2014-01-01

Homeomorphisms Between Pairs of Genus Two Handlebodies and Separating Circles in Their Boundaries

Daniel Orlando Vazquez
University of Texas at El Paso, dovazquez@miners.utep.edu

HOMEOMORPHISMS BETWEEN PAIRS OF GENUS TWO HANDLEBODIES AND SEPARATING CIRCLES IN THEIR BOUNDARIES

DANIEL VAZQUEZ

Department of Mathematics

APPROVED:

Luis Valdez-Sanchez, Chair, Ph.D.

Leslie Foged, Ph.D.

Ricardo Bernal, Ph.D.

Bess Sirmon-Taylor, Ph.D.
Interim Dean of the Graduate School

HOMEOMORPHISMS BETWEEN PAIRS OF GENUS TWO HANDLEBODIES AND SEPARATING CIRCLES IN THEIR BOUNDARIES

by

DANIEL VAZQUEZ

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Mathematics

THE UNIVERSITY OF TEXAS AT EL PASO

August 2014

Abstract

A known result in low-dimensional topology is that every incompressible surface properly embedded in a genus two handlebody is boundary compressible. This result produces a pair consisting of a genus two handlebody and a separating curve in its boundary. We study particular examples of genus two handlebodies and essential simple closed curves which separate their boundaries. The aim of this paper is to give our own construction of separating curves on the boundary of genus two handlebodies and compare them with previously studied pairs. These constructions involve taking $2n$ arcs on either side of a waist disk of a genus two handlebody and defining a mapping which takes the endpoints of one side and connects them to the other side. Before studying homeomorphisms with known pairs, we must first state and prove restrictions on the mapping of the arcs in our construction. Only particular cases of mappings will result in a single circle component and even fewer cases will result in homeomorphisms with previously studied pairs.

Table of Contents

Chapter 1

Introduction

A 3-*manifold* is any compact, Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^3$. A *handlebody* H *of genus* $g \geq 1$ is a 3-manifold homeomorphic to the regular neighborhood of a graph in $\mathbb{R}^3$ consisting of 1 vertex and g edges (also called a wedge of g circles). We may also view the genus of a handlebody as the number of 1-handles on the given surface. A solid torus is an example of a handlebody; it has genus one.

In this paper we study genus two handlebodies with certain separating circles in their boundaries. In particular we look at one pair which arises from a known result in low-dimensional topology which states that an incompressible surface which is properly embedded in a handlebody will automatically boundary compress. The proof of this statement as well as the construction of the handlebody-circle pair is given in Chapter 2. This pair is also seen in the paper of Yukihiro Tsutsumi [5] as an aid to prove that any essential simple closed curve in the boundary of a genus two handlebody will bound at most four mutually disjoint, non-parallel, genus one incompressible surfaces, or once-punctured tori, properly embedded in the genus two handlebody. This theorem, which Tsutsumi proves, gives rise to the question of which such separating circles will yield three once-punctured tori and which curves will yield four. In Chapter 3 we give an algorithm for the construction of a different handlebody-circle pair along with its properties which we then compare to the pair from Chapter 2. This process involves taking $2n$ arcs on either side of a waist disk of a genus two handlebody and defining a mapping which takes the endpoints of one side and connects them to the other side. We study specific cases in which our construction from Chapter 3 is homeomorphic to the pair from Chapter 2. In Chapter 4 we give a general classification to all such constructions. In doing so we may pose certain questions which

are easier understood in one pair rather than its homeomorphic representation. Chapter 5 ties our construction to the theorem in [5] and maps out future work which may prove exactly which handlebody-circle pairs bound exactly four mutually disjoint, non-parallel, once-punctured tori.

Chapter 2

Preliminaries

2.1 Fundamental Group

Given a topological space X, we define the fundamental group of X in terms of loops. For two points $a, b \in X$, we say a *path* in X from a to b is a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$, where we refer to a and b as *endpoints*.

