

5-2020

Why Some Powers Laws Are Possible And Some Are Not

Edgar Daniel Rodriguez Velasquez
Universidad de Piura, edgar.rodriguez@udep.pe

Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu

Olga Kosheleva
University of Texas at El Paso, olgak@utep.edu

Nguyen Hoang Phuong
Thang Long University, nhphuong2008@gmail.com

Follow this and additional works at: https://scholarworks.utep.edu/cs_techrep



Part of the [Mathematics Commons](#)

Comments:

Technical Report: UTEP-CS-20-47

Recommended Citation

Rodriguez Velasquez, Edgar Daniel; Kreinovich, Vladik; Kosheleva, Olga; and Phuong, Nguyen Hoang, "Why Some Powers Laws Are Possible And Some Are Not" (2020). *Departmental Technical Reports (CS)*. 1438. https://scholarworks.utep.edu/cs_techrep/1438

This Article is brought to you for free and open access by the Computer Science at ScholarWorks@UTEP. It has been accepted for inclusion in Departmental Technical Reports (CS) by an authorized administrator of ScholarWorks@UTEP. For more information, please contact lweber@utep.edu.

Why Some Powers Laws Are Possible And Some Are Not

Edgar Daniel Rodriguez Velasquez^{1,2}, Vladik Kreinovich³

Olga Kosheleva⁴, and Hoang Phuong Nguyen⁵

¹Department of Civil Engineering
Universidad de Piura in Peru (UDEP)
Av. Ramón Mugica 131, Piura, Peru
edgar.rodriguez@udep.pe

²Department of Civil Engineering

³Department of Computer Science

⁴Department of Teacher Education

University of Texas at El Paso, 500 W. University
El Paso, TX 79968, USA

edrodriguezvelasquez@miners.utep.edu

vladik@utep.edu, olgak@utep.edu

⁴Division Informatics, Math-Informatics Faculty

Thang Long University, Nghiem Xuan Yem Road

Hoang Mai District, Hanoi, Vietnam

nhphuong2008@gmail.com

Abstract

Many dependencies between quantities are described by power laws, in which y is proportional to x raised to some power a . In some application areas, in different situations, we observe all possible pairs (A, a) of the coefficient of proportionality A and of the exponent a . In other application areas, however, not all combinations (A, a) are possible: once we fix the coefficient A , it uniquely determines the exponent a . In such case, the dependence of a on A is usually described by an empirical logarithmic formula. In this paper, we show that natural scale-invariance ideas lead to a theoretical explanation for this empirical formula.

1 Formulation of the Problem

Power laws are ubiquitous. In many application areas, the dependence between two quantities x and y is described by power laws

$$y = A \cdot x^a \tag{1}$$

for constants a and A . Such dependencies are known as *power laws*.

Power laws are truly ubiquitous. Let us just give a few examples:

- power laws describe how the aerodynamic resistance force depends on the plane's velocity,
- they describe how the perceived signal depends on the intensity of the signal that we hear and see,
- they describe how the mass of celestial structures – ranging from small star clusters to galaxies to clusters of galaxies – depends on the structure's radius, etc.;

see, e.g., [3, 7].

Sometimes, not all power laws are possible. The parameters A and a have to be determined from the experiment. In some application areas, all pairs (A, a) are possible. In some other applications areas, however, not all such pairs are possible. Sometimes, a is fixed, and A can take all possible values. In other application areas, we have different values of A – but for each A , we can only have one specific value of a . One such example can be found in transportation engineering: it describes the dependence of number y of cycles until fatigue failure on the initial strain x ; see, e.g., [2, 4, 5, 6, 8].

In many such situations, the value of a corresponding to A is determined by the following empirical formula [2, 4, 5, 6, 8]:

$$a = c_0 + c_1 \cdot \ln(A). \tag{2}$$

Comment. The case when the value a is fixed can be viewed as a particular case of this empirical formula, corresponding to $c_1 = 0$.

Resulting challenge. How can we explain the formula (2)?

What we do in this paper. In this paper, we provide a theoretical explanation for this formula.

To come up with this explanation, we recall the reason why power laws are ubiquitous in the first place – because they correspond to scale-invariant dependencies. We then use the scale-invariance idea to explain the ubiquity of the formula (2).

