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Why Significant Wave Height And Rogue Waves Are So Defined: A Possible Explanation

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Abstract

Data analysis has shown that if we want to describe the wave pattern by a single characteristic, the best characteristic is the average height of the highest one third of the waves; this characteristic is called significant wave height. Once we know the value of this characteristic, a natural next question is: what is the highest wave that we should normally observe – so that waves higher than this amount would be rare ("rogue"). Empirically, it has been shown that rogue waves are best defined as the ones which are at least twice higher than the significant wave height. In this paper, we provide a possible theoretical explanation for these two empirical facts.

1 Formulation of the Problem

Empirical facts. In many practical problems, ranging from ocean navigation to the effect of waves on the ocean shores and on the shore-located constructions, we need to study the effect of the ocean waves. One of the most important characteristics of a wave is its height: the taller the wave, the larger its impact. How can we describe the wave heights?

Ocean waves are random. To fully describe the heights of waves, we need to describe the probability of different height values, i.e., the probability distribution on the set of possible wave heights. To describe a general distribution, it is necessary to specify infinitely many parameters – e.g., the values of the corresponding cumulative distribution function for different heights. In data processing, however, it is difficult to analyze too many parameters. Thus, it is desirable to come up with a single characteristic that best describes the difference between two different wave patterns. Empirical analysis has shown that the best such characteristic is the average height of the highest one third of the waves; this characteristic is known as the *significant wave height*; see, e.g., [2] and references therein.

Once we know this average characteristic, a natural next question is how to use this characteristic to describe the upper bound on the vast majority of actual waves – so that waves higher that this bound will be truly rare. In oceanography, such rare waves are known as *rogue waves*. Empirical analysis shows that the best way to set up this bound is at the level twice the significant wave height – so that waves which are at least twice higher than this height are considered rogue.

Problem. How can we explain these empirical facts?

What we do in this paper. In this paper, we provide a possible theoretical explanation for the two above-described empirical facts.

2 Our Explanations

How to explain the definition of significant wave height: idea. Usually, dangerous waves are rarer: the majority of the waves are not very dangerous, but others are. Even in severe storms, the majority of the waves are not that harmful, but the minority of really high and really strong waves leads to all the damage. So, to describe the overall effect of the waves, we need to concentrate on the higher waves.

From the above idea to explaining why we only consider the highest one third of the waves. According to this idea, we need to consider the higher waves, i.e., all the waves which are higher than some value h_0. All we know is that waves which are not that high form a majority, i.e., that the number n_+ of higher-than-h_0 waves is smaller than the number n_- of lower-than-h_0 waves.

If we fix the value n_-, then all we know about the value n_+ is that this value is smaller than n_-, i.e., that the number n_+ can take any values from the interval $[0, n_-)$.

We do not know which values from this interval are more probable and which values are less probable. Such situations are ubiquitous in applications. To deal with such situations, Laplace came up with a natural idea called Principle of Insufficient Reason: if we have no reason to believe that one event is more or less probable than another one, we assign to these two events equal probability. This idea is still actively and efficiently used, under the general name of Maximum Entropy approach; see, e.g., [1].

In particular, in the situation when all we know about some value is that this value is located in a given interval, we consider all the values from this interval to be equally probable, i.e., we assume that this value is uniformly distributed on the interval.

Which value from this interval should we choose? Informally, we want a value v which is the closest to different values from this interval. If we had only finitely many possible values $v_1, \ldots, v_n$, then we would say that we want to have $v \approx v_1$, $v \approx v_2$, $\ldots$, $v \approx v_n$. In other words, we want the tuples $(v, v, \ldots, v)$ and $(v_1, v_2, \ldots, v_n)$ to be as close to each other as possible.

How can we define "closest"? Each n-element tuple can be naturally represented as a point in the n-dimensional space. So, the distance between the two tuples can be represented by the usual definition of distance in the multi-D space:

$$d = \sqrt{\sum_{i=1}^{n} (v - v_i)^2}.$$

Minimizing this distance is equivalent to minimizing its square $d^2 = \sum\limits_{i=1}^{n} (v - v_i)^2$. The minimizing value of v can be easily found if we differentiate this expression for d^2 with respect to v and equate the resulting derivative to 0. Then, we get

$$2 \sum_{i=1}^{n} (v - v_i) = 0,$$

i.e., equivalently, $n \cdot v = \sum\limits_{i=1}^{n} v_i$ and

$$v = \frac{1}{n} \cdot \sum_{i=1}^{n} v_i.$$

Thus, if we have several possible values, it is reasonable to select their average – i.e., their mean value. For the uniform distribution on an interval, the average is the midpoint of this interval. Thus, from all the values from the interval $[0, n_-)$, it is reasonable to select a midpoint $n_+ = \dfrac{n_-}{2}$. Thus, with respect to the overall number of waves $n = n_- + n_+$, higher-than-h_0 waves form the proportion

$$\frac{n_+}{n} = \frac{n_+}{n_- + n_+} = \frac{\dfrac{n_-}{2}}{n_- + \dfrac{n_-}{2}} = \frac{\dfrac{1}{2}}{1 + \dfrac{1}{2}} = \frac{1}{3}.$$

This explains why we consider only the highest one third of the waves.

Why we should consider the average of the highest one third of the waves. Higher-than-h_0 waves can have different heights. We want to select a single numerical characteristic that would best capture all these heights. In other words, we want a single value v that best represents several possible height values.

In the previous subsection, we have already argued that the best value representing a group of values is the average. Thus, it makes sense to consider the average of the highest one third of the waves – which explains the empirical efficiency of this characteristic.

Why rogue waves are so defined. The only remaining question is why rogue waves are so defined. Once we know the average a of the highest one third of

the waves, how do we define a threshold r that separates rare (rogue) waves from not-so-high waves?

From the commonsense viewpoint, the only thing we know about the desired threshold r is that it should be larger than a: $a < r$. How can we transform this vague requirement into a precise choice of r? To do this, let us "flip" the problem. Let us assume that we already know the value r, and we are deciding on the best value a. In this case, all we know about a is that it can take any value from the interval $[0, r)$. Thus, arguing as above, we conclude:

- that it is reasonable to assume that the value a is uniformly distributed on this interval, and
- that the most reasonable value a is the average value with this respect to this distribution – i.e., the midpoint $a = \dfrac{r}{2}$ of this interval.

Now, we can flip this formula back and conclude that, once we know the significant wave height a, we can select the threshold r that satisfies the above equality – i.e., the threshold $r = 2a$. This explains why rogue waves are defined as waves which are at least twice higher than the significant wave height.

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