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Can We Preserve Physically Meaningful "Macro" Analyticity without Requiring Physically Meaningless "Micro" Analyticity?

Olga Kosheleva

Vladik Kreinovich

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Can We Preserve Physically Meaningful "Macro" Analyticity without Requiring Physically Meaningless "Micro" Analyticity?

Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso
500 W. University, El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

Abstract

Physicists working on quantum field theory actively used "macro" analyticity – e.g., that an integral of an analytical function over a large closed loop is 0 – but they agree that "micro" analyticity – the possibility to expand into Taylor series – is not physically meaningful on the micro level. Many physicists prefer physical theories with physically meaningful mathematical foundations. So, a natural question is: can we preserve physically meaningful "macro" analyticity without requiring physically meaningless "micro" analyticity? In the 1970s, an attempt to do it was made by using *constructive mathematics*, in which only objects generated by algorithms are allowed. This did not work out, but, as we show in this paper, the desired separation between "macro" and "micro" analyticity can be achieved if we limit ourselves to *feasible* algorithms.

1 "Macro" vs. "Micro" Analyticity: Formulation of the Problem

Smoothness in physics. On macro-level, we observe many non-smooth and even discontinuous phenomena:

- earthquakes,
- phase transitions, etc.

However, on the micro-level, all equations and all phenomena are smooth – and even analytical; see, e.g., [4, 10]. Some of these phenomena are very fast – so we perceive them as discontinuous.

Analyticity. For complex numbers, smoothness implies analyticity.

Analyticity has been successfully used in quantum field theory. For example, to compute the values of some integral expressions, it is convenient to use the fact that for an analytical function, a contour integral over a closed loop is 0:

$$\int_\gamma f(z)\,dz = 0,$$

or it is equal to an explicit expression in terms of the poles.

How this "macro" analyticity can help physics. By using a loop $[-N, N]\cup \gamma'$, we can replace a difficult-to-compute integral over real numbers $\int_{-N}^{N} f(x)\,dx$ with an often-easier-to-compute integral over the complex values $\int_{\gamma'} f(z)\,dz$. This idea – mostly pioneered by Nikolai Bogolyubov (see, e.g., [3] – led to many successful applications.

This "macro" analyticity has been confirmed by many experiments and makes perfect physical sense.

But what about "micro" analyticity? The problem is that in traditional mathematics, such "macro" analyticity is equivalent to "micro" one, that the corresponding dependencies can be expanded in Taylor series:

$$f(z) = a_0 + a_1 \cdot (z - z_0) + a_2 \cdot (z - z_0)^2 + \ldots + a_n \cdot (z - z_0)^n + \ldots$$

In the opinion of physicists, however, this "micro" analyticity does not make direct physical sense, since on the micro level, quantum uncertainty makes exact measurements impossible.

Can we preserve physically meaningful "macro" analyticity without requiring physically meaningless "micro" analyticity?

2 Khalfin's Idea: First Attempt

Maybe constructive mathematics can help? The equivalence between "macro" and "micro" analyticity holds in traditional mathematics, where, crudely speaking, we only care about the *existence* of different objects – but not about *algorithms* for computing these objects.

The algorithmic problems are important. So,to deal with these problems, researchers have come up with the idea of *constructive mathematics*, where we say that an object exists only if we have an algorithm for constructing this object; see, e.g., [1, 2, 11].

In constructive mathematics, some equivalence results of traditional mathematics hold – in the sense that equivalence is algorithmic – while other equivalence results do not hold. So, in early 1970s, Leonid Khalfin, a specialist in mathematical physics from St. Petersburg, Russia, suggested that maybe the use of constructive mathematics can help us preserve physically meaningful "macro" analyticity without requiring physically meaningless "micro" analyticity?

This did not help. By the early 1970s, specialist in constructive mathematics have thoroughly studied complex analysis; see, e.g., [1, 6, 8]. Actually, the 1972 talk of Vladimir Overkov (one of the constructive mathematics pioneers), the talk whose results later appeared in [8] – this talk inspired Khalfin's suggestion.

