11-2020

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Comments:

Technical Report: UTEP-CS-20-109

Published in *Applied Mathematical Sciences*, 2020, Vol. 14, No. 18, pp. 859-865.

Why Ancient Egyptians Preferred Some Sum-of-Inverses Representations of Fractions?

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Abstract

Ancient Egyptians represented a fraction as a sum of inverses of natural numbers, with the smallest possible number of terms. In our previous paper, we explained that this representation makes sense since it leads to the optimal way of solving a problem frequently mentioned in the Egyptian papyri: dividing bread between workers. However, this does not explain why ancient Egyptians preferred some representations with the same number of terms but not others. For example, to represent 2/3, they used the sum 1/2 + 1/6 but not the sum 1/3 + 1/3 with the same number of terms. In this paper, we use a more detailed analysis of the same dividing-bread problem to explain this preference. Namely, in our previous explanation, we assumed that each cut requires the same amount of time. If we take into account that in practice, each consequent cut of the same loaf – just like any other repetitive action – takes a little less time, we get the desired explanation of why ancient Egyptians preferred, e.g., 1/2 + 1/6.

1 Egyptian Fractions – What and Why: A Reminder

Egyptian fractions – what is it. In ancient Egypt, fractions were represented as the sum of inverse numbers with the smallest possible number of terms; see, e.g., [1, 2, 3] and references therein. For example, $\frac{5}{6}$ was represented as

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}. \tag{1}$$

Egyptian fractions – why. One of the possible explanations for the above representation is that it leads to the optimal solution of a problem actively discussed in the ancient Egyptian papyri: how to divide loaves of bread between workers by making the smallest possible number of cuts; see, e.g., [4].

Suppose that we have 6 workers, and we want to give each of them $\frac{5}{6}$ of the loaf. For this purpose, we need $6 \cdot \frac{5}{6} = 5$ loaves. A straightforward idea is to divide each loaf into 6 parts, and give 5 parts to each worker. To divide a loaf into 6 parts, we need to make $6 - 1 = 5$ cuts. Thus, overall, we need to make $5 \cdot 5 = 25$ cuts.

Alternatively, we can use representation (1) and give each worker $\frac{1}{2}$ and $\frac{1}{3}$ of a loaf. In this arrangement:

- To make six half-parts, we need to cut 3 loaves into 2 parts each. For each of these loaves, we need just one cut, so this requires 3 cuts.
- To make six one-thirds, we need to cut each of the remaining 2 loaves into 3 parts each. To divide a loaf into 3 parts, we need $3 - 1 = 2$ cuts, to the total of $2 \cdot 2 = 4$ cuts.

Thus, overall number of cuts in this arrangement is $3 + 4 = 7$, which is much smaller than 25.

In general, suppose that we represent a fraction $\frac{p}{q}$ as a sum of inverses:

$$\frac{p}{q} = \frac{1}{n_1} + \ldots + \frac{1}{n_k}. \tag{2}$$

Then, to make sure that each of N workers gets $\frac{p}{q}$-part of a loaf, we can:

- divide $N \cdot \frac{1}{n_1}$ loaves into n_1 parts each; each division requires $n_1 - 1$ cuts, to the total of $N \cdot \frac{n_1 - 1}{n_1}$;
- divide $N \cdot \frac{1}{n_2}$ loaves into n_2 parts each; each division requires $n_2 - 1$ cuts, to the total of $N \cdot \frac{n_2 - 1}{n_2}$, etc.

Thus, overall, we need the following number of cuts:

$$N \cdot \frac{n_1 - 1}{n_1} + \ldots + N \cdot \frac{n_k - 1}{n_k} = N \cdot \left(1 - \frac{1}{n_1} + \ldots + 1 - \frac{1}{n_k}\right) =$$

$$N \cdot \left(k - \left(\frac{1}{n_1} + \ldots + \frac{1}{n_k}\right)\right) = N \cdot \left(k - \frac{p}{q}\right).$$

So, to minimize the number of cuts, we need to minimize the number k of inverses, i.e., find the representation (2) with the smallest possible number of terms.

From this viewpoint, the Egyptian's representation (1) with two terms is preferable to the implicitly understood representation of the fraction $\frac{5}{6}$ as

$$\frac{5}{6} = \frac{1}{6} + \ldots + \frac{1}{6},$$

that requires five terms.

