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Abstract

In ancient Egypt, fractions were represented as the sum of inverses to natural numbers. Processing fractions in this representation is computationally complicated. Because of this complexity, traditionally, Egyptian fractions used to be considered an early inefficient approach. In our previous papers, we showed, however, that the Egyptian fractions actually provide an optimal solution to problems important for ancient Egypt – such as the more efficient distribution of food between workers. In these papers, we assumed, for simplicity, that we know the exact amount of food needed for each worker – and that this value must be maintained with absolute accuracy. In this paper, we show that the corresponding food distribution can become even more efficient if we make the setting more realistic by allowing "almost exact" (approximate) representations.

1 Formulation of the Problem

Egyptian fractions: reminder. In ancient Egypt, fractions were represented as a sum of fractions of the type $\dfrac{1}{n}$, with the smallest possible number of terms; see, e.g., [1, 2, 3, 4, 11] and references therein. For example, $\dfrac{5}{12}$ was represented as

$$\frac{5}{12} = \frac{1}{3} + \frac{1}{12}. \tag{1}$$

Traditional history-of-mathematics view on Egyptian fractions. Dealing with such fractions is not computationally easy: e.g., multiplying two 3-term Egyptian fractions would generate the sum of $3 \cdot 3 = 9$ term-by-term products, and we face a complex problem of how to find the representation of this product that uses the smallest number of terms. Because of this complexity, books on history of mathematics usually dismiss Egyptian fractions as a not-very-effective approach to representing fractions; see, e.g., [1, 4].

1

What we showed in our previous paper: a reminder. In our paper [6], we showed that Egyptian fractions actually provide an optimal solution to a practical problem described in many papyri from ancient Egypt: how to divide loaves of bread between multiple workers; see also [5, 7, 8, 9, 10].

Indeed, it is easy and fast to divide a rectangular-shaped bread into n equal pieces along its long side: this requires $n - 1$ cuts. So, if want to give each worker 5/12 of a loaf, a straightforward idea is to divide each loaf into 12 parts and give 5 such parts to each worker. In particular, if we have 12 workers, then we take 5 loaves and divide each of them into 12 parts. This requires 11 cuts per loaf, to the total of $5 \cdot 11 = 55$ cuts.

We can do all this cutting much faster if we take into account representation (1). For 12 workers, this means that we can:

- take $12 \cdot \dfrac{1}{3} = 4$ loaves and divide each loaf into 3 pieces; this requires $3 - 1 = 2$ cuts per loaf, to the total of $4 \cdot 2 = 8$ cuts, and

- take $12 \cdot \dfrac{1}{12} = 1$ loaf and divide it into $12 - 1 = 11$ cuts.

Overall, we need $11+8 = 19$ cuts, which is much fewer than the original 55 cuts.

A general description of what is optimal. Each fraction $\dfrac{p}{q}$ can be represented in many ways as a sum of different fractions, and each fraction $\dfrac{m}{n}$ in this sum can be represented as

$$\frac{m}{n} = \frac{1}{n} + \ldots + \frac{1}{n} \ (m \text{ times}).$$

Thus, we can represent each fraction $\dfrac{p}{q}$ as a sum of fractions of the type $\dfrac{1}{n}$:

$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}. \tag{2}$$

Without losing generality, we can assume that the integers n_i are listed in increasing order, i.e., that $n_1 \leq n_2 \leq \ldots \leq n_k$.

Let N be the least common multiple of all the values n_i. Then, the representation (2) means that to feed N workers, for each i from 1 to k, we take $N \cdot \dfrac{1}{n_i}$ loaves and cut each into n_i pieces. This requires $n_i - 1$ cuts per loaf, to the total of

$$N \cdot \frac{1}{n_i} \cdot (n_i - 1) = N \cdot \left(1 - \frac{1}{n_i}\right).$$

Thus, the overall number of cuts is

$$N \cdot \left(1 - \frac{1}{n_1}\right) + N \cdot \left(1 - \frac{1}{n_2}\right) + \ldots + N \cdot \left(1 - \frac{1}{n_k}\right) =$$

2

$$N \cdot \left(k - \sum_{i=1}^{k} \frac{1}{n_i}\right) = N \cdot \left(k - \frac{p}{q}\right).$$

So, the number of cuts per worker is

$$k - \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}\right) = k - \frac{p}{q}.$$

Thus, to minimize the number of cuts, we need to minimize the number k of terms in the representation (2) – which is exactly what the Egyptian fractions do!

