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Decision Making Under Interval Uncertainty: Towards (Somewhat) More Convincing Justifications for Hurwicz Optimism-Pessimism Approach

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Abstract

In the ideal world, we know the exact consequences of each action. In this case, it is relatively straightforward to compare different possible actions and, as a result of this comparison, to select the best action. In real life, we only know the consequences with some uncertainty. A typical example is interval uncertainty, when we only know the lower and upper bounds on the expected gain. How can we compare such interval-valued alternatives? A usual way to compare such alternatives is to use the optimism-pessimism criterion developed by Nobelist Leo Hurwicz. In this approach, we maximize a weighted combination of the worst-case and the best-case gains, with the weights reflecting the decision maker's degree of optimism. There exist several justifications for this criterion; however, some of the assumptions behind these justifications are not 100% convincing. In this paper, we propose new, hopefully more convincing justifications for Hurwicz's approach.

1 Formulation of the Problem

Need to make decisions under interval uncertainty. In many real-life situations, we need to make a decision, i.e., we need to select one of the possible alternatives. For example, we want to select the best investment strategy, we need to decide whether to accept a new job offer, etc.

In the ideal world, we should know the exact consequence of each possible alternative. In such an ideal case, we select an alternative which is the best for us. For example, if the goal of the investment is to save for retirement, then we should select the investment strategy that will bring us the larger amount of money by the expected retirement date.

In real world, there is uncertainty. We can rarely predict the exact consequences of each action. In the simplest case, instead of knowing the exact amount of money m resulting from each alternative, we only know that this amount will be somewhere between the values $\underline{m}$ and $\overline{m}$. In other words, we do not know the exact value m; instead, we only know the interval $[\underline{m}, \overline{m}]$ that contains the actual (not yet known) value m. Such a situation is known as the situation of *interval uncertainty.* If we know intervals corresponding to different alternatives, which alternative should we select?

In other cases, in addition to the bounds $\underline{m}$ and $\overline{m}$, we also have some information about which values from the corresponding interval are more probable and which are less probable. In other words, we have some information – usually partial – about the actual probability distribution on the interval $[\underline{m}, \overline{m}]$. Sometimes, we know the exact probability distribution. In this case, we can, e.g., select the alternative for which the expected gain is the largest – of, if we want to be cautious, e.g., the alternative for which the gain guaranteed with a certain probability (e.g., 80%) is the largest.

In practice, we rarely know the exact probability distribution. Even if we know that the distribution is, e.g., Gaussian, we still do not know the exact values of the corresponding parameters – from the observations, we can only determine parameters with some uncertainty. For different possible combinations of these parameters, the expected gain – or whatever else characteristic we use – may take different values. Thus, for each alternative, instead of the *exact* value m of the corresponding objective function (such as expected gain), we have a whole *interval* $[\underline{m}, \overline{m}]$ of possible values of this objective function. So, we face the exact same problem as in the simplest possible case – we need to select an alternative in a situation when for each alternative, we only know the interval of possible values of the objective function.

How decisions under interval uncertainty are currently made. As we have mentioned earlier, decision making under interval uncertainty is an important practical problem. Not surprisingly, methods for solving this problem have been known for many decades. Usually, practitioners use a solution proposed in the early 1950s by the future Nobelist Leo Hurwicz; see, e.g., [2, 3, 5]. According to this solution, a decision maker should:

- first, select a parameter α from the interval $[0, 1]$, and then

- select an alternative for which the following combination attains the largest possible value:
$$\alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u}.$$

This idea is known as the *optimism-pessimism* criterion, and the selected value α is known as the *optimism parameter*. The reason for these terms is straightforward:

- If $\alpha = 1$, this means that the decision maker simply selects the alternative with the largest possible value of $\overline{m}$. In other words, the decision maker completely ignores the possibility that the outcome of each alternative can be smaller than in the best possible case, and bases his/her decision exclusively on comparing these best possible consequences of different actions. This is clearly an extreme case of an optimist.

- Vice versa, if $\alpha = 0$, this means that the decision maker simply selects the alternative with the largest possible value of $\underline{m}$. In other words, the decision maker completely ignores the possibility that the outcome of each alternative can be better that in the worst possible case, and bases his/her decision exclusively on comparing these worst possible consequences of different actions. This is clearly an extreme case of a pessimist.

Both these situations are extreme. In real life, most people take into account both good and bad possibilities, i.e., in Hurwicz terms, they make decisions based on some intermediate value α – which is larger than the pessimist's 0 but smaller than the optimist's 1.

How can we explain the current approach to decision making under uncertainty. There exist reasonable explanations for Hurwicz criteria, both:

- for the case when the outcome of each alternative is simply monetary and

- for the case when the outcome is not monetary – in this case, decision theory helps us describe the user's preferences in terms of special values known as *utilities*; see, e.g., [1, 3, 5, 6, 7] and references therein.

Remaining problem and what we do in this paper. In both monetary and utility cases, to derive Hurwicz's formula, we need to make certain assumptions;

- some of these assumptions are more reasonable,

- some of these assumptions are slightly less convincing.

