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What If Not All Interval-Valued Fuzzy Degrees Are Possible?

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Abstract

One of the applications of intervals is in describing experts' degrees of certainty in their statements. In this application, not all intervals are realistically possible. To describe all realistically possible degrees, we end up with a mathematical question of describing all topologically closed classes of intervals which are closed under the appropriate minimum and maximum operations. In this paper, we provide a full description of all such classes.

1 Formulation of the Problem

Numerical and interval-valued fuzzy degrees: a brief reminder. One of the main ideas behind traditional logic is that every statement is either true or false. In the computer, "true" is usually represented as 1 and "false" as 0.

In practice, people are often not 100% confident in their statements. A natural way to describe a person's degree of confidence in a statement is to use numbers intermediate between the value 1 (corresponding to full confidence) and the value 0 (corresponding to the complete absence of confidence). This representation of degrees of confidence by numbers from the interval $[0,1]$ is one of the main ideas behind *fuzzy logic*, a methodology for transforming imprecise ("fuzzy") expert statements into precise computer-understandable form; see, e.g., [1, 3, 5, 7, 8, 9].

If we have two statements A_1 and A_2 with degrees of confidence a_1 and a_2, then our degree of confidence in a composite statement $A_1 \,\&\, A_2$ cannot exceed neither a_1 not a_2 and thus, must be smaller than or equal to $\min(a_1, a_2)$. Similarly, our degree of confidence in a composite statement $A_1 \vee A_2$ cannot be smaller than either a_1 or a_2 and thus, must be greater than or equal to $\max(a_1, a_2)$.

The most natural way to estimate the expert's degree of confidence is to ask the expert him/herself. The problem is that just like the expert is not certain about his/her statement, he/she is also not confident about the exact degree of confidence. From this viewpoint, it is more natural to characterize the degree of confidence not by a single value, but by an *interval* of possible values, i.e., by a subinterval of the interval $[0,1]$; see, e.g., [5].

The minimum and maximum operations can be naturally extended to such intervals, in the usual interval-computation way (see, e.g., [2, 4, 6]):

$$\min([a_1,b_1],[a_2,b_2]) \stackrel{\text{def}}{=} \{\min(v_1,v_2) : v_1 \in [a_1,b_1] \,\&\, v_2 \in [a_2,b_2]\};$$

$$\max([a_1,b_1],[a_2,b_2]) \stackrel{\text{def}}{=} \{\max(v_1,v_2) : v_1 \in [a_1,b_1] \,\&\, v_2 \in [a_2,b_2]\}.$$

Since both min and max are monotonic functions, these ranges are easy to compute:

$$\min([a_1,b_1],[a_2,b_2]) = [\min(a_1,a_2), \min(b_1,b_2)]; \tag{1}$$

$$\max[a_1,b_1],[a_2,b_2]) = [\max(a_1,a_2), \max(b_1,b_2)]. \tag{2}$$

Not all intervals are realistic. In principle, we can consider all possible subintervals of the interval $[0,1]$, but in practice, not all of them appear. For example, the interval $[0,1]$ would mean that the expert has no idea whether his/her statement is true or not – but in this case, the expert would not make this statement. It is therefore desirable to describe the class of all realistic intervals.

What are the natural properties of such a class? It is reasonable to require that this class be closed under the application of operations (1) and (2).

Also, since close values of degree are practically indistinguishable, when we have a sequence of realistic intervals that converges to a limit, this limit is indistinguishable from all sufficiently close elements of this sequence – so it makes sense to consider this limit interval realistic too. In other words, it makes sense to require that the class of realistic intervals be topologically closed.

In this paper, we describe all resulting classes. The result is simple, and the proof is straightforward.

2 Result

Definition. *By a* class of realistic intervals*, we mean the class $\mathcal{C}$ of subintervals of the interval $[0,1]$ which contains all numbers from this interval (i.e., all degenerate intervals $[a,a]$), which is topologically closed, and which is closed under minimum and maximum operations* (1) *and* (2)*.*

Proposition.

- *Every class $\mathcal{C}$ of realistic intervals has the form*

$$\mathcal{C} = \{[a,b] : a \le b \le f(a)\}, \tag{3}$$

where $f(a)$ is a monotonic right-continuous function for which $a \le f(a)$ for all a.

- *For each monotonic right-continuous function for which $a \le f(a)$ for all a, the class defined by the formula* (3) *is a class of realistic intervals.*

Reminder. Right-continuous means that for every sequence $a_n \to a$ for which $a_n \ge a$ for all n, we have $f(a_n) \to f(a)$.

Example. We can take $f(a) = \min(f(a) + \delta, 1)$ for some $\delta > 0$.

Proof.

1°. Let $\mathcal{C}$ be a class of realistic intervals. For each a, there are intervals $[a, b]$ in the class $\mathcal{C}$ – e.g., the degenerate interval $[a, a]$. Let us define

$$f(a) \stackrel{\text{def}}{=} \sup\{b : [a, b] \in \mathcal{C}\}. \tag{4}$$

The supremum $f(a)$ is a limit of values b_n for which $[a, b_n] \in \mathcal{C}$. Thus, since the class $\mathcal{C}$ is closed, the interval $[a, f(a)]$ also belongs to this class $\mathcal{C}$:

$$[a, f(a)] \in \mathcal{C}. \tag{5}$$

2°. By definition of the supremum, no interval $[a, b]$ with $b > f(a)$ belongs to the class $\mathcal{C}$. On the other hand, if $a \le b \le f(a)$, then the interval $[a, b]$ indeed belongs to the class $\mathcal{C}$, since this interval is equal to the result $\min([b, b], [a, f(a)])$ of applying the operation (1) to intervals $[b, b]$ and $[a, f(a)]$ from this class – and the class $\mathcal{C}$ is closed under this operation.

