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# Which Classes of Bi-Intervals Are Closed Under Addition?

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# Which Classes of Bi-Intervals Are Closed Under Addition?

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**Abstract**

In many practical situations, uncertainty with which we know each quantity is described by an interval. In processing such data, it is useful to know that the sum of two intervals is always an interval. In some cases, however, the set of all possible value of a quantity is described by a *bi-interval* – i.e., by a union of two intervals. It is known that the sum of two bi-intervals is not always a bi-interval. In this paper, we describe all the class of bi-intervals which are closed under addition – i.e., for which the sum of bi-intervals is a bi-interval.

## 1 Formulation of the Problem

**Interval uncertainty: a brief reminder.** In many real-life situations, our uncertainty about a quantity is described by an interval; see, e.g., [4].

For example, usually, the information about a physical quantity $x$ comes from measurements. As a result of the measurement, we get a value $\widetilde{x}$ which is, in general, different from the actual (unknown) value $x$: $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x \neq 0$. Often, the only information that we have about the measurement accuracy is the upper bound $\Delta$ on the absolute value $|\Delta x|$ of the measurement error $\Delta x$. In this case, after the measurement, the only information that we gain about the actual value $x$ is that $x$ belongs to the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$; see, e.g., [4].

In some cases, we only get the lower bound, in this case, we get an interval $[v, \infty)$. It is also possible to have intervals $(-\infty, v]$, and, of course, we can have the whole real line $(-\infty, \infty)$ corresponding to the case when we do not have any information about $x$.

The class of intervals is closed under many different operations; see, e.g., [1, 2, 3]. For example, if $A$ and $B$ are intervals, then their sum

$$A + B \stackrel{\text{def}}{=} \{a + b : a \in A \text{ and } b \in B\} \tag{1}$$

is also an interval, and for every two real numbers $p$ and $q$, the set

$$p \cdot A + q \stackrel{\text{def}}{=} \{p \cdot a + q : a \in A\} \tag{2}$$

is also an interval.

**Bi-intervals.** In some cases, the set of possible value of a physical quantity is a union of two intervals; we will call such unions *bi-intervals.* For example, if we measured the absolute value of the velocity of an object moving along a line, and the result is $[1,2]$, but we do not know the direction of the motion, then all we know about the actual velocity is that its value is in the union $[-2,-1] \cup [1,2]$.

**In general, the set of bi-intervals is not closed under addition.** It is easy to come witn an example when the sum of two bi-intervals is not a bi-interval: e.g., it is easy to check that

$$([0,1] \cup [5,6]) + ([0,1] \cup [5,6]) = [0,1] \cup [5,7] \cup [10,12].$$

**A natural question.** A natural question is: when is a class of bi-intervals closed under addition?

In this paper, we provide an answer to this question.

## 2 Definitions and the Main Result

**Definition 1.** *For an interval $\mathbf{a} = [\underline{a}, \overline{a}]$, its* width *is defined as $w(\mathbf{a}) = \overline{a} - \underline{a}$.*

**Definition 2.** *For two intervals $\mathbf{a}$ and $\mathbf{b}$, the* lower distance *$\underline{d}(\mathbf{a}, \mathbf{b})$ is the smallest possible value of $|a - b|$ when $a \in \mathbf{a}$ and $b \in \mathbf{b}$.*

*Comment.* If the intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$ intersect, then clearly the lower distance is 0. If they do not intersect, then, without losing generality, we can assume that $\overline{a} < \underline{b}$, then $\underline{d}(\mathbf{a}, \mathbf{b}) = \underline{b} - \overline{a}$.

**Definition 3.** *By a* bi-interval*, we mean a union of two intervals $[\underline{a}, \overline{a}] \cup [\underline{b}, \overline{b}]$.*

*Comment.* In particular, every interval is a bi-interval: it is sufficient to take $\mathbf{a} = \mathbf{b}$.

**Definition 4.** *We say that a bi-interval $\mathbf{a} \cup \mathbf{b}$ is* close *if $\underline{d}(\mathbf{a}, \mathbf{b}) \le \max(w(\mathbf{a}), w(\mathbf{b}))$.*

*Comment.* Since every infinite or semi-infinite interval has infinite width, this implies that every bi-interval in which one of the intervals is infinite or semi-infinite is close. In particular, every interval of the type $(-\infty, \overline{a}] \cup [\underline{b}, \infty)$ is close.

**Lemma 1.**

- *A bi-interval $\mathbf{a} \cup \mathbf{b}$ is close if and if its sum with itself is also a bi-interval.*

- *The sum of two close bi-intervals is always close.*
- *For each close bi-interval* $\mathbf{a} \cup \mathbf{b}$ *and for all* $p$ *and* $q$*, the bi-interval*

$$(p \cdot \mathbf{a} + q) \cup (p \cdot \mathbf{b} + q)$$

  *is also close.*

The first two statements of this lemma immediately lead to the following description of classes of bi-intervals which are closed under addition:

**Proposition 1.**

- *If* $C$ *is a class of bi-intervals which is closed under addition, then every bi-interval from the class* $C$ *is close.*
- *For every set* $S$ *of close bi-intervals, the set*

$$C = \{S_1 + \ldots + S_n : S_i \in S \textit{ for all } i\}$$

  *is a class of bi-intervals which is closed under addition.*

# 3 Proof of the Lemma

$1^\circ$. Let us first prove that a bi-interval $\mathbf{a} \cup \mathbf{b}$ is close if and if its sum with itself is also a bi-interval.

If the bi-interval is an interval, then its sum with itself is also an interval hence a bi-interval. So, it is sufficient to consider the case when the intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$ are disjoint. In this case, without losing generality, we can assume that $\overline{a} < \underline{b}$.

The sum $S$ of the bi-interval with itself has the following form (in which we sorted the three component intervals by their lower bounds):

$$[2\underline{a}, 2\overline{a}] \cup [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \cup [2\underline{b}, 2\overline{b}].$$

If

$$2\overline{a} < \underline{a} + \underline{b} \tag{3}$$

and

$$\overline{a} + \overline{b} < 2\underline{b}, \tag{4}$$

then this sum is a union of three disjoint intervals, i.e., not a bi-interval; otherwise, it is a union of two (or one) intervals, i.e., a bi-interval.

The inequality (3) is equivalent to $\overline{a} - \underline{a} < \underline{b} - \overline{a}$, i.e., to $w(\mathbf{a}) < \underline{d}(\mathbf{a}, \mathbf{b})$. Similarly, the inequality (4) is equivalent to $\overline{b} - \underline{b} < \underline{b} - \overline{a}$, i.e., to $w(\mathbf{b}) < \underline{d}(\mathbf{a}, \mathbf{b})$. Thus, both inequalities are satisfied if and only if $\max(w(\mathbf{a}), w(\mathbf{b})) < \underline{d}(\mathbf{a}, \mathbf{b})$, i.e., exactly if and only if the bi-interval is *not* close.

The statement is proven.

$2^\circ$. It is also easy to prove that the sum of two close by-intervals is always close.

Indeed, when we add intervals, their width increases (or at least nor decrease), while the lower distance only decreases (or at least remains the same), thus the inequality remains.

$3^\circ$. Finally, let us prove that for each close bi-interval $\mathbf{a} \cup \mathbf{b}$ and for all $p$ and $q$, the bi-interval $(p \cdot \mathbf{a} + q) \cup (p \cdot \mathbf{b} + q)$ is also close.

Indeed, a shift by $q$ does not change the lower distance and the widths, and multiplication by $p$ multiplies all these values by $|p|$. Thus, for the new intervals, the inequality describing closeness still remains.

The lemma is proven.

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