Natural questions are:

- Do we need these somewhat less convincing assumptions?

- Can we avoid them altogether – and, if not, can we replace them with somewhat more convincing assumptions?

These are the question that we will analyze – and answer – in this paper.

Structure of this paper. We will start the paper with the easier-to-describe and easier-to-analyze case of monetary alternatives. First, in Section 2, we describe the usual assumptions leading to the Hurwicz criterion, explain how the Hurwicz criterion can be derived from these assumptions (in this, we largely follow [4]), and why some of these assumptions may not sound fully convincing. Then, in Section 3, we present new – hopefully more convincing – assumptions, and show how Hurwicz criterion can be derived from the new assumptions.

Then, we deal with the utility case. In Section 4, we briefly remind the readers who are not familiar with all the technical details of decision theory, what is utility and what are the properties of utility. In Section 5, we describe the usual assumptions leading to Hurwicz criterion for the utility case (they are somewhat different from the monetary case), explain how Hurwicz criterion can be derived from these assumptions (in this, we also largely follow [4]), and why some of these assumptions may not sound fully convincing. Finally, in Section 6, we show that the Hurwicz criterion can be derived from the (hopefully) more convincing assumptions in the utility case as well.

2 Monetary Case: Usual Derivation of the Hurwicz Criterion and the Limitations of This Derivation

To make a decision, we need to have an exact numerical equivalent for each interval. We want to be able to compare different alternatives with interval uncertainty. In particular, for each interval-valued alternative $[\underline{m}, \overline{m}]$ and for each alternative with a known exact monetary value m, we need to be able to decide:

- whether the exact-valued alternative is better or
- whether the interval-valued alternative is better.

Of course, if $m < \underline{m}$, then no matter what is the actual value from the interval $[\underline{m}, \overline{m}]$, this value will be larger than m. Thus, in this case, the interval alternative is clearly better. We will denote this by $m < [\underline{m}, \overline{m}]$.

Similarly, if $m > \overline{m}$, then no matter what is the actual value from the interval $[\underline{m}, \overline{m}]$, this value will be smaller than m. Thus, in this case, the interval alternative is clearly worse: $[\underline{m}, \overline{m}] < m$.

If $m < [\underline{m}, \overline{m}]$ and $m' < m$, the clearly $m' < [\underline{m}, \overline{m}]$. Similar, if $[\underline{m}, \overline{m}] < m$ and $m < m'$, then $[\underline{m}, \overline{m}] < m'$.

One can show that because of this, there is a threshold value separating the two cases, namely, the value

$$\sup\{m : m < [\underline{m}, \overline{m}]\} = \inf\{m : [\underline{m}, \overline{m}] < m\}.$$

Let us denote this threshold value – depending on $\underline{m}$ and $\overline{m}$ – by $f(\underline{m}, \overline{m})$.

By definition, for every $\varepsilon > 0$, we have

$$f(\underline{m}, \overline{m}) - \varepsilon < [\underline{m}, \overline{m}] < f(\underline{m}, \overline{m}) + \varepsilon.$$

In particular, this property holds for an arbitrarily small ε, including such small ε that no one will notice the difference between the value m and the values $m-\varepsilon$ and $m+\varepsilon$. So, from the practical viewpoint, we can say that the interval $[\underline{m}, \overline{m}]$ is *equivalent* to the monetary value $f(\underline{m}, \overline{m})$. We will denote this equivalence by

$$[\underline{m}, \overline{m}] \equiv f(\underline{m}, \overline{m}).$$

From this viewpoint, all we need to do to describe decision making under interval uncertainty is to describe the corresponding function $f(\underline{m}, \overline{m})$.

The numerical value $f(\underline{m}, \overline{m})$ should always be between $\underline{m}$ and $\overline{m}$. Since, as we have mentioned earlier, for every value $m < \underline{m}$, we have $m < [\underline{m}, \overline{m}]$, the set $\{m : m < [\underline{m}, \overline{m}]\}$ contains all the values from the set $(-\infty, \underline{m})$. Thus, its supremum $f(\underline{m}, \overline{m})$ has to be greater than or equal to all the values $m < \underline{m}$, in particular, than all the values $m = \underline{m} - 1/n$. So, we must have

$$\underline{m} - 1/n < f(\underline{m}, \overline{m})$$

for all n. In the limit $n \to \infty$, we conclude that $\underline{m} \le f(\underline{m}, \overline{m})$.

Similarly, since for every value $m > \overline{m}$, we have $[\underline{m}, \overline{m}] > m$, the set

$$\{m : [\underline{m}, \overline{m}] < m\}$$

contains all the values from the set $(\overline{m}, \infty)$. Thus, its infimum $f(\underline{m}, \overline{m})$ has to be smaller than or equal to all the values $m > \overline{m}$ – in particular, than all the values $m = \overline{m} + 1/n$. So, we must have $f(\underline{m}, \overline{m}) < \overline{m} + 1/n$ for all m. In the limit $n \to \infty$, we conclude that $f(\underline{m}, \overline{m}) \le \overline{m}$.

