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How the Proportion of People Who Agree to Perform a Task Depends on the Stimulus: A Theoretical Explanation of the Empirical Formula

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Abstract

For each task, the larger the stimulus, the larger proportion of people agree to perform this task. In many economic situations, it is important to know how much stimulus we need to offer so that a sufficient proportion of the people will agree to perform the needed task. There is an empirical formula describing how this proportion increases as we increase the amount of stimulus. However, this empirical formula lacks a convincing theoretical explanation, as a result of which practitioners are somewhat reluctant to use it. In this paper, we provide a theoretical explanation for this empirical formula, thus making it more reliable – and hence, more useable.

1 Formulation of the Problem

The larger the stimulus, the more people agree to do the task. In economics, we need to entice people to perform certain tasks – whether it is planting crops or working on a factory or writing a software package. When the corresponding stimulus is too small, no one will agree to perform the task. When the stimulus is very high, everyone will agree. The proportion p of people who agree to perform a task will increase with the increase in the stimulus s.

It is desirable to know the exact amount of stimulus. A company wants certain tasks to be performed, so it has to use some stimulus. It is therefore desirable to find the exact amount of stimulus needed:

- if the stimulus is too low, no one will volunteer,
- if it is very high, the tasks will be performed, but the company will lose too much money.

How the amount of stimulus is usually determined now. In many cases, the selection of the right stimulus is done mostly by trial and error. This is, e.g., how airline companies, in an overbooked situation, ask for volunteers to give up their seats and fly the next day: they increase the award offered to potential volunteers until they get enough volunteers.

Formulas are needed, and there are such formulas. Trial-and-error is a lengthy process, difficult to predict. It is therefore desirable to have some analytical expressions that would help us select the right amount of stimulus.

Such expressions exist; see, e.g., [2] and references therein. The most empirically adequate expression is

$$p = \frac{s^q}{s^q + c}, \tag{1}$$

for some constants q and c.

This formula is purely empirical. One of the main limitation of this formula is that is it purely empirical, it does not have a convincing theoretical explanation. Practitioners are usually very suspicious of best-fit purely empirical formulas, they are reluctant so use these formulas, they prefer formulas for which some theoretical explanation exists – since purely empirical formulas often turn out to be wrong.

And in economics and related areas, such later-wrong empirical formulas are ubiquitous:

- when a country has a boom, empirical formulas predict exponential growth forever;
- when, in the 1920s, the number of telephone operators started growing exponentially, empirical formulas predicted that in a few decades, half of the population will be telephone operators;

there are many examples like that.

It is thus desirable to come up with a theoretical explanation for empirical formulas.

What we do in this paper. In this paper, we provide a theoretical explanation for the formula (1).

2 Main Idea and the Resulting Explanation

Let us reformulate the problem in terms of probabilities. In the above text, we talked about proportion of people who take on the task. From the mathematical viewpoint, a proportion is not something about which we know much.

But what is proportion? It is simply the probability that a randomly selected person will take on the task. So, whatever we said about proportions can be reformulated in terms of probabilities – and about probabilities, we know a lot!

Comment. Good news is that we do not even need to change the notation p, since both words "proportion" and "probability" start with the same letter p.

What do we know about probabilities? One of the most widely used facts about probabilities is that if we add new evidence E, the probability of each hypothesis H_i changes according to the Bayes formula (see, e.g., [3]), from the original value $p_0(H_i)$ to the new value

$$p(H_i \mid E) = \frac{p_0(H_i) \cdot p(E \mid H_i)}{\sum\limits_j p_0(H_j) \cdot p(E \mid H_j)}. \tag{2}$$

In our case, we have two hypotheses:

- the hypothesis H_0 that the person will take on the task whose probability is $p(H_0)$, and
- the hypothesis H_1 that the person will not take on the task; its probability is equal to $p(H_1) = 1 - p(H_0)$.

In this case, the general formula (2) takes the form

$$p(H_0 \mid E) = \frac{p_0(H_0) \cdot p(E \mid H_0)}{p_0(H_0) \cdot p(E \mid H_0) + (1 - p_0(H_0)) \cdot p(E \mid H_1)} =$$

$$\frac{p(H_0) \cdot p(E \mid H_0)}{p(H_0) \cdot (p(E \mid H_0) - p(E \mid H_1)) + p(E \mid H_1)}.$$

In other words, the change of the probability from the previous value $p = p_0(H_0)$ to the new value $p' = p(H_0 \mid E)$ is described by the formula

$$p' = \frac{p \cdot p(E \mid H_0)}{p \cdot (p(E \mid H_0) - p(E \mid H_1)) + p(E \mid H_1)}. \tag{3}$$

If we divide both the numerator and the denominator of the formula (3) by $p(E \mid H_1)$, then we get the following expression:

$$p' = \frac{p \cdot \dfrac{p(E \mid H_0)}{p(E \mid H_1)}}{1 + p \cdot \left(\dfrac{p(E \mid H_0)}{p(E \mid H_1)} - 1\right)},$$

i.e., the expression

$$p' = \frac{a \cdot p}{1 + (1 - a) \cdot p}, \tag{4}$$

where we denoted

$$a \stackrel{\text{def}}{=} \frac{p(E \mid H_0)}{p(E \mid H_1)}.$$

In other words, from the mathematical viewpoint, the change of the probability from the previous value p to the new value p' is thus described by a fractional-linear formula (4).

Here comes our idea. Our idea is that when we increase the stimulus, the resulting change of the probability should follow the formula (4).