Comment. Ancient Egyptians had additional restrictions on representation of type (2): e.g., they required that all denominators n_i be different. This additional requirement is not related to our optimality result, so we do not consider it in this paper.

By the way, it would be interesting to understand the motivation behind this additional requirement.

What we did in our previous paper and what are the remaining problems. In our previous paper, we describe an algorithm for computing the shortest possible representation of a fraction in the form (2). From the theoretical viewpoint, this solves the problem – although, of course, it is always desirable to look for a more efficient algorithm.

The remaining problem is that for some fractions, the number of cuts is still too large. The only reason why we need this many cuts is that we wanted to *exactly* represent the original fraction. But from the practical viewpoint, it may be beneficial to make slightly larger bread portions – we will spend slightly more money on bread but save on cuts. In other words, in addition to the problem of *representing* fractions in the form (2), we need to consider the problem of *approximating* the fractions by expressions of type (2).

Let us formulate this approximation problem in precise terms.

Formulating the approximation problem in precise terms. We are given two positive numbers:

- the price b of a loaf of bread, and

- the per-cut wages c that we need to pay the bread cutters.

We are also given the fractional part $f \in (0, 1)$ of a loaf that needs to be given to each worker. What we then need to do is minimize the overall extra expenses: i.e., the cutting expenses plus extra-bread expenses per worker. In other words, we need to find, among the tuples $(n_1, \ldots, n_k)$ for which

$$f \leq \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}, \tag{3}$$

the tuple for which the overall expenses

$$c \cdot \left(k - \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k}\right)\right) + b \cdot \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} - f\right) \tag{4}$$

are the smallest possible.

What we do in this paper. In this paper, we describe an algorithm for solving the above optimal approximation problem.

2 Solution to the Problem

Let us first simplify the objective function. The above expression (4) can be represented as

$$c \cdot k + (b - c) \cdot \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} - f \right) + c \cdot f. \tag{4a}$$

Adding a constant $c \cdot f$ to all the values of the objective function does not change which values are larger and which are smaller. Thus, minimizing the expression (4a) is equivalent to minimizing the expression

$$c \cdot k + (b - c) \cdot \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} - f \right). \tag{4b}$$

Similarly, dividing all the values of the objective function by the same constant $b - c$ does not change wich value is larger and which value is smaller. So, minimizing the expression (4b) is equivalent to minimizing the expression (4b) divided by $b - c$, i.e., the expression

$$r \cdot k + \left(\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} - f \right), \tag{4c}$$

where we denoted $r \overset{\text{def}}{=} \dfrac{c}{b - c}$.

To solve the corresponding problem, it is sufficient to solve an auxiliary problem for several values k: idea. Our idea is that for each value $k = 1, 2, \cdot$, we find the values $n_1, \ldots, n_k$ that minimize the expression (4c). Then, we find k for which the corresponding minimum is the smallest.

For each k, the term $r \cdot k$ does not depend on the choice of n_i, so minimizing the expression (4c) is equivalent to finding the values $n_1, \ldots, n_k$ that minimize the difference

$$\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} - f; \tag{5}$$

(provided that this difference is non-negative).

Let d_k denote the smallest possible value of this difference corresponding to the given value k (we assume that for the given value k, there exist integers n_i that satisfy the condition (3)). Then, the smallest possible value of the expression (4c) for given k is equal to

$$e_k \overset{\text{def}}{=} r \cdot k + d_k. \tag{6}$$

Thus, once we know all the values d_k, we must find the value k that minimizes the expression (6).

Each value f can be approximated, with any given accuracy, by a rational number, and each rational number can be represented, for some k, in the form (2). Thus if we approximate f with approximation error $< r$, then, for some k, we will get $d_k < r$ and thus, $e_k < r \cdot k + r = r \cdot (k+1)$. Once we reach the value k for which $d_k < r$, considering larger values k does not make sense: already the first term in the expression (6) will be larger than the current value e_k.