Based on the two above examples, we should always have $\underline{m} \le f(\underline{m}, \overline{m}) \le \overline{m}$.

Let us prepare for the usual derivation of Hurwicz criterion. In order to explain the usual derivation of Hurwicz criterion from several assumptions, let us first provide the usual motivation for these assumptions.

Monotonicity. Let us assume that we start with an interval $[\underline{m}, \overline{m}]$, and then we:

- delete all the lowest-value options – i.e., options for which $m \le \underline{m}'$ for some $\underline{m}' > \underline{m}$, and/or:
- add several higher-value options, with $m > \overline{m}$, e.g., all the values from $\overline{m}$ to some larger value $\overline{m}' > \overline{m}$.

After this, we get a clearly better interval $[\underline{m}', \overline{m}']$. Thus, we conclude that the function $f(\underline{m}, \overline{m})$ should be *monotonic*: if $\underline{m} \le \underline{m}'$ and $\overline{m} \le \overline{m}'$, then $f(\underline{m}, \overline{m}) \le f(\underline{m}', \overline{m}')$.

Additivity. Suppose that we have two situations:

- in the first situation, we can get any value from $\underline{a}$ to $\overline{a}$, and
- in the second situation, we can get any value from $\underline{b}$ to $\overline{b}$.

By definition of the function $f(\underline{m}, \overline{m})$, we are willing to pay the value $f(\underline{a}, \overline{a})$ to participate in the first situation, and the value $f\left(\underline{b}, \overline{b}\right)$ to participate in the second situation.

What if we consider these two choices as a single situation? In this case, the smallest possible value that we get overall – in both situations – is when we get the smallest possible value $\underline{a}$ in the first situation and the smallest possible value $\underline{b}$ in the second situation. In this case, the overall value is $\underline{a} + \underline{b}$.

Similarly, the largest possible value that we get overall – in both situations – is when we get the largest possible value $\overline{a}$ in the first situation and the largest possible value $\overline{b}$ in the second situation. In this case, the overall value is $\overline{a} + \overline{b}$.

Thus, when we consider these two choices as a single situation, the interval of possible monetary gains has the form $\left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$. So, the equivalent monetary value of the two choices treated as a single situation is $f\left(\underline{a} + \underline{b}, \overline{a} + \overline{b}\right)$.

It is reasonable to require that the price that we pay for two choices sold together should be equal to the sum of the prices that we pay for two choices taken separately, i.e., that $f\left(\underline{a} + \underline{b}, \overline{a} + \overline{b}\right) = f(\underline{a}, \overline{a}) + f\left(\underline{b}, \overline{b}\right)$. This property is known as *additivity.*

The usual derivation of Hurwicz criterion. Now, we are ready to describe the usual derivation of Hurwicz criterion.

Definition 1.

- *By a* value function, *we mean a function $f(\underline{m}, \overline{m})$ that assigns, to each pair $(\underline{m}, \overline{m})$ of real numbers for which $\underline{m} \le \overline{m}$, a real number $f(\underline{m}, \overline{m})$ for which $\underline{m} \le f(\underline{m}, \overline{m}) \le \overline{m}$.*
- *We say that a value function $f(\underline{m}, \overline{m})$ is* monotonic *if whenever $\underline{m} \le \underline{m}'$ and $\overline{m} \le \overline{m}'$, then $f(\underline{m}, \overline{m}) \le f(\underline{m}', \overline{m}')$.*
- *We say that a value function $f(\underline{m}, \overline{m})$ is* additive *if for all possible values $\underline{a} \le \overline{a}$ and $\underline{b} \le \overline{b}$, we have $f\left(\underline{a} + \underline{b}, \overline{a} + \overline{b}\right) = f(\underline{a}, \overline{a}) + f\left(\underline{b}, \overline{b}\right)$.*
- *We say that a value function $f(\underline{m}, \overline{m})$ has a* Hurwicz *form if it has the form $f(\underline{m}, \overline{m}) = \alpha \cdot \overline{m} + (1 - \alpha) \cdot \underline{m}$ for some $\alpha \in [0, 1]$.*

Proposition 1. *For a value function $f(\underline{m}, \overline{m})$, the following two conditions are equivalent to each other:*

- *the value function is monotonic and additive;*
- *the value function has the Hurwicz form.*

Proof. It is easy to prove that a Hurwicz-form value function is monotonic and additive.

Vice versa, let us assume that a value function $f(\underline{m}, \overline{m})$ is monotonic and additive. Let us denote $\alpha \stackrel{\text{def}}{=} f(0,1)$.

Due to additivity, for every natural number n, we have

$$[0,1/n] + \ldots + [0,1/n] \ (n \text{ times}) = [0,1],$$

thus

$$f(0,1/n) + \ldots + f(0,1/n) \ (n \text{ times}) = n \cdot f(0,1/n) = f(0,1) = \alpha,$$

hence $f(0,1/n) = \alpha \cdot (1/n)$.

