiently small integers b , the inverse $1/b$ is pre-computed. For fractions, when both a and b are integers, this algorithm requires only one multiplication. Can we make the procedure even faster by not using multiplication at all? To do this, we need to represent each fraction as the sum of inverses – which, interestingly, is how ancient Egyptians represented fractions.

Mathematics Subject Classification: 01A16, 68M07

Keywords: Egyptian fractions, computer architecture

1 Formulation of the Problem

In computers, all computations are reduced to arithmetic operations. Computers perform many important computations. We can use them to compute the value of complex functions, we can use them to solve complex partial differential equations, etc. However, the only operations which are supported on the hardware level are arithmetic operations. All other computations – no matter how complex they are – are performed, in a computer, as a sequence of arithmetic operations.

In some cases, a translation of a computational task into a sequence of arithmetic operations is straightforward. For example, to compute $\exp(x)$ or $\sin(x)$, we expand these functions in Taylor series and keep several first terms in this expansion. As a result, the desired function is approximated by a polynomial, and a polynomial is, by definition, what can be obtained by variables and constants by using addition, subtraction, and multiplications.

How arithmetic operations are implemented in a computer. In general, there are four basic arithmetic operations – addition, subtraction, multiplication, and division. For us – and for a computer – the simplest are addition and subtraction. Multiplication is more complex because, e.g., for binary numbers, the usual multiplication algorithm – in which we multiply one of the numbers by digits and then add the result – reduced to several additions: indeed, binary digits are 0 and 1, so multiplying by these digits is trivial.

Division is even more complex than multiplication, since the usual way – that we learn at school – requires several multiplications. In a computer, division a/b is implemented in a more efficient way: as $a/b = a \cdot (1/b)$, where the inverses $1/b$ are:

- pre-recorded for small integers b , and
- computed based on these pre-recorded values for non-integer denominators b .

This way, computing each rational number – i.e., each fraction a/b in which both a and b are natural numbers – requires only one multiplication.

Natural question: can we do better? Can we make this computation even faster? We have already went from the traditional division algorithm – that requires several multiplications – to an algorithm that needs only one multiplication. So, the only way to further speed up computations is to find a way to avoid multiplication altogether.

What we do in this paper. In this paper, we show that this may indeed be possible – if we use an idea going back several thousand years, to ancient Egyptian fractions.

2 Egyptian Fractions and the Resulting Idea

What we want. We want to avoid multiplication. We still need to use $1/b$ values – otherwise, if we only use addition and subtraction, we will never get beyond integers. We do not want to apply multiplication to these fractions, we only want to apply addition (and subtraction, but subtracting $1/b$ is equivalent

to adding $1/(-b)$). In other words, we want to represent the fraction a/b as the sum of inverses:

$$\frac{a}{b} = \frac{1}{n_1} + \dots + \frac{1}{n_k}. \quad (1)$$

This is exactly what ancient Egyptians did. Interestingly, this is exactly how people in ancient Egypt represented fractions: as sums of inverses; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

How exactly we can derive such a representation: idea. Of course, we can always represent the fraction a/b as

$$\frac{a}{b} = \frac{1}{b} + \dots + \frac{1}{b} \text{ (} a \text{ times)}, \quad (2)$$

but for large a , this would require too many additions – and the resulting computation time will be large. We therefore need to find a representation (1) with a small number of terms.

A natural idea is thus – after we sort the inverses in order, so that

$$n_1 \leq n_2 \leq \dots \leq n_k$$

and thus, $1/n_1 \geq \dots \geq 1/n_k$ – is to select the first term $1/n_1$ to be as large as possible – i.e., to select the value n_1 to be as large as possible.

The condition $a/b \geq 1/n_1$ is equivalent to $n_1 \geq b/a$. Thus, the smallest possible n_1 with this property is the smallest possible integer which is larger than or equal to b/a . Such an integer is known as a *ceiling* of the value b/a ; it is denoted by $n_1 = \lceil b/a \rceil$.

If a divides b , i.e., if $b = c \cdot a$ for some c , then $a/b = 1/c$, i.e., we get the representation (1) with just one value $n_1 = c$. In general, a does not divide b , i.e., we have $b = c \cdot a + r$ for some non-zero remainder $0 < r < a$. In this case, $b/a = c + r/a$, where $0 < r/a < 1$, and so $n_1 = \lceil b/a \rceil = c + 1$.

Now, the remaining part $a/b - 1/c$ has the form

$$\frac{a}{b} - \frac{1}{c} = \frac{a}{c \cdot a + r} - \frac{1}{c+1} = \frac{(c+1) \cdot a - (c \cdot a + r)}{b \cdot (c+1)} = \frac{a - r}{b \cdot (c+1)}.$$

The numerator $a - r$ of this remaining part is smaller than the original numerator a .

The remainder r can take any value between 0 and a , so its average value is $a/2$. Thus, on average, the numerator of the remaining part is twice smaller than the original numerator a . In 2 steps, we get a numerator $a/4$, in k steps, we get a numerator of order $a/2^k$. So, in $k = \log_2(a)$ steps, when $2^k = a$, we get the numerator equal to $a/2^k = 1$ – i.e., the remaining fraction is the inverse of an integer. Thus, on average, we only need $\log_2(a)$ terms in the

expression (1). This number is much smaller than a terms needed for the naive representation (2).