So, we arrive at the following reduction algorithm.

Reduction: algorithm. To solve the original optimization problems, for $k = 0, 1, 2, \ldots$ we compute d_k and e_k. We stop when $d_k < r$. We then:

- select the value k for which e_k is the smallest, and

- for the selected value k, we find the values $n_1, \ldots, n_k$ that minimize the difference (5).

Now, all we need to do is to show how to compute d_k.

Proposition 1. *There exists an algorithm that, given f and k, computes the smallest possible value d_k.*

Proof. Since $n_1 \leq n_2 \leq \ldots \leq n_k$, we have

$$\frac{1}{n_k} \leq \ldots \leq \frac{1}{n_2} \leq \frac{1}{n_1},$$

so the inequality (3) implies that

$$f \leq k \cdot \frac{1}{n_1},$$

which is equivalent to

$$n_1 \leq \frac{k}{f}.$$

Thus, we need to consider only finitely many values n_1.

For each of these values n_1, (3) implies that

$$f - \frac{1}{n_1} \leq \frac{1}{n_2} + \ldots + \frac{1}{n_k}$$

hence

$$f - \frac{1}{n_1} \leq (k-1) \cdot \frac{1}{n_2}$$

and

$$n_2 \leq \frac{k-1}{f - \dfrac{1}{n_1}}.$$

(If the difference is 0, then $d_1 = 0$, so further computations are not needed.) Thus, for each n_1, we only need to consider finitely many values n_2.

5

In general, once we have selected the values $n_1, \ldots, n_i$, then (3) implies that

$$f - \left(\frac{1}{n_1} + \ldots + \frac{1}{n_i} \right) \leq \frac{1}{n_{i+1}} + \ldots + \frac{1}{n_k}$$

hence

$$f - \left(\frac{1}{n_1} + \ldots + \frac{1}{n_i} \right) \leq (k - i) \cdot \frac{1}{n_{i+1}}$$

and

$$n_{i+1} \leq \frac{k - i}{f - \left(\dfrac{1}{n_1} + \ldots + \dfrac{1}{n_i} \right)}.$$

So overall, we need to consider a finite set of possible tuples $(n_1, \ldots, n_k)$. We then select the tuple for which the difference (4) is the smallest. This is all we need to do to compute the desired smallest value d_k.

The proposition is proven.

How accurately can we thus represent a number? A natural question is: what accuracy can we achieve by such an approximation? In this paper, we provide the answer to this question for $k = 1$, $k = 2$, and $k = 3$.

Definition 1. *For each positive integer k, by an* accuracy a_k *of approximation by a $\leq k$-term Egyptian fraction, we mean the smallest number a_k for which: for every number $f \in (0, 1)$, there exist integers $n_1, \ldots, n_j$, $j \leq k$ for which*

$$f \leq \frac{1}{n_1} + \ldots + \frac{1}{n_j} \leq f + a_k,$$

i.e., for which

$$0 \leq \frac{1}{n_1} + \ldots + \frac{1}{n_j} - f \leq a_k.$$

Proposition 2. *For $k = 1$, the accuracy a_1 of approximation by a ≤ 1-term Egyptian fraction is $a_1 = \dfrac{1}{2}$.*

Proof. The largest possible value of the expression $\dfrac{1}{n} < 1$ corresponds to the smallest possible value $n = 2$. Thus, all values $f > \dfrac{1}{2}$ have to be approximated from above by the number 1 (in this case, no cuts are needed). For each $\varepsilon > 0$, for the value $f = \dfrac{1}{2} + \varepsilon$, the difference $1 - f$ is equal to $\dfrac{1}{2} - \varepsilon$, thus a_1 cannot be smaller than $\dfrac{1}{2}$.

The value $a_1 = \dfrac{1}{2}$ satisfies the desired condition:

- for values $f \leq \dfrac{1}{2}$, we take $n_1 = 2$ (i.e., we cut each loaf in half and give each worked half of a loaf), and

6

- for values $f > \dfrac{1}{2}$, we take $n_1 = 1$ (i.e., we give each worker the whole loaf and do not cut anything at all).

The proposition is proven.