Similarly, for every m and n, we have

$$f(0,m/n) = f(0,1/n) + \ldots + f(0,1/n) \ (m \text{ times}) = m \cdot f(0,1/n) = \alpha \cdot (m/n).$$

For every real number r, we have $m/n \le r \le (m+1)/n$, where $m \stackrel{\text{def}}{=} \lfloor r \cdot n \rfloor$. Thus, due to monotonicity, we have $f(0,m/n) \le f(0,r) \le f(0,(m+1)/n)$, i.e., $\alpha \cdot (m/n) \le f(0,r) \le \alpha \cdot (m+1)/n$. Here, $0 \le r - m/n \le 1/n$, so in the limit $n \to \infty$, we have $m/n \to r$ and $(m+1)/n \to r$. Thus, the above inequality leads to $f(0,r) = \alpha \cdot r$.

In particular, for every $\underline{m} \le \overline{m}$, we have $f(0, \overline{m} - \underline{m}) = \alpha \cdot (\overline{m} - \underline{m})$. By the property of a value function, we have $\underline{m} \le f(\underline{m}, \underline{m}) \le \underline{m}$, i.e., $f(\underline{m}, \underline{m}) = \underline{m}$. Thus, due to additivity,

$$f(\underline{m}, \overline{m}) = f(\underline{m} + 0, \underline{m} + (\overline{m} - \underline{m})) = f(\underline{m}, \underline{m}) + f(0, \overline{m} - \underline{m}) = \underline{m} + \alpha \cdot (\overline{m} - \underline{m}).$$

One can easily check that this is indeed the Hurwicz expression.

Limitations. The above motivations are reasonably reasonable, but they may not be 100% convincing.

Indeed, we argued that if the worst-case scenario is possible for each of the two situations, then it is possible that we have the worst-case scenario in both situations. This may sound reasonable, but it is not in full agreement with common sense. Indeed, e.g., when we fly from point A to point B, we understand:

- that there may an unexpected delay at the airport A,
- that a plane may have a problem in flight and we will have to get back,
- that there may a problem at the airport B and we will get stuck on the plane, etc.,

but we honestly do *not* believe that all these low-probable disasters will happen at the same – this only happens in comedies describing lovable losers who always get into trouble.

We can raise another issues about the additivity requirement: that additivity assumes that for the combination of two items, we always pay the same price as for the two items separately. Sometimes, this is true, but often, this is not true: there *are* discounts if you buy several items (or several objects of the same type) at the same time.

What should we do? Since the arguments that we used above to justify the assumptions are not 100% convincing, maybe we can find somewhat more convincing arguments in favor of Hurwicz formula – or, alternatively, maybe these more convincing arguments can lead us to a different formula?

This is what we will analyze in the next section.

3 Monetary Case: New, Hopefully More Convincing, Derivation of the Hurwicz Criterion

Shift-invariance. Suppose that we offer a user a package deal in which he/she gets m dollars cash *and* an alternative in which he/she gets between $\underline{m}$ and $\overline{m}$. The equivalent value for the interval-value alternative is $f(\underline{m}, \overline{m})$, so the overall value for this package is $m + f(\underline{m}, \overline{m})$.

On the other hand, if we consider this a package deal, then in this deal, we get any amount between $m + \underline{m}$ and $m + \overline{m}$. Thus, the value of this package deal should be equal to $f(m + \underline{m}, m + \overline{m})$. It is reasonable to require that these two valuations should lead to the same result, i.e., that we should have $m + f(\underline{m}, \overline{m}) = f(m + \underline{m}, m + \overline{m})$. In mathematical terms, this property is known as *shift-invariance.*

Discussion. At first glance shift-invariant is very similar to additivity. Indeed, it can be viewed as a particular case of additivity, in which the first interval is simply the interval $[m, m]$ consisting of a single number m.

But good news is that both above objections to general additivity do not apply here. Indeed, we are not talking about a combination of rare events, so the first objection is not applicable. The second objection is also not applicable, since while we may expect a discount if we buy two big bottles of milk, no one expects a discount if we buy a bottle of milk and a fixed amount of money (e.g., when we ask to change a big banknote when paying).

Need for additional assumptions. If we limit ourselves only to shift-invariance, we will get too many possibilities in addition to Hurwicz formula: specifically, one can see that we can have a more general expression

$$f(\underline{m}, \overline{m}) = \underline{m} + F(\overline{m} - \underline{m}),$$

where $F(z)$ is a monotonic function defined for all $z \geq 0$ for which $F(z) \leq z$ for all z – e.g., $F(z) = z/(1 + z)$. (By the way, it is possible to show that the above expression is the most general form of a monotonic shift-invariant value function.)

To narrow down the class of possible value functions, we need to make additional reasonable assumptions. We will describe one such assumption right away.

A new assumption: transitivity. Let us start with the same interval $[0,1]$ with which we started the proof of Proposition 1. Similarly to this proof, let us denote the value $f(0,1)$ corresponding to this interval by α.

