

1-1-2021

Distributions on an Interval as a Scale-Invariant Combination of Scale-Invariant Functions: Theoretical Explanation of Empirical Marchenko-Pastur-Type Distributions

Vladik Kreinovich

The University of Texas at El Paso, vladik@utep.edu

Kevin Alvarez

The University of Texas at El Paso, kalvarez9@miners.utep.edu

Chon Van Le

International University of Ho-Chi-Minh City, lvchon@hcmiu.edu.vn

Follow this and additional works at: https://scholarworks.utep.edu/cs_techrep



Part of the [Computer Sciences Commons](#), and the [Mathematics Commons](#)

Comments:

Technical Report: UTEP-CS-21-04

Recommended Citation

Kreinovich, Vladik; Alvarez, Kevin; and Le, Chon Van, "Distributions on an Interval as a Scale-Invariant Combination of Scale-Invariant Functions: Theoretical Explanation of Empirical Marchenko-Pastur-Type Distributions" (2021). *Departmental Technical Reports (CS)*. 1537.

https://scholarworks.utep.edu/cs_techrep/1537

This Article is brought to you for free and open access by the Computer Science at ScholarWorks@UTEP. It has been accepted for inclusion in Departmental Technical Reports (CS) by an authorized administrator of ScholarWorks@UTEP. For more information, please contact lweber@utep.edu.

Distributions on an Interval as a Scale-Invariant Combination of Scale-Invariant Functions: Theoretical Explanation of Empirical Marchenko-Pastur-Type Distributions

Vladik Kreinovich, Kevin Alvarez, and Chon Van Le

Abstract In many practical situations, we know the lower and upper bounds \underline{x} and \bar{x} on possible values of a quantity x . In such situations, the probability distribution of this quantity is also located on the corresponding interval $[\underline{x}, \bar{x}]$. In many such cases, the empirical probability distribution has the form

$$\rho(x) = \text{const} \cdot (x - \underline{x})^{\alpha_-} \cdot (\bar{x} - x)^{\alpha_+} \cdot x^\alpha.$$

In the particular case $\alpha_- = \alpha_+ = 0.5$ and $\alpha = -1$, we get the Marchenko-Pastur distribution that describes the distribution of the eigenvalues of a random matrix. However, in some cases, the empirical distribution corresponds to different values of α_- , α_+ , and α . In this paper, we show that by using the general idea of scale-invariance, we can provide a theoretical explanation for the ubiquity of such Marchenko-Pastur-type distributions.

1 Formulation of the Problem

Some distributions are located on an interval. For many physical and economic quantities x :

- there are is a lower bound \underline{x} on its possible values and
- there is an upper bound \bar{x} on its possible values.

Kevin Alvarez. Vladik Kreinovich,
University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA
e-mail: vladik@utep.edu, kalvarez9@miners.utep.edu,

Chon Van Le
International University of Ho-Chi-Minh City, Vietnam, e-mail: lvchon@hcmiu.edu.vn

This means that all possible values on the quantity x are located on the interval $[\underline{x}, \bar{x}]$. In particular, this means that the probability distribution of this quantity is located on this interval.

Empirical fact. In economics, as shown, e.g., in [4], many such distributions have the form

$$\rho(x) = \text{const} \cdot (x - \underline{x})^{\alpha_-} \cdot (\bar{x} - x)^{\alpha_+} \cdot x^{\alpha}. \quad (1)$$

In particular, for $\alpha_- = \alpha_+ = 0.5$ and $\alpha = -1$, we get the Marchenko-Pastur distribution – the distribution of eigenvalues of a random matrix [3]. For example, this is how the eigenvalues of the cross-correlation matrix of different stocks are distributed [4].

However, in other cases, we have distributions of type (1) with different values of α_- , α_+ , and α .

A natural question. How can we explain the ubiquity of such Marchenko-Pastur-type distributions?

What we do in this paper. In this paper, we use the idea of scale invariance to provide a theoretical explanation for this empirical family of distributions.

2 Analysis of the Problem and the Resulting Explanation

Scale-invariance: a brief reminder. The numerical value of a quantity depends on the choice of a measuring unit. For example, we can describe the price of a financial instrument in Euros, in US Dollars, in Japanese Yen, or in any other currency. The instrument is the same in all cases, but for different currencies, we will get different numerical representations of the same price.

In many situations, there is no reason to select this or that measuring unit, the choice of the unit is just a matter of convention. In such situations, it makes sense to require that the formula $y = f(x)$ describing the dependence between quantities x and y should not change if we replace the original measuring unit for x . If we replace the original measuring unit for x by a new unit which is λ times smaller, then all the numerical values of this quantity will be multiplied by λ :

$$x \rightarrow x' = \lambda \cdot x.$$

Of course, for the formula $y = f(x)$ to be valid in the new units, we need to appropriately change the unit for y . For example, the formula $y = x^2$ that describes how the area of a square depends on its size does not depend on the choice of units, but if we replace, e.g., meters with centimeters, we need to also replace square meters with square centimeters.

In general, scale-invariance of a function $f(x)$ takes the following form: *For every $\lambda > 0$, there exists a value $\mu(\lambda) > 0$ for which $y = f(x)$ implies $y' = f(x')$, where $x' \stackrel{\text{def}}{=} \lambda \cdot x$ and $y' \stackrel{\text{def}}{=} \mu(\lambda) \cdot y$.*

Which functions are scale-invariant? Substituting the expressions $x'=\lambda \cdot x$ and $y'=\mu(\lambda) \cdot y$ into the formula $y' = f(x')$, we get $\mu(\lambda) \cdot y = f(\lambda \cdot x)$. Here, $y = f(x)$, so we get

$$f(\lambda \cdot x) = \mu(\lambda) \cdot f(x). \quad (2)$$

It is known that every continuous (and even every measurable) function $f(x)$ that satisfies the equation (2) for all x and λ has the form

$$f(x) = c \cdot x^a, \quad (3)$$

for some constants c and a ; see, e.g., [1].