Proposition 3. *For $k = 2$, the accuracy a_2 of approximation by $a \leq 2$-term Egyptian fraction is $a_2 = \dfrac{1}{6}$.*

Proof. Let us first find the largest possible value $f < 1$ that can be represented as

$$f = \frac{1}{n_1} + \frac{1}{n_2}.$$

In general, we could have $n_1 = 2$ or $n_1 \geq 3$.

- For $n_1 = 2$, to have $\dfrac{1}{n_1} + \dfrac{1}{n_2} < 1$, we must have $\dfrac{1}{n_2} < \dfrac{1}{2}$, i.e., we must have $n_2 > 2$. The largest value of this fraction corresponds to the smallest value of n_2 that satisfies this inequality, i.e. to the value $n_2 = 3$. In this case,
$$\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

- For $n_1 \geq 3$, due to $n_1 \leq n_2$, we have
$$\frac{1}{n_1} + \frac{1}{n_2} \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

The upper bound $\dfrac{2}{3}$ for cases $n_1 \geq 3$ is smaller than the value $\dfrac{5}{6}$ achievable for $n_1 = 2$. Thus the largest values $f < 1$ that can be represented by a ≤ 2-term Egyptian fraction is $\dfrac{5}{6}$.

So, each value f which is larger than $\dfrac{5}{6}$ has to be approximated by number 1. Hence, the value a_2 cannot be smaller than the differences

$$1 - \left(\frac{5}{6} + \varepsilon\right) = \frac{1}{6} - \varepsilon$$

for any $\varepsilon > 0$. Thus, we must have $a_2 \geq \dfrac{1}{6}$.

To complete the proof, we need to show that every number f from the interval $(0, 1)$ can indeed be approximated by a ≤ 2-term Egyptian fraction with accuracy $\dfrac{1}{6}$. Indeed:

- values $v \in \left(0, \dfrac{1}{6}\right]$ can be approximated by $\dfrac{1}{6}$;

- values $v \in \left(\dfrac{1}{6}, \dfrac{1}{3}\right]$ can be approximated by $\dfrac{1}{3}$;

- values $v \in \left(\dfrac{1}{3}, \dfrac{1}{2} \right]$ can be approximated by $\dfrac{1}{2}$;

- values $v \in \left(\dfrac{1}{2}, \dfrac{2}{3} \right]$ can be approximated by $\dfrac{2}{3} = \dfrac{1}{2} + \dfrac{1}{6}$;

- values $v \in \left(\dfrac{2}{3}, \dfrac{5}{6} \right]$ can be approximated by $\dfrac{5}{6} = \dfrac{1}{2} + \dfrac{1}{3}$;

- values $v \in \left(\dfrac{5}{6}, 1 \right)$ can be approximated by 1.

Thus indeed, $a_2 = \dfrac{1}{6}$. The proposition is proven.

Proposition 4. *For $k = 3$, the accuracy a_3 of approximation by $a \leq 3$-term Egyptian fraction is $a_3 = \dfrac{1}{42}$.*

Proof. Let us first find the largest possible value $f \in (0, 1)$ that can be represented as the sum $\dfrac{1}{n_1} + \dfrac{1}{n_2} + \dfrac{1}{n_3}$ of three inverses.

In general, we can have $n_1 = 2$ or $n_1 > 2$.

If $n_1 = 2$, then we cannot have $n_2 = 2$ – this would lead to the sum equal to 1. Thus, we must have $n_2 \geq 3$.

- If $n_2 = 3$, then to get the sum smaller than 1, we must have $n_3 > 6$. The smallest such value is $n_3 = 7$ for which

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42} = 0.976\ldots$$

  By the way, for $n_3 = 8$, we get

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{23}{24} = 0.958\ldots$$

- If $n_2 = 4$, then to get the sum smaller than 1, we must have $n_3 > 4$. The smallest such value is $n_3 = 5$ for which

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} = 0.95.$$

  This is smaller than the previous value, so the largest representable f cannot be reached for this value of n_2.

- If $n_2 \geq 5$, then we have $n_3 \geq n_2 \geq 5$, hence

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{5} = \frac{9}{10} = 0.9,$$

  which is also smaller than what we had for $n_2 = 3$.