What can we conclude that from the fact that $f(0,1)=\alpha$? Well, due to shift invariance, we can conclude that for every x, we have $f(x,1+x)=\alpha+x$. From the mathematical viewpoint, this is all that we can conclude. However, from the common sense viewpoint, we can make yet another conclusion.

Indeed, e.g., for each x from the interval $[0,1]$, the alternative corresponding to the interval $[x,1+x]$ is equivalent to getting a monetary amount $\alpha+x$: $[x,1+x]\equiv\alpha+x$. If we do not know which of these intervals the alternative corresponds to – but we know that it corresponds to one of these alternatives, this means that the actual gain can take any value from the *union* of these intervals. Each of these intervals is equivalent to the value $\alpha+x$, thus, the union of these intervals is equivalent to the set of all possible values $\alpha+x$ when $x\in[0,1]$:

$$\bigcup_{x\in[0,1]}[x,1+x]\equiv\{\alpha+x:x\in[0,1]\}.$$

Let us estimate the left-hand side and the right-hand side of this equality.

- The smallest possible value in the left-hand side is when we take the smallest value from the interval $[x,1+x]$ – i.e., the value x – for the smallest possible value x from the interval $[0,1]$ (i.e., for the value $x=0$). Thus, the smallest possible value in the left-hand side is equal to 0.

- The largest possible value in the left-hand side is when we take the largest value from the interval $[x,1+x]$ – i.e., the value $1+x$ – for the largest possible value x from the interval $[0,1]$ (i.e., for the value $x=1$). Thus, the largest possible value in the left-hand side is equal to $1+1=2$.

So, the left-hand side of the above equality is the interval $[0,2]$.

Similarly:

- The smallest possible value in the right-hand side is when we take the smallest possible value x from the interval $[0,1]$, i.e., the value $x=0$. Thus, the smallest possible value in the right-hand side is equal to $\alpha+0=\alpha$.

- The largest possible value in the right-hand side is when we take the largest possible value x from the interval $[0,1]$, i.e., the value $x=1$. Thus, the smallest possible value in the right-hand side is equal to $\alpha+1$.

So, the left-hand side of the above equality is the interval $[\alpha,1+\alpha]$.

Thus, the above equivalent takes the form $[0,2]\equiv[\alpha,1+\alpha]$. Good news is that we already known – as a particular case of shift-invariance – that the interval $[\alpha,1+\alpha]$ is equivalent to the value $\alpha+\alpha=2\alpha$. Thus, by transitivity of equivalence, we conclude that the interval $[0,2]$ is equivalent to 2α, i.e., that $f(0,2)=2\alpha$. Then, by shift-invariance, we will get $f(x,2+x)=2\alpha+x$ for each x.

By similarly combining intervals $[x, 1+x]$ corresponding to $x \in [0,2]$, we conclude that $[0,3] \equiv [\alpha, 2+\alpha]$, and since we already know that $[\alpha, 2+\alpha] \equiv 2\alpha + \alpha$, by transitivity, we will have $f(0,3) = 3\alpha$.

Instead of stacking intervals of width 1, we could similarly stack intervals of a different width w.

New derivation of Hurwicz formula. It turns out that this way, we can indeed get a new derivation of Hurwicz formula. Let us describe all this in precise terms.

Definition 2.

- *We say that a value function $f(\underline{m}, \overline{m})$ is* shift-invariant *if for every m and for all $\underline{m} \le \overline{m}$, we have $m + f(\underline{m}, \overline{m}) = f(m + \underline{m}, m + \overline{m})$.*

- *We say that a value function is* transitive *if for each w and for all $\underline{m} \le \overline{m}$, we have $f\left(\underline{\ell}, \overline{\ell}\right) = f(\underline{r}, \overline{r})$, where*

$$\left[\underline{\ell}, \overline{\ell}\,\right] \stackrel{\text{def}}{=} \bigcup_{m \in [\underline{m}, \overline{m}]} [m, w+m]$$

and

$$[\underline{r}, \overline{r}] \stackrel{\text{def}}{=} \{f(m, m+w) : m \in [\underline{m}, \overline{m}]\}.$$

Comment. In this definition, we only described transitivity for the case when all combined intervals have the exact same width. Our main motivation for this restriction is that, as we will show, only such transitivity is needed – and in derivations, it is always desirable to avoid unnecessarily general assumptions and to limit ourselves only to weakest possible assumptions – weakest possible among those that will lead to the desired derivation.

There is another reason for this limitation: as we how later in this section, if we generalize this property too much, then there will be no realistic value function at all that would satisfy thus generalized property.

Proposition 2. *For a value function $f(\underline{m}, \overline{m})$, the following two conditions are equivalent to each other:*

- *the value function is monotonic, shift-invariant, and transitive;*

- *the value function has the Hurwicz form.*

Proof. Similarly to our above arguments, we can see that $\left[\underline{\ell}, \overline{\ell}\,\right] = [\underline{m}, \overline{m} + w]$, so $f\left(\underline{\ell}, \overline{\ell}\right) = f(\underline{m}, \overline{m} + w)$.