We say two paths f_1, f_2 are *homotopic* if they share the same endpoints and there is a homotopy between them which fixes endpoints. That is, $f_1(0) = f_2(0) = a$ and $f_1(1) = f_2(1) = b$. The equivalence class of a path f is said to be the *homotopy class of f* and is denoted as $[f]$.

Now, a *loop* in a topological space X is a path f such that $f(0) = x_0 = f(1)$ for some $x_0 \in X$. This starting and ending point x_0 is called the *basepoint*.

Given two paths $f, g : [0, 1] \to x$ such that $f(1) = g(0)$, we define the *product path $f \cdot g$* by the formula

$$(f \cdot g)(x) = \begin{cases} f(2x) & 0 \leq x \leq \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

The *fundamental group* of X with basepoint $x_0 \in X$ is the set of equivalence classes of closed paths with endpoints at x_0 under the above product operation. We denote it as $\pi_1(X, x_0)$. To see that this operation forms a group, see [4]. It is also a known result in [4] that, up to isomorphism, the fundamental group of X is independent of the basepoint chosen whenever X is connected, in which case we may refer to the fundamental group of X as $\pi_1(X)$. These groups interest us since they give algebraic conditions for two spaces to be homeomorphic.

2.2 Hyperbolic Knots

In the late 20^{th} century William Thurston was able to classify all knots in S^3 into either torus knots, satellite knots, or hyperbolic knots. Torus knots are well understood and are those whose fundamental groups have nontrivial center. Satellite knots are those which have an incompressible non-boundary parallel torus in their exterior; this torus may actually be used to break the exterior of the knot into smaller pieces. The last type of knot is one which is neither a torus knot nor a satellite knot; these knots are called hyperbolic knots, and can also be defined as a knot K whose complement $S^3 \setminus K$ is a non-compact hyperbolic 3-manifold of finite volume. It is known that every compact orientable 3-manifold contains a knot such that its complement is hyperbolic [1]. These hyperbolic structures give rise to hyperbolic metrics and certain geometric invariants which can be used to distinguish between manifolds.

2.3 Surfaces in 3-Manifolds

We say a compact surface S which is embedded in a 3-manifold M is *properly embedded* in M if $\partial S \subseteq \partial M$ and S is transverse to ∂M. Unless otherwise stated, we will assume all manifolds to be connected and orientable and all surfaces to be properly embedded.

In a surface S, we say a circle is *trivial* if it bounds a disk in S. Similarly, an arc is said to be *trivial* in S if it cobounds a disk in S with another arc in ∂S.

Let S be a surface in a 3-manifold M which is not a 2-sphere nor a disk. S is *compressible* in M if there is a disk D in M such that $D \cap S = \partial D$ and ∂D does not bound a disk in S; we say D is a *compressing disk* of S. If no such D exists, S is said to be *incompressible.*

The surface S is *boundary compressible* in M if there is a disk D in M where $\partial D = \alpha \cup \beta$ with α and β being connected arcs, $\alpha \cap \beta = \partial\alpha \cap \partial\beta$, $D \cap S = \alpha$, $D \cap \partial M = \beta$, α is a nontrivial arc in S, and β is not parallel to ∂S in ∂M. If S is boundary compressible, we say D is a *boundary compressing disk* of S. S is *boundary incompressible* if no such D

exists.

The following theorems and corollaries are well known results in three-dimensional topology and can be found in [4].

Theorem 2.3.1 (Dehn's Lemma)**.** *Suppose M^3 is a 3-manifold and $f : D^2 \to M^3$ is a map of a disk with no singularities on ∂D^2 (i.e. $x \in \partial D^2$, $x \neq y \in D^2 \implies f(x) \neq f(y)$). Then there exists an embedding $g : D^2 \to M^3$ with $g(\partial D) = f(\partial D)$.*

Corollary 2.3.2. *Suppose $J \subset \partial M$ is a simple closed curve in the boundary of a 3-manifold M and that J is homotopically trivial in M. Then J bounds a properly embedded disk D in M (i.e. $\partial D \subset \partial M$, and the interior of D is properly contained in the interior of M).*

Theorem 2.3.3 (Loop Theorem)**.** *If M is a 3-manifold and the inclusion homomorphism $\pi_1(\partial M) \to \pi_1(M)$ has nontrivial kernel, then there exists a 2-disk $D \subset M$ such that ∂D lies in ∂M and represents a nontrivial element of $\pi_1(\partial M)$.*

We say a 3-manifold M is *irreducible* if every 2-sphere in M bounds a 3-ball.