If $n_1 = 3$, then we can have $n_2 \geq 3$.

- If $n_2 = 3$, then, to get the sum smaller than 1, we must have $n_3 > 3$. The smallest such value is $n_3 = 4$ for which

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{11}{12} = 0.916\ldots;$$

this is smaller than what we had earlier;

- If $n_2 \geq 4$, then $n_3 \geq 4$, hence

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6} = 0.833...,$$

also smaller.

Finally, if we have $n_1 \geq 4$, then $n_2 \geq 4$, $n_3 \geq 4$, hence

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} = 0.75,$$

still smaller.

So, the largest number representable as a sum of three Egyptian terms is $\frac{41}{42}$. Thus, every largest number has to be approximated by 1, so a_3 cannot be smaller than

$$1 - \frac{41}{42} = \frac{1}{42} = 0.0238\ldots$$

Let us prove that can indeed approximate any value $f \in (0,1)$ with this accuracy. Most of the numbers can be approximated with an even better accuracy $\frac{1}{60}$:

- values $v \in \left(0, \frac{1}{60}\right]$ can be approximated by $\frac{1}{60}$;

- values $v \in \left(\frac{1}{60}, \frac{2}{60}\right]$ can be approximated by $\frac{2}{60} = \frac{1}{30}$;

- values $v \in \left(\frac{2}{60}, \frac{3}{60}\right]$ can be approximated by $\frac{3}{60} = \frac{1}{20}$;

- values $v \in \left(\frac{3}{60}, \frac{4}{60}\right]$ can be approximated by $\frac{4}{60} = \frac{1}{15}$;

- values $v \in \left(\frac{4}{60}, \frac{5}{60}\right]$ can be approximated by $\frac{5}{60} = \frac{1}{12}$;

- values $v \in \left(\frac{5}{60}, \frac{6}{60}\right]$ can be approximated by $\frac{6}{60} = \frac{1}{10}$;

- values $v \in \left(\dfrac{6}{60}, \dfrac{7}{60}\right]$ can be approximated by $\dfrac{7}{60} = \dfrac{1}{10} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{7}{60}, \dfrac{8}{60}\right]$ can be approximated by $\dfrac{8}{60} = \dfrac{2}{15} = \dfrac{1}{10} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{8}{60}, \dfrac{9}{60}\right]$ can be approximated by $\dfrac{9}{60} = \dfrac{3}{20} = \dfrac{1}{10} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{9}{60}, \dfrac{10}{60}\right]$ can be approximated by $\dfrac{10}{60} = \dfrac{1}{6}$;

- values $v \in \left(\dfrac{10}{60}, \dfrac{11}{60}\right]$ can be approximated by $\dfrac{11}{60} = \dfrac{1}{6} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{11}{60}, \dfrac{12}{60}\right]$ can be approximated by $\dfrac{12}{60} = \dfrac{1}{5}$;

- values $v \in \left(\dfrac{12}{60}, \dfrac{13}{60}\right]$ can be approximated by $\dfrac{13}{60} = \dfrac{1}{5} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{13}{60}, \dfrac{14}{60}\right]$ can be approximated by $\dfrac{14}{60} = \dfrac{7}{30} = \dfrac{1}{5} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{14}{60}, \dfrac{15}{60}\right]$ can be approximated by $\dfrac{15}{60} = \dfrac{1}{4}$;

- values $v \in \left(\dfrac{15}{60}, \dfrac{16}{60}\right]$ can be approximated by $\dfrac{16}{60} = \dfrac{4}{15} = \dfrac{1}{4} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{16}{60}, \dfrac{17}{60}\right]$ can be approximated by $\dfrac{17}{60} = \dfrac{1}{4} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{17}{60}, \dfrac{18}{60}\right]$ can be approximated by $\dfrac{18}{60} = \dfrac{3}{10} = \dfrac{1}{4} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{18}{60}, \dfrac{19}{60}\right]$ can be approximated by $\dfrac{19}{60} = \dfrac{1}{4} + \dfrac{1}{15}$;

- values $v \in \left(\dfrac{19}{60}, \dfrac{20}{60}\right]$ can be approximated by $\dfrac{20}{60} = \dfrac{1}{3}$;