2 Power Laws and Scale Invariance: A Brief Reminder

Scaling. The main purpose of data processing is to deal with physical quantities. However, in practice, we only deal with the numerical values of these quantities.

What is the difference? The difference is that to get a numerical value, we need to select a measuring unit for measuring the quantity. If we replace the original measuring unit with a new one which is λ times smaller, then all numerical values are multiplied by λ : $x \rightarrow X = \lambda \cdot x$. For example, if we move from meters to centimeters, all the numerical values will be *re-scaled*: multiplied by 100, e.g., 1.7 m becomes $1.7 \cdot 100 = 170$ cm.

Scale-invariance. In many application areas, there is no fixed measuring unit, the choice of the measuring unit is rather arbitrary. In such situations, it is reasonable to require that the dependence $y = f(x)$ between the quantities x and y not depend on the choice of the unit.

Of course, this does not mean that $y = f(x)$ imply $y = f(X) = f(\lambda \cdot x)$ for the exact same function $f(x)$ – that would mean that $f(\lambda \cdot x) = f(x)$ for all x and λ , i.e., that $f(x)$ is a constant and thus, that there is no dependence.

What we need to do to keep the same dependence is to accordingly re-scale y , to $Y = \mu \cdot y$ for some μ depending on λ . For example, the area y of a square is equal to the square of its size $y = x^2$. This formula is true if we use meters to measure length and square meters to measure area. The same formula holds if we use centimeters instead of meters – but then, we should use square centimeters instead of square meters. In this case, $\lambda = 100$ corresponds to $\mu = 10000$.

So, we arrive at the following definition of scale-invariance: for every $\lambda > 0$ there exists a value $\mu > 0$ for which, for every x and y , the relation $y = f(x)$ implies that $Y = f(X)$ for $X = \lambda \cdot x$ and $Y = \mu \cdot y$.

Scale-invariance and power laws. It is easy to check that every power law is scale-invariant. Indeed, it is sufficient to take $\mu = \lambda^a$. Then, from $y = A \cdot x^a$ we get $\lambda^a \cdot y = \lambda^a \cdot A \cdot x^a = A \cdot (\lambda \cdot x)^a$, i.e., indeed $Y = f(X)$.

It turns out that, vice versa, the only continuous scale-invariance dependencies are power laws; see, e.g., [1]. For differentiable functions $f(x)$, this can be easily proven. Indeed, by definition, scale-invariance means that

$$\mu(\lambda) \cdot f(x) = f(\lambda \cdot x). \tag{3}$$

Since the function $f(x)$ is differentiable, the function

$$\mu(\lambda) = \frac{f(\lambda \cdot x)}{f(x)}$$

is also differentiable, as the ratio of two differentiable functions. Since both functions $f(x)$ and $\mu(\lambda)$ are differentiable, we can differentiate both sides of the equality (3) with respect to λ :

$$\mu'(\lambda) \cdot f(x) = x \cdot f'(\lambda \cdot x),$$

where f' , as usual, means the derivative. In particular, for $\lambda = 1$, we get $\mu_0 \cdot f(x) = x \cdot f'(x)$, where we denoted $\mu_0 \stackrel{\text{def}}{=} \mu'(1)$, i.e.,

$$\mu_0 \cdot f = x \cdot \frac{df}{dx}.$$

We can separate the variables x and f if we divide both sides by $x \cdot f$ and multiply both sides by dx , then we get

$$\frac{df}{f} = \mu_0 \cdot \frac{dx}{x}.$$

Integrating both sides, we get $\ln(f) = \mu_0 \cdot \ln(x) + c$, where c is the integration constant. Thus, for $f = \exp(\ln(f))$, we get

$$f(x) = \exp(\mu_0 \cdot \ln(x) + c) = A \cdot x^a,$$

where we denoted $A \stackrel{\text{def}}{=} \exp(c)$ and $a \stackrel{\text{def}}{=} \mu_0$.

3 Main Idea and Resulting Explanation

Main idea. Since, in principle, for the corresponding application areas, we can have different values A and a , this means that the value of the quantity y is not uniquely determined by the value of the quantity x , there must be some other quantity z that influences y . In other words, we should have

$$y = F(x, z). \tag{4}$$

for some function $F(x, z)$. Different situations – i.e., different pairs (A, a) – are characterized by different values of the auxiliary quantity z .