Remaining problem. The above argument explains why ancient Egyptians used their representation of fractions. However, from this viewpoint, there is no difference between, e.g., two different two-term representations of the fraction $\frac{2}{3}$:

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{1}{3} + \frac{1}{3}.$$

However, ancient Egyptians actively used the first representation and never used the second one. Why?

In this paper, we provide a possible explanation of why ancient Egyptians preferred some sum-of-inverses representations of fractions.

2 Main Idea and Resulting Explanation

Main idea. In the above argument, we simply counted the overall number of cuts. This implicitly assumed that each cut requires the same effort and takes the same time.

In practice, when we perform the same task several times in a row, we kind of get into the rhythm and start doing things faster and faster – not drastically faster (that would require a lot of experience), but somewhat faster.

For example, if we make 5 cuts needed to cut a loaf into 6 pieces, then the second cut takes a little bit less time than the first one, the third cut takes a little less time than the second cut, etc. When we finish cutting the loaf and move to the next loaf, we break the rhythm and get back to the original cutting times.

Let us describe this idea in precise terms. For each $c \geq 1$, let us denote the time needed to perform c cuts on the same loaf by $T(c)$. Clearly, $T(0) = 0$ – if we do not need to perform any cuts, we do not need to spend any time.

For any $c \geq 1$, the time $T(c)$ of making c cuts consists of the time $T(c-1)$ that is needed to make first $c-1$ cuts, and the time needed to make the last cut. Thus, the time needed to make the c-th cut is equal to the difference

$$T(c) - T(c-1).$$

In these terms, the requirement that this time decreases with c means that for every c, we have

$$T(c+1) - T(c) < T(c) - T(c-1). \tag{3}$$

Towards a convenient equivalent description. We have defined the function $T(c)$ only for non-negative integer values c. To simplify our analysis, we can apply linear interpolation between every two consequent integers and thus, get a function which is defined for all non-negative real values c. For this function, on each interval $(c, c+1)$, the derivative is equal to $T(c+1) - T(c)$.

In these terms, the formula (3) means that the derivative is a (non-strictly) decreasing function of c. Thus, the function $T(c)$ is *concave*; see, e.g., [5].

So which representation should we prefer? When we divide $N \cdot \frac{1}{n_i}$ loaves into $n_i - 1$ parts each, we need time $T(n_i - 1)$ to process each loaf. Thus, the time to process all these loaves is equal to $N \cdot \frac{T(n_i - 1)}{n_i}$, and the overall time for processing all the loaves is equal to the sum of these expressions:

$$N \cdot \frac{T(n_1 - 1)}{n_1} + \ldots + N \cdot \frac{T(n_k - 1)}{n_k} = N \cdot \left(\frac{T(n_1 - 1)}{n_1} + \ldots + \frac{T(n_k - 1)}{n_k} \right).$$

We want to select a procedure that minimizes this time, i.e., equivalently, that minimizes the time-per-worker

$$T = \frac{T(n_1 - 1)}{n_1} + \ldots + \frac{T(n_k - 1)}{n_k}. \quad (4)$$

Case of $k = 2$. In general, which representation is better depends on the specifics of the concave function $T(c)$. However, as we will show, in the important case $k = 2$, when

$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2}, \quad (5)$$

the optimal representation does not depend on the selection of the function $T(c)$ – and coincides with the choices that the ancient Egyptians made.

Indeed, the value (4) can be described as $T = \frac{p}{q} \cdot S$, where

$$S \stackrel{\text{def}}{=} \frac{q}{p} \cdot T = \frac{q}{p \cdot n_1} \cdot T(n_1 - 1) + \frac{q}{p \cdot n_2} \cdot T(n_2 - 1). \quad (6)$$

For each fraction $\frac{p}{q}$, minimizing time T is equivalent to minimizing $S = \frac{q}{p} \cdot T$.

Due to (5), we have

$$\frac{p}{q} = \frac{n_1 + n_2}{n_1 \cdot n_2},$$

hence

$$\frac{q}{p} = \frac{n_1 \cdot n_2}{n_1 + n_2},$$

and

$$\frac{q}{p \cdot n_1} = \frac{1}{n_1} \cdot \frac{q}{p} = \frac{1}{n_1} \cdot \frac{n_1 \cdot n_2}{n_1 + n_2} = \frac{n_2}{n_1 + n_2}.$$