- values $v \in \left(\dfrac{20}{60}, \dfrac{21}{60}\right]$ can be approximated by $\dfrac{21}{60} = \dfrac{7}{20} = \dfrac{1}{3} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{21}{60}, \dfrac{22}{60}\right]$ can be approximated by $\dfrac{22}{60} = \dfrac{11}{30} = \dfrac{1}{3} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{22}{60}, \dfrac{23}{60}\right]$ can be approximated by $\dfrac{23}{60} = \dfrac{1}{3} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{23}{60}, \dfrac{24}{60}\right]$ can be approximated by $\dfrac{24}{60} = \dfrac{2}{5} = \dfrac{1}{3} + \dfrac{1}{15}$;

- values $v \in \left(\dfrac{24}{60}, \dfrac{25}{60}\right]$ can be approximated by $\dfrac{25}{60} = \dfrac{5}{12} = \dfrac{1}{5} + \dfrac{1}{12}$;

- values $v \in \left(\dfrac{25}{60}, \dfrac{26}{60}\right]$ can be approximated by $\dfrac{26}{60} = \dfrac{13}{30} = \dfrac{1}{3} + \dfrac{1}{10}$;

- values $v \in \left(\dfrac{26}{60}, \dfrac{27}{60}\right]$ can be approximated by $\dfrac{27}{60} = \dfrac{9}{20} = \dfrac{1}{4} + \dfrac{1}{5}$;

- values $v \in \left(\dfrac{27}{60}, \dfrac{28}{60}\right]$ can be approximated by $\dfrac{28}{60} = \dfrac{7}{15} = \dfrac{1}{3} + \dfrac{1}{10} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{28}{60}, \dfrac{29}{60}\right]$ can be approximated by $\dfrac{29}{60} = \dfrac{1}{3} + \dfrac{1}{10} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{29}{60}, \dfrac{30}{60}\right]$ can be approximated by $\dfrac{30}{60} = \dfrac{1}{2}$;

- values $v \in \left(\dfrac{30}{60}, \dfrac{31}{60}\right]$ can be approximated by $\dfrac{31}{60} = \dfrac{1}{2} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{31}{60}, \dfrac{32}{60}\right]$ can be approximated by $\dfrac{32}{60} = \dfrac{8}{15} = \dfrac{1}{2} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{32}{60}, \dfrac{33}{60}\right]$ can be approximated by $\dfrac{33}{60} = \dfrac{11}{20} = \dfrac{1}{2} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{33}{60}, \dfrac{34}{60}\right]$ can be approximated by $\dfrac{34}{60} = \dfrac{17}{30} = \dfrac{1}{2} + \dfrac{1}{15}$;

- values $v \in \left(\dfrac{34}{60}, \dfrac{35}{60}\right]$ can be approximated by $\dfrac{35}{60} = \dfrac{7}{12} = \dfrac{1}{2} + \dfrac{1}{12}$;

- values $v \in \left(\dfrac{35}{60}, \dfrac{36}{60}\right]$ can be approximated by $\dfrac{36}{60} = \dfrac{3}{5} = \dfrac{1}{2} + \dfrac{1}{10}$;

- values $v \in \left(\dfrac{36}{60}, \dfrac{37}{60}\right]$ can be approximated by $\dfrac{37}{60} = \dfrac{1}{2} + \dfrac{1}{10} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{37}{60}, \dfrac{38}{60}\right]$ can be approximated by $\dfrac{38}{60} = \dfrac{19}{30} = \dfrac{1}{2} + \dfrac{1}{10} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{38}{60}, \dfrac{39}{60}\right]$ can be approximated by $\dfrac{39}{60} = \dfrac{13}{20} = \dfrac{1}{2} + \dfrac{1}{10} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{39}{60}, \dfrac{40}{60}\right]$ can be approximated by $\dfrac{40}{60} = \dfrac{2}{3} = \dfrac{1}{2} + \dfrac{1}{6}$;

- values $v \in \left(\dfrac{40}{60}, \dfrac{41}{60}\right]$ can be approximated by $\dfrac{41}{60} = \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{41}{60}, \dfrac{42}{60}\right]$ can be approximated by $\dfrac{42}{60} = \dfrac{7}{10} = \dfrac{1}{2} + \dfrac{1}{5}$;