Starting point can also be different. For many quantities – e.g., for time – we can also select different starting points. If we replace the original starting point with a new starting point which is x_0 units before, then all the numerical values x of this quantity are replaced by new values:

$$x' = x + x_0.$$

If we have a scale-invariant dependence $f(x) = c \cdot (x')^a$ in the new scale, then, in the old scale, this dependence takes the form $y = c \cdot (x + x_0)^a$.

What are natural starting points for functions located on an interval $[\underline{x}, \bar{x}]$. If we know that a quantity x is always located on an interval $[\underline{x}, \bar{x}]$, then we have two natural starting points: \underline{x} and \bar{x} . Thus, in addition to the original scale-invariant functions $f(x) = c \cdot x^a$, we also get functions

$$f(x) = c_- \cdot (x - \underline{x})^{a-} \quad (4)$$

and

$$f(x) = c_+ \cdot (\bar{x} - x)^{a+}. \quad (5)$$

Since we need a single function, we need to combine these functions.

How can we use scale-invariance to combine different functions? We want to combine several functions $y_1 = f_1(x), \dots, y_n = f_n(x)$ into a single quantity $y = F(y_1, \dots, y_n)$. In view of the above, it makes sense to do it in scale-invariant way. In other words, we want to find a function $F(y_1, \dots, y_n)$ that has the following property: *For every combination of possible values $\lambda_1 > 0, \dots, \lambda_n > 0$, there should exist a value $\mu(\lambda_1, \dots, \lambda_n)$ for which $y = F(y_1, \dots, y_n)$ implies that $y' = F(y'_1, \dots, y'_n)$, where $y'_1 \stackrel{\text{def}}{=} \lambda_1 \cdot y_1, \dots, y'_n \stackrel{\text{def}}{=} \lambda_n \cdot y_n$, and $y' \stackrel{\text{def}}{=} \mu(\lambda_1, \dots, \lambda_n) \cdot y$.*

Which combination operations are scale-invariant? Substituting the expressions $y'_i = \lambda_i \cdot y_i$ and $y' = \mu(\lambda_1, \dots, \lambda_n) \cdot y$ into the formula $y' = F(y'_1, \dots, y'_n)$, we get

$$\mu(\lambda_1, \dots, \lambda_n) \cdot y = F(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n).$$

Here, $y = F(y_1, \dots, y_n)$, so we get

$$F(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n) = \mu(\lambda_1, \dots, \lambda_n) \cdot F(y_1, \dots, y_n). \quad (6)$$

It is known that every continuous (and even every measurable) function $F(y_1, \dots, y_n)$ that satisfies the equation (6) for all y_1, \dots, y_n and $\lambda_1, \dots, \lambda_n$ has the form

$$y = F(y_1, \dots, y_n) = C \cdot y_1^{a_1} \cdot \dots \cdot y_n^{a_n}, \quad (7)$$

for some constants C, a_1, \dots, a_n ; see, e.g., [1].

Resulting expression. In our case, we combine three expression:

- the expression for y_1 described by the formula (4),
- the expression for y_2 described by the formula (5),
- the expression for y_3 described by the formula (3).

Substituting $n = 3$ and the expressions (4), (5), and (3) for y_i into the formula (7), we conclude that

$$y = c_0 \cdot (x - \underline{x})^{\alpha_-} \cdot (\bar{x} - x)^{\alpha_+} \cdot x^\alpha, \quad (8)$$

where we denoted $c_0 \stackrel{\text{def}}{=} C \cdot c^{a_1} \cdot c_-^{\alpha_-} \cdot c_+^{\alpha_+}$, $\alpha_- \stackrel{\text{def}}{=} a_2 \cdot a_-$, $\alpha_+ \stackrel{\text{def}}{=} a_3 \cdot a_+$, and $\alpha \stackrel{\text{def}}{=} a_1 \cdot a$.

For the case when y is the probability density, this is exactly the desired formula (1). So, we have indeed explained the empirical formula (1) by using scale-invariance.

Comment. Our explanation is more general that explaining the empirical dependence (1) of the distribution located on an interval.

It also explains, e.g., why in many cases, Bernstein polynomials, i.e., sums of monomials of the type $(x - \underline{x})^{\alpha_-} \cdot (\bar{x} - x)^{\alpha_+}$, provide a good approximation to functions located on an interval; see, e.g., [2].

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes). It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

References

1. J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 2008.
2. G. G. Lorentz, *Bernstein Polynomials*, American Mathematical Society, Providence, Rhode Island, 2012.
3. V. A. Marchenko and L. A. Pastur, "Distribution of eigenvalues for some sets of random matrices", *Mathematics of the USSR-Sbornik*, 1967, Vol. 1, No. 4, pp. 457–483.

4. Hieu T. Nguyen, Phuong N. U. Tran, and Quang Nguyen, “An analysis of eigenvectors of a stock market cross-correlation matrix”, In: Ly H. Anh, Le Si Dong, V. Kreinovich, and Nguyen Ngoc Thach (eds.), *Econometrics for Financial Applications*, Springer Verlag, Cham, Switzerland, 2018, pp. 504–513.