Now let S be an arbitrary surface in the 3-manifold M, and let $N(S)$ be a regular neighborhood of S in M. We say that the manifold $\overline{M \setminus N(S)}$ is the result of cutting M along the surface S; observe that the original manifold M can be recovered as $\overline{M \setminus N(S)} \cup N(S)$, where the union is taken along the *frontier* $\overline{\partial N(S) \setminus \partial M}$ of $N(S)$ in M.

Given a 3-manifold M and a disk $D \subset M$, we define a *full Dehn twist* as the self-homeomorphism taken by cutting M along a regular neighborhood of D, giving one side of $M \setminus N(D)$ a full twist in some specified direction, and reattaching both sides of $M \setminus N(D)$ along the same neighborhood of D.

2.4 Handlebodies

A *complete disk system* $\mathcal{D}$ of a handlebody H of genus $g \geq 1$ is a set of disjoint compression disks of ∂H in H such that the manifold obtained by cutting H along $\cup\mathcal{D}$ is a 3-ball. It

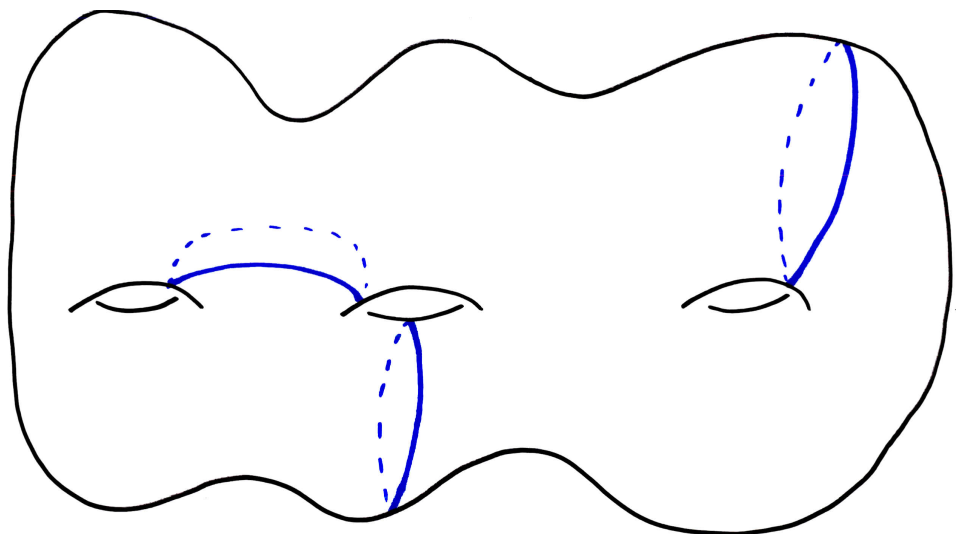

Figure 2.1: A genus three handlebody with one of its complete disk systems.

follows that each disk in $\mathcal{D}$ is nonseparating, and hence that the collection $\mathcal{D}$ contains g disks. Fig. 2.1 shows a genus three handlebody with a complete disk system.

Being a submanifold of $\mathbb{R}^3$, a handlebody inherits several properties of $\mathbb{R}^3$. For instance, in a handlebody, any 2-sphere bounds a 3-ball, and any closed surface separates in the handlebody. Surfaces with boundary in a handlebody can be incompressible but must be boundary compressible, as shown in the next result.