It is not clear whether the new algorithm is always better. Whether the use of Egyptian fraction will indeed be faster than multiplication is not clear. Indeed:

- The use of Egyptian fractions requires $\log_2(a)$ additions.
- On the other hand, multiplication of a binary number by a means adding as many terms as there are binary digits in a – i.e., also, an order or $\log_2(a)$.

So, to decide which method is faster we need to implement the new method and check the results.

How to derive the desired representation: algorithm. If a divides b , i.e., if $b = c \cdot a$ for some c , then $a/b = 1/c$ is the desired 1-term representation of type (1).

If a does not divide b , then we take $n_1 = \lceil b/a \rceil$. To the remaining fraction $a/b - 1/n_1$, we apply the same algorithm, etc., until we get a representation of type (1).

Examples. Let us consider, as examples, all irreducible fractions a/b with $b \leq 7$ and $1 < a < b$.

For $b = 3$, the only such fraction is $2/3$. Here, $\lceil 3/2 \rceil = 2$, so $n_1 = 2$. The difference $2/3 - 1/2$ is equal to $1/6$, so we get

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}.$$

For $b = 4$, the only such fraction is $3/4$. By applying the above algorithm, we get

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}.$$

For $b = 5$, we have three such fractions: $2/5$, $3/5$. and $4/5$. Here,

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15},$$

and

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}.$$

For $4/5$, we get $n_1 = 2$. Here, $4/5 - 1/2 = 3/10$. For $3/10$, we get $n_2 = \lceil 10/3 \rceil = 4$, and $3/10 - 1/4 = 1/20$, so

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}.$$

For $b = 6$, the only such fraction is $5/6$:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}.$$

For $b = 7$, we get

$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28},$$

$$\frac{3}{7} = \frac{1}{3} + \frac{2}{21} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231};$$

$$\frac{4}{7} = \frac{1}{2} + \frac{1}{14},$$

$$\frac{5}{7} = \frac{1}{2} + \frac{3}{14} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70},$$

$$\frac{6}{7} = \frac{1}{2} + \frac{5}{14} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes). It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

References

- [1] C. B. Boyer and U. C. Merzbach, *A History of Mathematics*, Wiley, New York, 1991.
- [2] D. Eppstein, Egyptian fractions website
<http://www.ics.uci.edu/~eppstein/numth/egypt/>
- [3] M. Gardner, “Puzzles and number-theoretic problems arising from the curious fractions of Ancient Egypt”, *Scientific American*, October 1978.
- [4] P. Hoffman, *The man who loved only numbers: the story of Paul Erdős and the search for mathematical truth*, Hyperion, New York, 1998.
- [5] O. Kosheleva and V. Kreinovich, “Egyptian fractions revisited”, *Abstracts of the 2005 Meeting of the Southwestern Section of the Mathematical Association of America (MAA)*, April 1–2, 2005, p. 6.

- [6] O. Kosheleva and V. Kreinovich, “Egyptian fractions revisited”, *Informatics in Education*, 2009, Vol. 8, No. 1, pp. 35–48.
- [7] O. Kosheleva and V. Kreinovich, “Why ancient Egyptians preferred some sum-of-inverses representations of fractions?”, *Applied Mathematical Sciences*, 2020, Vol. 14, No. 18, pp. 859–865.
- [8] O. Kosheleva and V. Kreinovich, “Yet another possible explanation of Egyptian fractions: motivated by fairness”, *Applied Mathematical Sciences*, 2020, Vol. 14, No. 19, pp. 919–924
- [9] O. Kosheleva, V. Kreinovich, and F. Zapata, “Egyptian fractions re-revisited”, *Russian Digital Libraries Journal*, 2019, Vol. 22, No. 6, pp. 763–768.
- [10] O. Kosheleva and I. Lyublinskaya, “Teaching Fractions with the Help of Egyptian Papyrus and Technology”, *Abstracts of the Teachers Teaching with Technology T3 Regional Conference “Using Technology to Engage Students in Discovery Learning”*, Staten Island, New York, November 3–4, 2006.
- [11] O. Kosheleva and I. Lyublinskaya, “Can Egyptian papyrus enrich our students’ understanding of fractions?”, *Abstracts of the Annual Meeting of the National Council of Teachers of Mathematics NCTM “Mathematics: Representing the Future”*, Atlanta, Georgia, March 21–24, 2007, p. 40.
- [12] O. Kosheleva and I. Lyublinskaya, “Using innovative fraction activities as a vehicle for examining conceptual understanding of fraction concepts in pre-service elementary teachers mathematical education”, In: T. Lamberg and L. R. Wiest (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education PME-NA 2007*, Stateline (Lake Tahoe), Nevada, October 25–28, 2007, University of Nevada Publ., Reno, 2007, pp. 36–38.
- [13] L. Streefland, *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

Received: January 23, 2021