Due to the monotonicity of the value function, we have

$$[\underline{r}, \overline{r}] = [f(\underline{m}, \overline{m}), f(\underline{m} + w, \overline{m} + w)].$$

Due to shift-invariance, we have

$$[\underline{r}, \overline{r}] = [f(\underline{m}, \overline{m}), w + f(\underline{m} + w, \overline{m} + w)],$$

so, again due to shift-invariance – this time in relation to a shift by $f(\underline{m}, \overline{m})$ – we get $[\underline{r}, \overline{r}] = f(\underline{m}, \overline{m}) + [0, w]$.

Thus, again due to shift-invariance, $f(\underline{r}, \overline{r}) = f(\underline{m}, \overline{m}) + f(0, w)$. Therefore, transitivity means that

$$f(\underline{m}, \overline{m} + w) = f(\underline{m}, \overline{m}) + f(0, w).$$

One can easily see that the Hurwicz formula is shift-invariant and satisfies the above property for all w and for all $\underline{m} \le \overline{m}$.

Vice versa, let us assume that we have a value function that satisfies this property for all w and for all $\underline{m} \le \overline{m}$. In particular, for $\underline{m} = 0$, this means that $f(0, \overline{m} + w) = f(0, \overline{m}) + f(0, w)$. This is exactly the particular case of the additivity property that we used (as well as monotonicity) in the proof of Proposition 1 to prove that $f(0, r) = \alpha \cdot r$ for all real numbers r. From this formula, in that proof, we used, in effect, shift-invariance to prove that the Hurwicz formula is indeed true for all $\underline{m} \le \overline{m}$. Since we still assume shift-invariance, this means that we have a derivation of the Hurwicz formula in this case as well.

The proposition is proven.

Discussion: we cannot generalize the transitivity property too much. Let us show that the transitivity assumption cannot be realistically generalized too much, to cases when united intervals have different widths.

Definition 3. *We say that a value function is* fully transitive *if for each family of intervals* $\{[\underline{m}(a), \overline{m}(a)]\}_{a \in A}$ *for which both sets* $\bigcup\limits_{a \in A} [\underline{m}(a), \overline{m}(a)]$ *and* $\{f(\underline{m}(a), \overline{m}(a)) : a \in A\}$ *are intervals, we have* $f\left(\underline{\ell}, \overline{\ell}\right) = f(\underline{r}, \overline{r})$, *where we denoted*

$$\left[\underline{\ell}, \overline{\ell}\,\right] = \bigcup_{a \in A} [\underline{m}(a), \overline{m}(a)]$$

and

$$[\underline{r}, \overline{r}] = \{f(\underline{m}(a), \overline{m}(a)) : a \in A\}.$$

Proposition 3. *For a value function* $f(\underline{m}, \overline{m})$, *the following two conditions are equivalent to each other:*

- *the value function is monotonic, shift-invariant, and fully transitive;*
- *the value function has the Hurwicz form with* $\alpha = 0$ *or* $\alpha = 1$.

Discussion. So, full transitivity is satisfied only in the two extreme (and unrealistic) cases:

- when $\alpha = 0$ – the case of full pessimism, and
- when $\alpha = 1$ – the case of full optimism.

Proof. One can easily check that both extreme value functions $f(\underline{m}, \overline{m}) = \underline{m}$ (that corresponds to $\alpha = 0$) and $f(\underline{m}, \overline{m}) = \overline{m}$ (that corresponds to $\alpha = 1$) are fully transitive.

Let us prove that, vice versa, every monotonic shift-invariant and fully transitive value function coincides with one of the two extreme functions. Indeed, since the general condition should be satisfied for all possible families of intervals $[\underline{m}(a), \overline{m}(a)]$, in particular, it should be satisfied for all the families from Definition 2. Thus, due to Proposition 2, the value function should have the Hurwicz form.

Now, for the family $[0, a]$, where $a \in A = [0, 1]$, the union $\left[\underline{\ell}, \overline{\ell}\right]$ is simply equal to $[0, 1]$, so $f\left(\underline{\ell}, \overline{\ell}\right) = f(0, 1) = \alpha$.

On the other hand, here, $f(0, a) = \alpha \cdot a$, so

$$[\underline{r}, \overline{r}] = \{\alpha \cdot a : a \in [0, 1]\} = [0, \alpha],$$

thus $f(\underline{r}, \overline{r}) = \alpha \cdot \alpha = \alpha^2$. Thus, the generalized transitivity is satisfied only when $\alpha = \alpha^2$, i.e., when either $\alpha = 0$ or $\alpha = 1$.

The proposition is proven.

4 What Is Utility and What Are the Properties of Utility: A Brief Reminder

What is utility. To apply computer-based number-oriented tools for making decisions in a non-monetary case, we need to describe the user's preferences in numerical terms. In decision theory (see, e.g., [1, 3, 5, 6, 7]), this is done as follows.