Main assumption. The very fact that for each fixed z , the dependence of y on x is described by a power law means that when the value of z is fixed, the dependence of y on x is scale-invariant.

It is therefore reasonable to conclude that, vice versa, for each fixed value x , the dependence of y on z is also scale-invariant.

This assumption leads to the desired explanation of the above empirical formula. Let us show that this assumption indeed explains the formula (2).

Indeed, the fact that for each z , the dependence of y on x is described by the power law, with coefficients A and a depending on z , can be described as

$$F(x, z) = A(z) \cdot x^{a(z)}. \tag{5}$$

Similarly, the fact that the dependence of y on z is scale-invariant means that for each x , the dependence of y on z can also be described by the power law, with the coefficients depending on x :

$$F(x, z) = B(x) \cdot z^{b(x)}, \tag{6}$$

for appropriate coefficients $B(x)$ and $b(x)$. By equating two different expressions (5) and (6) for $F(x, z)$, we conclude that

$$A(z) \cdot x^{a(z)} = B(x) \cdot z^{b(x)} \tag{7}$$

for all x and z .

In particular, for $x = 1$, the formula (7) implies that

$$A(z) = B(1) \cdot z^{b(1)}. \quad (8)$$

Similarly, for $z = 1$, the formula (7) implies that

$$B(x) = A(1) \cdot x^{a(1)}. \quad (9)$$

Substituting expressions (8) and (9) into the formula (7), we conclude that

$$B(1) \cdot z^{b(1)} \cdot x^{a(z)} = A(1) \cdot x^{a(1)} \cdot z^{b(x)}. \quad (10)$$

In particular, for $x = e$, we get

$$B(1) \cdot z^{b(1)} \cdot e^{a(z)} = A(1) \cdot e^{a(1)} \cdot z^{b(e)},$$

hence

$$\exp(a(z)) = \frac{A(1) \cdot \exp(a(1))}{B(1)} \cdot z^{b(e)-b(1)}. \quad (11)$$

From the formula (8), we conclude that

$$z^{b(1)} = \frac{A}{B(1)},$$

and thus,

$$z = \frac{A^{1/b(1)}}{B(1)^{1/b(1)}}. \quad (12)$$

Substituting the expression (12) into the formula (11), we conclude that

$$\exp(a) = \frac{A(1) \cdot \exp(a(1))}{B(1)} \cdot \frac{1}{B(1)^{(b(e)-b(1))/b(1)}} \cdot A^{(b(e)-b(1))/b(1)},$$

i.e., that $\exp(a) = C_0 \cdot A^{c_1}$ for some values C_0 and c_1 . Taking logarithms of both sides, we now get the desired dependence $a = c_0 + c_1 \cdot \ln(A)$, where we denoted $c_0 \stackrel{\text{def}}{=} \ln(C_0)$.

So, we indeed have the desired derivation.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science) and HRD-1242122 (Cyber-ShARE Center of Excellence).

References

- [1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, UK, 1989.
- [2] K. E. Cooper and P. S. Pell, *The Effect of Mix Variables on the Fatigue Strength of Bituminous Materials*, UK Transport and Road Research Laboratory, Report LR 663, 1974.
- [3] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.
- [4] J. Myre, “Fatigue of asphalt pavements”, *Proceedings of the Third International Conference on Bearing Capacity of Roads and Airfields*, Norwegian Institute of Technology, 1990.
- [5] A. L. Priest and D. H. Timm, *Methodology and Calibration of Fatigue Transfer Functions for Mechanistic-Empirical Flexible Pavement Design*, US National Center for Asphalt Technology (NCAT), Auburn, Alabama, NCAT Report 06-03, December 2006.
- [6] J. B. Rauhut, R. L. Lytton, and M. I. Darter, *Pavement Damage Functions for Cost Allocations. Vol. 2, Descriptions of Detailed Studies*, Federal Highway Administration (FHWA), Report FHWA-RD-84-019, Washington, D.C., 1984.
- [7] K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2017.
- [8] D. H. Timm, D. E. Newcomb, and B. Birgisson, *Mechanistic-Empirical Flexible Pavement Thickness Design: The Minnesota Method*, Minnesota Department of Transportation, Staff Paper MN/RC-P99-10, St. Paul, Minnesota, 1999.