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Abstract

At present, we mostly use decimal (base-10) number system, but in the past, many other systems were used: base-20, base-60 – which is still reflected in how we divide an hour into minutes and a minute into seconds – and many others. There is a known explanation for the base-60 system: 60 is the smallest number that can be divided by 2, by 3, by 4, by 5, and by 6. Because of this, e.g., half an hour, one-third of an hour, all the way to one-sixth of an hour all correspond to a whole number of minutes. In this paper, we show that a similar idea can explain all historical number systems, if, instead of requiring that the base divides *all* numbers from 2 to some value, we require that the base divides all but one (or all but two) of such numbers.

1 Formulation of the Problem

A problem. Nowadays, everyone use a decimal (base-10) system for representing integers – of course, with the exception of computers which use a binary (base-2) system. However, in the past, many cultures used different number systems; see, e.g., [1, 3, 4, 5, 9, 11, 12, 13, 15, 16, 17, 18]. Some of these systems have been in use until reasonably recently (and are still somewhat used in colloquial speech, e.g., when we count in dozens). Other systems are only known directly from historical records or from indirect sources – such as linguistics.

An interesting question is: why some numb systems (i.e., some bases) were used and some similar bases were not used?

Case when an explanation is known. One of the known reasons for selecting a base comes from base-60 system $\mathbf{B = 60}$ used by the ancient Babylonians; see, e.g., [2, 6, 8]. We still have a trace of that system – which was widely used throughout the ancient world – in our division of the hour into 60 minutes and a minute into 60 seconds.

A natural explanation for the use of this system is that it makes it easy to divide by small numbers: namely, when we divide 60 by 2, 3, 4, 5, and 6, we still get an integer. Thus, if we divide the hours into 60 minutes as we do, 1/2, 1/3, 1/4, 1/5, and 1/6 of the hour are all represented by a whole number of minutes – which makes it much easier for people to handle. And one can easily show that 60 is the smallest integer which is divisible by 2, 3, 4, 5, and 6.

Our idea. Let us use this explanation for the base-60 system as a sample, and see what we can get if make a similar assumption of divisibility, but for fewer numbers, or with all numbers but one or but two.

It turns out that many historically used number systems can indeed be explained this way.

2 Which Bases Appear If We Consider Divisibility by All Small Numbers from 1 to Some k

Let us consider which bases appear if we consider divisibility by all small natural numbers – i.e., by all natural numbers from 1 to some small number k. We will consider this for all values k from 1 to 7, and we will explain why we do not go further.

Case when $k = 2$. In this case, the smallest number divisible by 2 is the number 2 itself, so we get the binary (base-2) system $\mathbf{B} = \mathbf{2}$ used by computers.

Some cultures used powers of 2 as the base – e.g., $\mathbf{B} = \mathbf{4}$ or $\mathbf{B} = \mathbf{8}$ (see, e.g., [1]). This, in effect, is the same as using the original binary system – since, e.g., the fact that we have a special word for a hundred $100 = 10^2$ does not mean that we use a base-100 system.

Case when $k = 3$. The smallest number divisible by 2 and 3 is $\mathbf{B} = \mathbf{6}$. The base-6 number system has indeed been used, by the Morehead-Maro language of Southern New Guinea; see, e.g., [10, 14].

Case when $k = 4$. The smallest number divisible by 2, 3, and 4 is $\mathbf{B} = \mathbf{12}$. The base-12 number system has been used in many cultures; see, e.g., [2, 11, 16], and the use of dozens in many languages is an indication of this system's ubiquity.

Case when $k = 5$. The smallest number divisible by 2, 3, 4, and 5 is $B = 60$, the familiar Babylonian base. Since this number is also divisible by 6, the case $k = 6$ leads to the exact same base and thus, does not need to be considered separately.

Case when $k = 7$. The smallest number which is divisible by 2, 3, 4, 5, 6, and 7 is $B = 420$. This number looks too big to serve as the base of a number system, so we will not consider it. The same applied to larger values $k > 7$.

Thus, in this paper, we only consider values $k \leq 6$.

3 What If We Can Skip One Number

What happens if we consider bases which are divisible not by all, but by all-but-one numbers from 1 to k?

Of course, if we skip the number k itself, this is simply equivalent to being divisible by all the small numbers from 1 to $k-1$ – and we have already analyzed all such cases. So, it makes sense to skip a number which is smaller than k.