Similarly,

$$\frac{q}{p \cdot n_2} = \frac{n_1}{n_1 + n_2}.$$

So, the coefficients at $T(n_1 - 1)$ and $T(n_2 - 1)$ in the formula (6) add up to 1. Thus, the formula (6) can be presented as

$$S = \alpha \cdot T(x_1) + (1 - \alpha) \cdot T(x_2), \text{ where } \alpha \stackrel{\text{def}}{=} \frac{n_2}{n_1 + n_2} \text{ and } x_i \stackrel{\text{def}}{=} n_i - 1. \quad (7)$$

Here,

$$\alpha \cdot x_1 + (1 - \alpha) \cdot x_2 = \frac{n_2 \cdot (n_1 - 1)}{n_1 + n_2} = \frac{n_1 \cdot (n_2 - 1)}{n_1 + n_2} =$$

$$2 \cdot \frac{n_1 \cdot n_2}{n_1 + n_2} - \frac{n_1 + n_2}{n_1 + n_2} = 2 \cdot \frac{q}{p} - 1.$$

Let us denote this value by $x_0 \stackrel{\text{def}}{=} 2 \cdot \dfrac{q}{p} - 1$. Then, the question becomes:

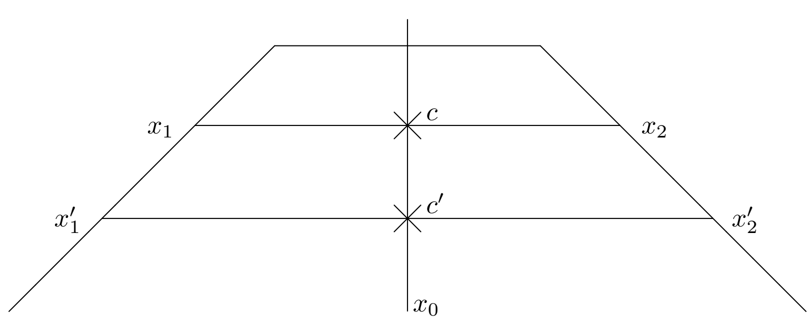
- we have a value x_0, we have a convex function $T(x)$;
- out of possible values x_1 and x_2 for which $\alpha \cdot x_1 + (1 - \alpha) \cdot x_2 = x_0$, we must select the values for which the convex combination

  $$c = \alpha \cdot T(x_1) + (1 - \alpha) \cdot T(x_2)$$

  is the smallest.

Without login generality, we can take $n_1 \le n_2$ and thus, $x_1 \le x_2$. When we decrease n_1 (i.e., equivalently, decrease x_1, then n_2 (and thus x_2) increases. So, whenever we compare two different pairs (x_1, x_2) and (x_1', x_2'), one of the corresponding intervals $[x_1, x_2]$ and $[x_1', x_2']$ is a subset of another. Without losing generality, we can assume that $[x_1, x_2] \subseteq [x_1', x_2']$. In this case, as one can easily see, for concave functions, the corresponding convex combination of the values of the function $T(x)$ is smaller for a larger interval:

$$c' \stackrel{\text{def}}{=} \alpha' \cdot T(x_1') + (1 - \alpha') \cdot T(x_2') < c \stackrel{\text{def}}{=} \alpha \cdot T(x_1) + (1 - \alpha) \cdot T(x_2):$$



Conclusion. Thus, we should select the pair (n_1, n_2) for which the value $x_1 = n_1 - 1$ is the smallest, i.e., equivalently, for which n_1 is the smallest.

In particular, for the fraction $\dfrac{2}{3}$, the smallest possible value of n_1 is $n_1 = 2$. This explains why the ancient Egyptians used the representation $\dfrac{2}{3} = \dfrac{1}{2} + \dfrac{1}{6}$, but not a more natural representation $\dfrac{2}{3} = \dfrac{1}{3} + \dfrac{1}{3}$.

Comment. We can easily algorithmically find such smallest n_1.

Indeed, since $n_1 \le n_2$, we have $\dfrac{1}{n_1} \ge \dfrac{1}{n_2}$ hence $\dfrac{1}{n_1} \ge \dfrac{1}{2} \cdot \dfrac{p}{q}$, and $n_1 \le \dfrac{2q}{p}$. Thus, there are finitely many possible values n_1, so, by trying all of them, we can algorithmically find the smallest n_1 for which $\dfrac{1}{n_1} + \dfrac{1}{n_2} = \dfrac{p}{q}$ for some natural number n_2, i.e., for which the value

$$\frac{1}{\dfrac{p}{q} - \dfrac{1}{n_1}}$$

is an integer.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes).

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