Thus, the class $\mathcal{C}$ indeed has the form (3).

3°. Let us prove that the function $f(a)$ defined by the formula (4) is indeed monotonic, right-continuous, and satisfies the condition $a \le f(a)$.

3.1°. The last condition is the easiest to prove, since the class $\mathcal{C}$ contain an interval $[a, f(a)]$, and the upper endpoint of an interval is always great than or equal to its lower endpoint.

3.2°. Monotonicity is also easy: if $a \le a'$, then, since $[a, f(a)] \in \mathcal{C}$, we have $\max([a', a'], [a, f(a)]) = [a', \max(a', f(a)] \in \mathcal{C}$. Thus, by definition of $f(a')$ as the largest b for which $[a', b] \in \mathcal{C}$, we conclude that $\max(a', f(a)) \le f(a')$. Since $f(a) \le \max(a', f(a))$, we thus get $f(a) \le f(a')$.

3.3°. Let us now prove right continuity. Suppose that $a_n \to a$ and $a_n \ge a$ for all n. Since the function $f(a)$ is monotonic, we have $f(a) \le f(a_n)$ for all n and thus, in the limit:

$$f(a) \le \liminf f(a_n). \tag{6}$$

Due to (3), for each n, we have $[a_n, f(a_n)] \in \mathcal{C}$. For a subsequence $f(a_{n_k})$ that converges to the upper limit $\limsup f(a_n)$, due to topological closeness, we have $[a, \limsup f(a_n)] \in \mathcal{C}$. By definition (4) of the function $f(a)$, this means that

$$\limsup f(a_n) \le f(a). \tag{7}$$

Since we always have $\liminf f(a_n) \le \limsup f(a_n)$, inequalities (6) and (7) imply that

$$f(a) \le \liminf f(a_n) \le \limsup f(a_n) \le f(a),$$

thus

$$\liminf f(a_n) = \limsup f(a_n) = f(a).$$

Since the lower and upper limits coincide, this means that the sequence $f(a_n)$ indeed has a limit, and this limit is equal to $f(a)$. Right continuity is now proven.

4°. To complete the proof, let us show that for every a monotonic right-continuous function $f(a)$ for which $a \le f(a)$ for all a, the formula (3) defines a class of realistic intervals, i.e., that the resulting class contains all numbers from this interval, is topologically closed, and is closed under minimum and maximum operations.

Let us prove these properties one by one.

4.1°. Since $a \le f(a)$, each interval $[a, a]$ belongs to the class $\mathcal{C}$.

4.2°. Let us prove that the class $\mathcal{C}$ is topologically closed, i.e., that if $[a_n, b_n] \to [a, b]$ and $[a_n, b_n] \in \mathcal{C}$ for all n, then $[a, b] \in \mathcal{C}$.

By definition of the class (3), for each n, we have $b_n \le f(a_n)$. Let us consider two cases.

4.2.1°. If for infinitely many values a_{n_k}, we have $a_{n_k} \le a$, then, due to monotonicity, we have $f(a_{n_k}) \le f(a)$, so for $b_{n_k} \le f(a_{n_k})$, we also have $b_{n_k} \le f(a)$, and in the limit, we get $b \le f(a)$, i.e., $[a, b] \in \mathcal{C}$.

4.2.2°. In the opposite case, when for all but finitely many indices n, we have $a < a_n$, then, due to right continuity, we have $f(a_n) \to f(a)$. Thus, from $b_n \le f(a_n)$, in the limit, we get $b \le f(a)$, i.e., also $[a, b] \in \mathcal{C}$.

4.3°. Let us show that the class $\mathcal{C}$ is closed under the minimum operation.

Indeed, let us assume that $[a_1, b_1] \in \mathcal{C}$ and $[a_2, b_2] \in \mathcal{C}$, i.e., $b_1 \le f(a_1)$ and $b_2 \le f(a_2)$. Let us prove that for $[a, b] \stackrel{\text{def}}{=} [\min(a_1, a_2), \min(b_1, b_2)]$, we also have $[a, b] \in \mathcal{C}$. Without losing generality, we can assume that $a_1 \le a_2$, so $a = \min(a_1, a_2) = a_1$. In this case, $b = \min(b_1, b_2) \le b_1 \le f(a_1) = f(a)$, so indeed $[a, b] \in \mathcal{C}$.

4.4°. Finally, let us show that the class $\mathcal{C}$ is closed under the maximum operation.

Indeed, let us assume that $[a_1, b_1] \in \mathcal{C}$ and $[a_2, b_2] \in \mathcal{C}$, i.e., $b_1 \le f(a_1)$ and $b_2 \le f(a_2)$. Let us prove that for $[a, b] \stackrel{\text{def}}{=} [\max(a_1, a_2), \max(b_1, b_2)]$, we also have $[a, b] \in \mathcal{C}$. Without losing generality, we can assume that $a_1 \le a_2$, so $a = \max(a_1, a_2) = a_2$. Here, $b_2 \le f(a_2)$, and since $b_1 \le f(a_1)$ and $f(a)$ is a monotonic function, $f(a_1) \le f(a_2)$ and thus, $b_1 \le f(a_2)$. From $b_1 \le f(a_2)$ and $b_2 \le f(a_2)$, we conclude that $b = \max(b_1, b_2) \le f(a_2) = f(a)$, so indeed $[a, b] \in \mathcal{C}$.

The proposition is proven.

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