Theorem 2.4.1. *Any incompressible surface in a handlebody is boundary compressible.*

Proof. Let $S \neq S^2, D^2$ be an incompressible surface in a handlebody H. Consider a complete disk system $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ of H and let $\mathcal{D}^* = D_1 \cup D_2 \cup \cdots \cup D_n$. Isotope $\mathcal{D}^*$ so that it is transverse to S; then the intersection $|\mathcal{D}^* \cap S|$ consists of arcs and circles properly embedded in S, and we use $|\mathcal{D}^* \cap S|$ to denote the number of components of $\mathcal{D}^* \cap S$. Among all complete disk systems $\mathcal{D}$, select one for which $|\mathcal{D}^* \cap S|$ is minimal.

Suppose $|\mathcal{D}^* \cap S| = 0$. Then S is disjoint from $\mathcal{D}^*$ and hence S lies in the 3-ball obtained by cutting H along $\mathcal{D}^*$. Since any surface properly embedded in a 3-ball is compressible, the surface S must compress in H, contradicting our hypothesis. Therefore $|\mathcal{D}^* \cap S| > 0$.

Suppose $\mathcal{D}^* \cap S$ has circle components, and let C be any one of them; thus C is a subset of some disk D_r of $\mathcal{D}$. If C is nontrivial in S then C bounds a subdisk of D_r and so the map

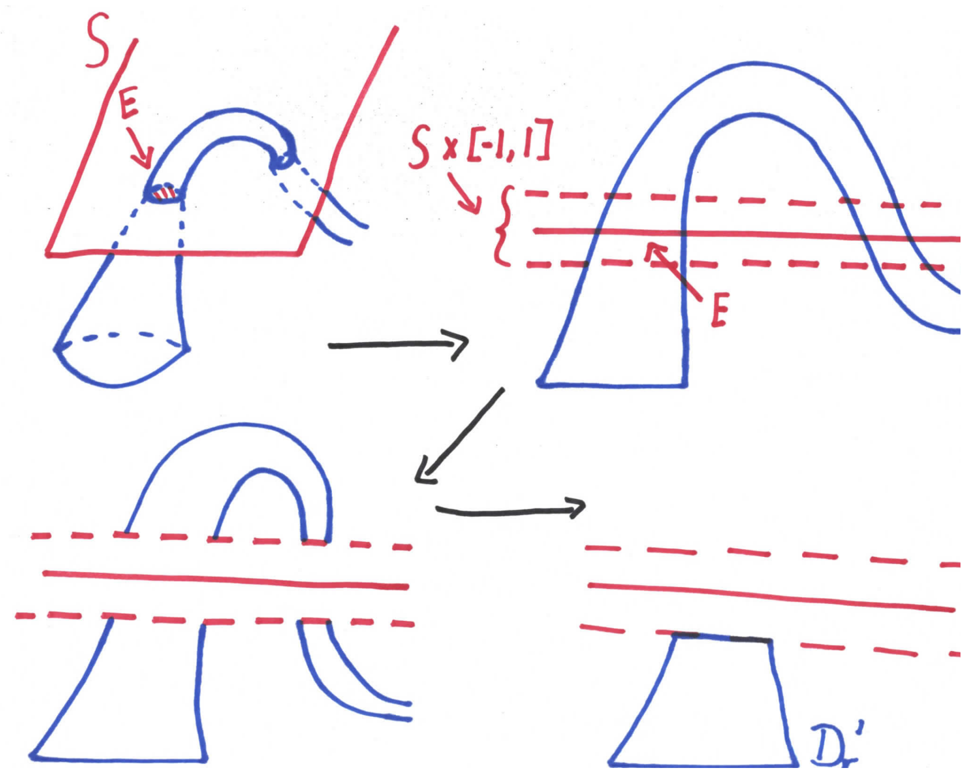

Figure 2.2: The construction of D_r' from the disk D_r.