- values $v \in \left(\dfrac{42}{60}, \dfrac{43}{60}\right]$ can be approximated by $\dfrac{43}{60} = \dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{43}{60}, \dfrac{44}{60}\right]$ can be approximated by $\dfrac{44}{60} = \dfrac{11}{15} = \dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{44}{60}, \dfrac{45}{60}\right]$ can be approximated by $\dfrac{45}{60} = \dfrac{3}{4} = \dfrac{1}{2} + \dfrac{1}{4}$;

- values $v \in \left(\dfrac{45}{60}, \dfrac{46}{60}\right]$ can be approximated by $\dfrac{46}{60} = \dfrac{23}{30} = \dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{15}$;

- values $v \in \left(\dfrac{46}{60}, \dfrac{47}{60}\right]$ can be approximated by $\dfrac{47}{60} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{47}{60}, \dfrac{48}{60}\right]$ can be approximated by $\dfrac{48}{60} = \dfrac{4}{5} = \dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{10}$;

- values $v \in \left(\dfrac{48}{60}, \dfrac{49}{60}\right]$ can be approximated by $\dfrac{49}{60} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{15}$;

- values $v \in \left(\dfrac{49}{60}, \dfrac{50}{60}\right]$ can be approximated by $\dfrac{50}{60} = \dfrac{5}{6} = \dfrac{1}{2} + \dfrac{1}{3}$;

- values $v \in \left(\dfrac{50}{60}, \dfrac{51}{60}\right]$ can be approximated by $\dfrac{51}{60} = \dfrac{17}{20} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{60}$;

- values $v \in \left(\dfrac{51}{60}, \dfrac{52}{60}\right]$ can be approximated by $\dfrac{52}{60} = \dfrac{13}{20} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{30}$;

- values $v \in \left(\dfrac{52}{60}, \dfrac{53}{60}\right]$ can be approximated by $\dfrac{53}{60} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{20}$;

- values $v \in \left(\dfrac{53}{60}, \dfrac{54}{60}\right]$ can be approximated by $\dfrac{54}{60} = \dfrac{9}{10} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{15}$;

- values $v \in \left(\dfrac{54}{60}, \dfrac{55}{60}\right]$ can be approximated by $\dfrac{55}{60} = \dfrac{11}{12} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{12}$;

- values $v \in \left(\dfrac{55}{60}, \dfrac{56}{60}\right]$ can be approximated by $\dfrac{56}{60} = \dfrac{14}{15} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{10}$;

- values $v \in \left(\dfrac{56}{60}, \dfrac{57}{60}\right]$ can be approximated by $\dfrac{57}{60} = \dfrac{19}{20} = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{5}$;

- values $v \in \left(\dfrac{57}{60}, \dfrac{23}{24} \right]$ can be approximated by $\dfrac{23}{24} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{8}$;

- values $v \in \left(\dfrac{23}{24}, \dfrac{41}{42} \right]$ can be approximated by $\dfrac{41}{42} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{7}$;

- values $v \in \left(\dfrac{41}{42}, 1 \right)$ can be approximated by 1.

The proposition is proven.

3   Conclusion

Ancient Egyptians represented each fraction as a sum of inverses of integers, e.g., $\dfrac{5}{6}$ was represented as $\dfrac{1}{2} + \dfrac{1}{3}$. In our previous paper, we showed that such representations correspond to the optimal solution to a problem that is mentioned several times in Egyptian papyri: how to divide bread loaves between workers. Egyptian fractions correspond to the smallest number of cuts needed for this division.

Sometimes, however, the attempt to provide the exact amount of bread to each worker leads to too many cuts. In many such cases, if we give every worker a little bit more bread, we will spend more on bread but our overall expenses will be lower, since we will need fewer cuts and thus, we will need to hire fewer bread cutters. In this paper, we show how to find the solution that minimizes the overall expenses. Depending on how many cuts per worker we allow, we can make sure that the resulting portion of a loaf is close to the original one. For example, if we allow 3 cuts, we can get the accuracy of $\dfrac{1}{42} \approx 2.5\%$.

Acknowledgments

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