How can we formalize this idea. What does it mean "increase the stimulus"? Intuitively, it means that we increase all the previous stimuli the same way.

What does that mean? If we simply add \$10 to all the previous stimulus values, this does not mean that we increases all the stimuli the same way. For example:

- if the previous stimulus was \$5, this is a drastic 3-times increase, but
- if the previous stimulus was \$1000, this is a barely noticeable 1% increase.

From the economic viewpoint, it makes more sense to increase all the previous stimulus values proportionally; e.g.:

- increase all the values by 1%, or
- increase all the values by 10%, or
- increase all the values by a factor of three.

With such an increase, instead of previous stimulus value s, we get a new stimulus value $\lambda \cdot s$, where, e.g.:

- an over-the-board 1% increase means $\lambda = 1.01$,
- an over-the-board 10% increase means $\lambda = 1.1$, and
- an over-the-board 3-times increase means $\lambda = 3$.

In these terms, the main idea takes the following form.

Resulting formulation. We want to find an increasing function $p(s)$ for which $p(0) = 0$, $p(s) \to 1$ as $s \to \infty$, and for every $\lambda > 0$, there exists $a(\lambda)$ for which, for all s, we have

$$p(\lambda \cdot s) = \frac{a(\lambda) \cdot p(s)}{1 + (a(\lambda) - 1) \cdot p(s)}. \tag{5}$$

Our main result. Our main result is that every function $p(s)$ satisfying the above conditions has exactly the form (1), for some values q and c.

This is exactly what we wanted. Thus, we indeed have the desired justification of the empirical formula (1).

3 Proof of the Main Result

Let us reformulate the formula (5) in terms of odds. For this proof, it is convenient to reformulate probabilities p in terms of the *odds*, i.e., in terms of the ratio

$$o = \frac{p}{1-p}.$$

Let us first find the odds corresponding to the new probability $p(\lambda\cdot s)$. From the formula (5), we get

$$1 - p(\lambda\cdot s) = 1 - \frac{a(\lambda)\cdot p(s)}{1+(a(\lambda)-1)\cdot p(s)} =$$

$$\frac{1+a(\lambda)\cdot p(s) - p(s) - a(\lambda)\cdot p(s)}{1+(a(\lambda)-1)\cdot p(s)} = \frac{1-p(s)}{1+(a(\lambda)-1)\cdot p(s)}. \quad (6)$$

Dividing (5) by (6), we get

$$d(\lambda\cdot s) = \frac{p(\lambda\cdot s)}{1-p(\lambda\cdot s)} = \frac{a(\lambda)\cdot p(s)}{1-p(s)} = a(\lambda)\cdot\frac{p(s)}{1-p(s)}.$$

Since the ratio in the right-hand side is exactly the odds $o(s)$ corresponding to the probability $p(s)$, we thus conclude that

$$o(\lambda\cdot s) = a(\lambda)\cdot o(s). \quad (7)$$

Now, we can use the known solution to the functional equation (7). According to [1], every monotonic solution of the equation (7) has the form

$$o(s) = C\cdot s^q \quad (8)$$

for some values C and q.

The general proof of this statement is somewhat complicated, but it becomes very straightforward if we make an additional natural assumption that the function $p(s)$ is differentiable. In this case, the ratio $o(s)$ is also differentiable. Due to the equation (7), the function $a(\lambda)$ is equal to the ratio of two differentiable functions

$$a(\lambda) = \frac{o(\lambda\cdot s)}{o(s)}$$

and is, thus, also differentiable. Thus, we can differentiate both sides of the equation (7) with respect to λ and get

$$s\cdot o'(\lambda\cdot s) = a'(\lambda)\cdot o(s),$$

where, as usual, $f'(x)$ denotes the derivative. In particular, for $\lambda = 1$, we get $s\cdot o'(s) = q\cdot o(s)$, where we denoted $q \stackrel{\text{def}}{=} a'(1)$. In other words, we have

$$s\cdot\frac{do}{ds} = q\cdot o.$$

We can separate the variables s and o if we divide both sides of the equation by $s \cdot o$ and multiply both sides by ds, then we get

$$\frac{do}{o} = q \cdot \frac{ds}{s}.$$

Integrating both sides, we get $\ln(o) = q \cdot \ln(s) + C_0$, where C_0 is an integration constant. By applying $\exp(x)$ to both sides, we then get $o(s) = C \cdot s^q$, where we denoted $C \stackrel{\text{def}}{=} \exp(C_0)$.

From the equality (8) to the desired formula (1). According to the formula (8), we have

$$o(s) = \frac{p(s)}{1 - p(s)} = C \cdot s^q.$$

By taking the inverse of both sides, we get

$$\frac{1 - p(s)}{p(s)} = 1 - \frac{1}{p(s)} = C^{-1} \cdot s^{-q},$$

thus

$$\frac{1}{p(s)} = 1 - C^{-1} \cdot s^{-q}$$

and therefore,

$$p(s) = \frac{1}{1 - C^{-1} \cdot s^{-q}}.$$

Multiplying both the numerator and the denominator by s^q, we get

$$p(s) = \frac{s^q}{s^q - C^{-1}}.$$

Probabilities are always smaller than or equal to 1, thus $s^q \le s^q - C^{-1}$, i.e., $C^{-1} < 0$. If we denote $c \stackrel{\text{def}}{=} -C^{-1}$, we will get the desired formula (1).

The main result is thus proven.

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