Unfortunately, these constructive mathematics results showed that in constructive mathematics, "macro" analyticity still implies the "micro" one; this was pointed out almost right away by Vladimir Lifschitz – another pioneer of constructive complex analysis, the authors of a paper [6]. He pointed out that each coefficient a_n of the Taylor series can be determined by the following formula:

$$a_n = \frac{1}{2\pi \cdot \mathrm{i}} \cdot \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \tag{1}$$

and in constructive mathematics, an integral of a computably continuous function is computable [1, 2, 11].

3 Problem Revisited

Main idea. The above derivation of "micro" analyticity from the "macro" one is based on the usual constructive mathematics. In this approach, existence of an object means, in effect, the existence of an algorithm producing more and more accurate approximations to this object – irrespective to how long this algorithm may take.

A more realistic idea is to only allow *feasible* (= polynomial-time) algorithms; see, e.g., [5, 9]. It turns out that in this case, Khalfin's dream *can* be materialized. Namely:

- wile there exists an algorithm computing, for each computable macro analytical function, all the terms in its Taylor series expansion,
- it turns out that the computation time of this algorithm seems to grow exponentially with the number n of the term – so such computations are probably not feasible.

Let us provide arguments in favor of this conclusion.

Explanation. We have a computable function $f(z)$. This means that we can, given z, compute $f(z)$.

For simplicity, we can also assume that we know the upper bound D on $|f'(z)|$: $|f'(z)| \le D$.

Computation of the n-th Taylor coefficient a_n is based on the formula (1). Here, the simplest possible loop γ around the point z_0 is a circle of some small radius $r < 1$. For this loop, $|z - z_0| = r$.

We want to compute a_n with a given accuracy $\varepsilon > 0$. This means that we need to compute the corresponding integral with accuracy $\varepsilon' = 2\pi \cdot \varepsilon$.

By definition, an integral is a limit of integral sums. So, in general, a natural way to compute an integral $\int g(z)\, dz$ is to consider the corresponding integral sum

$$\sum g(z_i) \cdot \Delta z, \text{ with } |z_{i+1} - z_i| = h \text{ for some small } h.$$

In this approximation, we approximate $g(z)$ with $g(z_i)$ on each arc of length h for which $|z - z_i| \le h/2$.

The inaccuracy of this approximation is

$$|g(z) - g(z_i)| \le \left(\max_z |g'(z)| \cdot |z - z_i|\right) \le \max_z |g'(z)| \cdot (h/2).$$

Here, $g(z) = \dfrac{f(z)}{(z - z_0)^{n+1}} \approx \dfrac{f(z)}{r^{n+1}}$. Thus, $\max_z |g'(z)| \le \dfrac{\max |f'(z)|}{r^{n+1}} = \dfrac{D}{r^{n+1}}$.

So, the approximation accuracy is $\dfrac{D}{r^{n+1}} \cdot (h/2)$. To get accuracy ε', we need to take h for which

$$\frac{D}{r^{n+1}} \cdot (h/2) = \varepsilon', \text{ i.e., } h = 2\frac{\varepsilon'}{D} \cdot r^{n+1}.$$

The whole loop γ of length $2\pi \cdot r$ should be covered by intervals of length h. These intervals correspond to values z_i at which we compute $f(z)$. Thus, we need to compute $f(z)$ for $N = \dfrac{2\pi \cdot r}{h}$ points.

Substituting the above expression for h, we conclude that we need to compute $f(z)$ at

$$N = \frac{2\pi \cdot r \cdot D}{2\varepsilon' \cdot r^{n+1}} \sim r^{-n} \text{ points.}$$

Since $r < 1$, this number indeed grows exponentially with n. This is exactly what we wanted to show.

4 Possible Applications

This result will probably be of interest to theoreticians (like Khalfin) – who are interested in providing physical theories with physically meaningful mathematical foundations.

This result may also have practical applications if we take into account that many times when we encountered a physical process whose properties are difficult to compute, it became possible to use this process to speed up computations. Successes of quantum computing are the latest example of this phenomenon; see, e.g., [7].

From this viewpoint, maybe measurement of the corresponding Taylor coefficients can lead to yet another efficient quantum computing scheme?

Acknowledgments

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