Let us analyze all the previous cases $k = 1, \ldots, 6$ from this viewpoint.

Case when $k = 2$. In this case, there is nothing to skip, so we still get a binary system.

Case when $k = 3$. In this case, the only number that we can skip is the number 2. The smallest integer divisible by 3 is the number 3 itself, so we get the ternary (base-3) system $\mathbf{B = 3}$; see, e.g., [5].

There is some evidence that people also used powers of 3, such as 9; see, e.g., [7, 15]

Case when $k = 4$. For $k = 4$, in principle, we could skip 2 or we could skip 3. Skipping 2 makes no sense, since if the base is divisible by 4, it is of course also divisible by 2 as well. Thus, the only number that we can meaningfully skip is the number 3. In this case, the smallest number which is divisible by the remaining numbers 2 and 4 is the number 4. As we have mentioned, the base-4 system is, in effect, the same as binary system – one digit of the base-4 system contains two binary digits, just like to more familiar base-8 and base-16 system, one digit corresponds to 3 or 4 binary digits.

Case when $k = 5$. In this case, we can skip 2, 3, or 4.

- Skipping 2 does not make sense, since then 4 remains, and, as we have mentioned earlier, if the base is divisible by 4, it is divisible by 2 as well.
- Skipping 3 leads to $\mathbf{B = 20}$, the smallest number divisible by 2, 4, and 5. Base-20 numbers have indeed been actively used, e.g., by the Mayan civilization; see, e.g., [2, 3, 4, 6, 8]. In Romance languages still 20 is described in a different way than 30, 40, and other similar numbers.
- Skipping 4 leads to $\mathbf{B = 30}$, the smallest number divisible by 2, 3, and 5. This seems to be the only case when the corresponding number system was not used by anyone.

Case when $k = 6$. In this case, in principle, we can skip 2, 3, 4, and 5. Skipping 2 or 3 does not make sense, since any number divisible by 6 is also divisible by 2 and 3. So, we get meaningful examples, we only consider skipping 4 or 5.

- If we skip 4, we get the same un-used base $B = 30$ that we have obtained for $k = 5$.
- If we skip 5, then the smallest number divisible by 2, 3, 4, and 6 is the base $B = 12$ which we already discussed earlier.

4 What If We Can Skip Two Numbers

What happens if we consider bases which are divisible by all-but-two numbers from 1 to k? Of course, to describe new bases, we need to only consider skipped numbers which are smaller than k.

Cases when $k = 2$ or $k = 3$. In these cases, we do not have two intermediate numbers to skip.

Case when $k = 4$. In this case, we skip both intermediate numbers 2 and 3 and consider only divisibility by 4. The smallest number divisible by 4 is the number 4 itself, and we have already considered base-4 numbers.

Case when $k = 5$. In this case, we have three intermediate numbers: 2, 3, and 4. In principle, we can form three pairs of skipped numbers: $(2, 3)$, $(2, 4)$, and $(3, 4)$. Skipping the first pair makes no sense, since then 4 still remains, and if the base is divisible by 4, then it is automatically divisible by 2 as well. Thus, we have only two remaining options:

- We can skip 2 and 4. In this case, the smallest number divisible by the two remaining numbers 3 and 5 is $\mathbf{B = 15}$. Historically, there is no direct evidence of base-15 systems, but there is an indirect evidence: e.g., Russia used to have 15-kopeck coins, a very unusual nomination.

- We can skip 3 and 4. In this case, the smallest number divisible by the two remaining numbers 2 and 5 is $\mathbf{B = 10}$. This is our usual decimal system.

Case when $k = 6$. In this case, we have four intermediate values 2, 3, 4, and 5. Skipping 2 or 3 makes no sense: if the base is divisible by 6, it is automatically divisible by 2 and 3. Thus, the only pair of values that we can skip is 4 and 5. In this case, the smallest number divisible by 2, 3, and 6 is the value $B = 6$, which we have already considered earlier.

5 What If We Can Skip Three or More Numbers

What if we skip three numbers? What happens if we consider bases which are divisible but by all-but-three numbers from 1 to k? Of course, to describe new bases, we need to only consider skipped numbers which are smaller than k.

Cases when $k = 2$, $k = 3$, or $k = 4$. In these cases, we do not have three intermediate numbers to skip.