Let us select the two extreme alternatives:

- a very bad alternative A_- which is worse than anything that we will actually encounter, and
- a very good alternative A_+ which is better than anything that we will actually encounter.

For each real number p from the interval $[0, 1]$, we can form a lottery – we will denote this lottery by $L(p)$ – in which:

- we get the very good alternative A_+ with probability p, and
- we get the very bad alternative A_- with the remaining probability $1 - p$.

To find how valuable is each alternative A for the decision maker, we ask him/her to compare the alternative A with lotteries $L(p)$ corresponding to different probabilities p. Here:

- when p is small, close to 0, the lottery $L(p)$ is similar to the very bad alternative A_- and is, thus, worse than A; we will denote this by $A_- < A$;

- when p is close to 1, the lottery $L(p)$ is similar to the very good alternative A_+ and is, thus, better than A: $A < L(p)$.

Also, the smaller the probability p of getting a very good alternative, the worse the lottery $L(p)$. Thus:

- if $L(p) < A$ and $p' < p$, then $L(p') < A$, and
- if $A < L(p)$ and $p < p'$, then $A < L(p')$.

Thus, similarly to the monetary case, there exists a threshold value

$$\sup\{p : L(p) < A\} = \inf\{p : A < L(p)\};$$

we will denote this threshold value by $u(A)$. This threshold value is known as the *utility* of the alternative A.

Similarly to the monetary case, for every $\varepsilon > 0$, we have $L(u(A) - \varepsilon) < A < L(u(A) + \varepsilon)$. This is true for arbitrarily small ε, in particular, for the values ε for which the difference in probabilities between $u(A) - \varepsilon$, $u(A)$, and $u(A) + \varepsilon$ is practically unnoticeable. So, we can conclude that from the practical viewpoint, the alternative A is equivalent to the lottery $L(u(A))$. We will denote this equivalence by $A \equiv L(u(A))$.

Utility is defined modulo a linear transformation. The numerical value of the utility $u(A)$ depends not only on the alternative A, it also depends on which pair (A_-, A_+) we select. What if we select a different pairs (A'_-, A'_+) – e.g., a pair for which $A_- < A'_- < A'_+ < A_+$? How will that change the numerical value of utility?

If an alternative A has utility $u'(A)$ with respect to the pair (A'_-, A'_+), this means that this alternative is equivalent to the lottery $L'(u'(A))$, in which:

- we get A'_+ with probability $u'(A)$, and
- we get A'_- with the remaining probability $1 - u'(A)$.

Since $A_- < A'_- < A_+$, we can find a utility value $u(A'_-)$ for which the alternative A'_- is equivalent to the lottery $L(u(A'_-))$, in which:

- we select A_+ with probability $u(A'_-)$, and
- we select A_- with probability $1 - u(A'_-)$.

Similarly, we have $A'_+ \equiv L(u(A'_+))$. Thus, the original alternative A is equivalent to a two-stage lottery, in which:

- first, we select either A'_+ (with probability $u'(A)$) or A'_- (with probability $1 - u'(A)$);
- then, we select either A_+ or A_- with probabilities depending on what we selected on the first stage: if we selected A'_+ on the first stage, then we select A_+ with probability $u(A'_+)$ and A_- with probability $1 - u(A'_+)$, and if we selected A'_- on the first stage, then we select A_+ with probability $u(A'_-)$ and A_- with probability $1 - u(A'_-)$.

As a result of this two-stage lottery, we get either A_+ or A_-, and the probability of selecting A_+ is equal to

$$u'(A) \cdot u(A'_+) + (1 - u'(A)) \cdot u(A'_-).$$

By definition, this probability is the utility $u(A)$ of the alternative A with respect to the pair (A_-, A_+), thus

$$u(A) = u'(A) \cdot u(A'_+) + (1 - u'(A)) \cdot u(A'_-).$$

The right-hand side is a linear expression in terms of $u'(A)$. So, we conclude that utilities corresponding to different pairs can be obtained from each other by a linear transformation.

In other words, the numerical value of the utility is defined modulo a generic linear transformation – just like the numerical value of time and temperature, where the corresponding linear transformations mean selecting a different starting point and/or a different measuring unit.

5 Utility Case: Usual Derivation of the Hurwicz Criterion and the Limitations of This Derivation

Formulation of the problem. As as we have mentioned earlier, in many practical situations, we do not know the exact consequence of each action, and thus, we do not know the exact value of the corresponding utility. Instead, for such situations, we only know the interval $[\underline{u}, \overline{u}]$ of possible utility values. According to the general idea of utility, to describe the decision maker's preferences for such interval-valued situations, we must assign, to each such interval, an appropriate utility value. Similarly to the monetary case, we will denote this utility value by $f(\underline{u}, \overline{u})$, and we will call the corresponding function a value function. Clearly, we must have $\underline{u} \le f(\underline{u}, \overline{u}) \le \overline{u}$, and clearly, if $\underline{u}$ and/or $\overline{u}$ increase, the interval-valued alternative becomes better – i.e., the value function should be monotonic.

What are other natural properties of the value function?