Case when $k = 5$. In this case, skipping all three intermediate numbers 2, 3, and 4 leave us with $\mathbf{B = 5}$. The base-5 system has actually been used; see, e.g., [3].

Case when $k = 6$. In this case, we have four intermediate numbers, so skipping 3 of them means that we keep only one. It does not add to the list of bases if we keep 2 or 3, since then the smallest number divisible by 6 and by one of them is still 6 – and we have already considered base-6 systems. Thus, the only options are keeping 4 and keeping 5.

If we keep 4, then the smallest number divisible by 4 and 6 is $B = 12$ – our usual counting with dozens, which we have already considered.

If we keep 5, then the smallest number divisible by 5 and 6 is $B = 30$, which we have also already considered.

What if we skip more than three intermediate numbers. The only numbers $k \leq 6$ that have more than three intermediate numbers are $k = 5$ and $k = 6$.

For $k = 5$, skipping more than three intermediate numbers means skipping all fours of them, so the resulting base is $B = 5$, which we already considered.

For $k = 6$, for which there are five intermediate numbers, skipping more than three means either skipping all of them – in which case we have $B = 6$ – or keeping one of the intermediate numbers. Keeping 2 or 3 still leaves us with $B = 6$, keeping 4 leads to $B = 12$, and keeping the number 5 leads to $B = 30$. All these bases have already been considered.

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References

[1] M. Ascher, *Ethnomathematics: A Multicultural View of Mathematical Ideas*, Routledge, Milton Park, Abingdon, Oxfordshire, UK, 1994.

[2] C. B. Boyer and U. C. Merzbach, *A History of Mathematics*, Wiley, New York, 1991.

[3] T. L. Heath, *A Manual of Greek Mathematics*, Dover, New York, 2003.

[4] G. Ifrah, *The Universal History of Numbers: From Prehistory to the Invention of the Computer*, John Wiley & Sons, Hoboken, New Jersey, 2000.

[5] D. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, 3rd edition, Addison Wesley, Boston, Massachusetts, 1998.

[6] O. Kosheleva, "Mayan and Babylonian arithmetics can be explained by the need to minimize computations", *Applied Mathematical Sciences*, 2012, Vol. 6, No. 15, pp. 697–705.

[7] O. Kosheleva and V. Kreinovich, "Was there a pre-biblical 9-ary number system?", *Mathematical Structures and Modeling*, 2019, Vol. 50, pp. 87–90.

[8] O. Kosheleva and K. Villaverde, *How Interval and Fuzzy Techniques Can Improve Teaching*, Springer Verlag, Cham, Switzerland, 2018.

[9] A. L. Kroeber, *Handbook of the Indians of California*, Bulletin 78, Bureau of American Ethnology of the Smithsonian Institution, Washington, DC, 1919.

[10] G. Lean, *Counting Systems of Papua New Guinea*, Vols. 1–17, Papua New Guinea University of Technology, Lae, Papua New Guinea, 1988–1992.

[11] A. R. Luria and L. S. Vygotsky, *Ape, Primitive Man, and Child: Essays in the History of Behavior*, CRC Press, Boca Raton, Florida, 1992.

[12] J. P. Mallory and D. Q. Adams, *Encyclopedia of Indo-European Culture*, Fitzroy Dearborn Publsishers, Chicago and London, 1997.

[13] H. J. Nissen, P. Damerow, and R. K. Englund, *Archaic Bookkeeping, Early Writing, and Techniques of Economic Administration in the Ancient Near East*, University of Chicago Press, Chicago, Illinois, 1993.

[14] K. Owens, "The work of Glendon Lean on the counting systems of Papua New Guinea and Oceania", *Mathematics Education Research Journal*, 2001, Vol. 13, No. 1, pp. 47–71.

[15] M. Parkvall, *Limits of Language: Almost Everything You Didn't Know about Language and Languages*, William James & Company, Portland, Oregon, 2008.

[16] P. Ryan, *Encyclopaedia of Papua and New Guinea*, Melbourne University Press and University of Papua and New Guinea, Melbourne, 1972.

[17] D. Schmandt-Besserat, *How Writing Came About*, University of Texas Press, Austin, Texas, 1996.

[18] C. Zaslavsky, *Africa Counts: Number and Pattern in African Cultures*, Chicago Review Press, Chicago, Illinois, 1999.