We cannot reuse assumptions from the monetary case. We cannot simply use the same properties as in the monetary case. For example, additivity makes no sense:

- it makes perfect sense to add dollar amounts, but
- it makes no sense to add probabilities (and utilities, as we have explained, *are* probabilities).

So, we need alternative assumptions.

Assumptions used in the usual derivation of the utility-case Hurwicz formula. Since utility is defined modulo a general linear transformation, it makes sense to require that the formulas transforming the bounds $\underline{u}$ and $\overline{u}$ into an equivalent utility should remain the same if we linearly "re-scale" all utility values. In particular:

- if we have $f(\underline{u},\overline{u}) = u$, then after shifting all the utility values by u_0 we should retain the same relation between the shifted utilities $\underline{u}' = \underline{u} + u_0$, $\overline{u}' = \overline{u} + u_0$, and $u' = u + u_0$: $f(\underline{u}',\overline{u}') = u'$;

- similarly, if we have $f(\underline{u},\overline{u}) = u$, then after re-scaling all the utility values by a factor $c > 0$, we should retain the same relation between the shifted utilities $\underline{u}' = c \cdot \underline{u}$, $\overline{u}' = c \cdot \overline{u}$, and $u' = c \cdot u$: $f(\underline{u}',\overline{u}') = u'$.

In the shift case, if we substitute the values $\underline{u}' = \underline{u} + u_0$, $\overline{u}' = \overline{u} + u_0$, and $u' = u + u_0 = f(\underline{u},\overline{u}) + u_0$ into the desired equality $f(\underline{u}',\overline{u}') = u'$, we get the requirement $f(\underline{u} + u_0, \overline{u} + u_0) = f(\underline{u},\overline{u}) + u_0$. One can see that this is exactly the property that we called shift-invariance.

In the re-scaling case, if we substitute the values $\underline{u}' = c \cdot \underline{u}$, $\overline{u}' = c \cdot \overline{u}$, and $u' = c \cdot u = c \cdot f(\underline{u},\overline{u})$ into the desired equality $f(\underline{u}',\overline{u}') = u'$, we get the requirement $f(c \cdot \underline{u}, c \cdot \overline{u}) = c \cdot f(\underline{u},\overline{u})$. We will call this property *scale-invariance.*

Definition 4. *We say that a value function $f(\underline{u},\overline{u})$ is* scale-invariant *if for every $c > 0$ and for all $\underline{u} \le \overline{u}$, we have $f(c \cdot \underline{u}, c \cdot \overline{u}) = c \cdot f(\underline{u},\overline{u})$.*

Proposition 4. *For a value function $f(\underline{u},\overline{u})$, the following two conditions are equivalent to each other:*

- *the value function is monotonic, shift-invariant, and scale-invariant;*

- *the value function has the Hurwicz form.*

Proof. It is easy to check that the Hurwicz formula is monotonic, shift-invariant, and scale-invariant. Let us show that, vice versa, every monotonic value function which is shift- and scale-invariant has the Hurwicz form.

Indeed, as in the proof of Proposition 1, let us denote $\alpha \stackrel{\text{def}}{=} f(0,1)$. For all $\underline{u} < \overline{u}$, due to shift-invariance with $u_0 = \underline{u}$, we have $f(\underline{u},\overline{u}) = \underline{u} + f(0, \overline{u} - \underline{u})$. Now, due to scale-invariance with $c = \overline{u} - \underline{u}$, we get $f(0, \overline{u} - \underline{u}) = (\overline{u} - \underline{u}) \cdot f(0,1) = (\overline{u} - \underline{u}) \cdot \alpha$.

Thus, $f(\underline{u},\overline{u}) = \underline{u} + f(0, \overline{u} - \underline{u}) = \underline{u} + (\overline{u} - \underline{u}) \cdot \alpha$, which is exactly the Hurwicz formula.

The proposition is proven.

Limitations. Shift-invariance is indeed reasonable, but scale-invariance is not fully convincing. Yes, indeed, we can have different units for measuring utility – just like we can use different units for measuring money, but it is not very convincing to expect that people will make the same choices involving 100 US dollars as in situations involving 100 Pesos (which at present, in 2020, represents a 20 times smaller amount of money).

It is therefore desirable to replace scale-invariance with a more convincing assumption.

6 Utility Case: New, Hopefully More Convincing, Derivation of the Hurwicz Criterion

We cannot simply dismiss scale-invariance. We cannot simply dismiss scale-invariance and keep only shift-invariance: we already considered this scenario when discussing the monetary case, and we showed that in this case, there are too many values functions satisfying these requirements.

So, we need additional assumptions – assumptions which are more convincing that scale-invariance.

What we propose. What we propose is the above-described transitivity property. The arguments in favor of this property apply verbatim to the utility case. And we already know – from Proposition 2 – that if we require shift-invariance and transitivity, then the only value functions we get are Hurwicz ones.

Thus, indeed, we get a new, (hopefully) more convincing derivation of the Hurwicz criterion in the utility case as well.

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