$\pi_1(S) \to \pi_1(M)$ is not injective. But then, by Dehn's Lemma, the surface S compresses in M, contradicting the hypothesis. Therefore any circle component of $\mathcal{D}^* \cap S$ bounds a disk in S, so we may assume C is *innermost* in S, that is, C bounds a disk E in S whose interior is disjoint from $\mathcal{D}^*$, as shown in Fig. 2.2.

On D_r, C also bounds a subdisk F such that $\partial F = C = \partial E$; we construct a new disk $D_r' = (D_r \backslash F) \cup E$, which we push away from S via a regular neighborhood of S. Since $\partial D_r' = \partial D_r$, a new complete disk system $\mathcal{E}$ for H is produced by replacing D_r with D_r' in $\mathcal{D}$ such that $|\mathcal{E}^* \cap S| < |\mathcal{D}^* \cap S|$, again contradicting our hypothesis on $\mathcal{D}$. Therefore $\mathcal{D}^* \cap S$ has no circle components.

We now claim that every arc of $D_r \cap S$ is nontrivial in S.

For suppose $\mathcal{D}^* \cap S$ contains an arc γ which is trivial in S, say $\gamma \subset D_r \cap S$ for some $D_r \in \mathcal{D}$. Without loss of generality we may assume γ is *outermost* in S, that is, γ cobounds

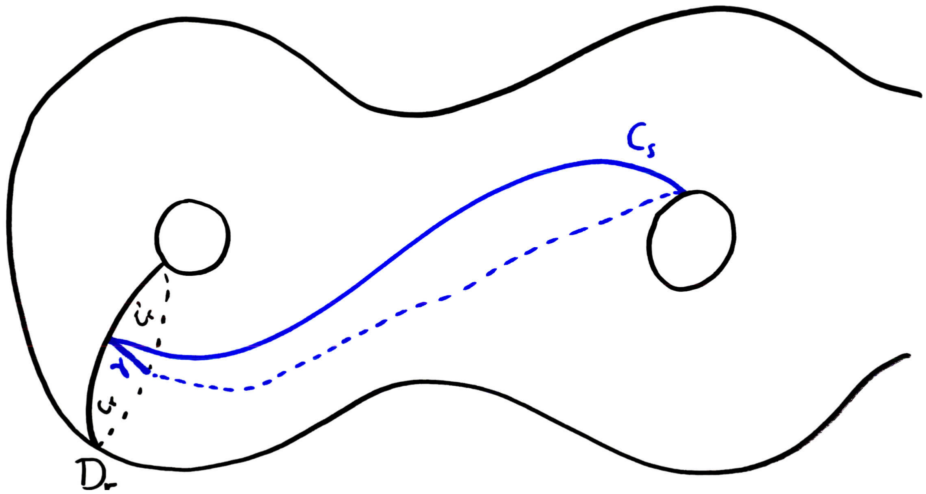

Figure 2.3: The outermost arc γ in $S \cap \mathcal{D}^* \subset S$.

a subdisk $C_s \subseteq S$ with ∂S such that and $C_s \cap \mathcal{D}^* = \gamma$, as show in Fig. 2.3.

Let C_r, C_r' be the closures of the components of $D_r \setminus \gamma$, and consider the disks $E = C_r \cup_\gamma C_s$ and $E' = C_r' \cup_\gamma C_s$ which are properly embedded in H, as Fig. 2.4 illustrates. Our goal is to replace D_r with one of the disks E or E' and obtain a new complete disk system of H with fewer intersections with S. Observe that, since $\overline{\partial H \setminus \mathcal{D}^*}$ is a connected surface (a punctured 2-sphere), it is possible to construct a circle $\alpha \subset \partial H$ that intersects D_r transversely at one point distinct from the endpoints of γ, is disjoint from all other disks in $\mathcal{D}$, and intersects the arc $\partial C_s \cap \partial H$ transversely, as shown in Fig. 2.5. It follows that α intersects E and E' transversely and that the relation $|E \cap \alpha| + |E' \cap \alpha| = 2|C_s \cap \alpha| + 1$ holds, which implies that either E or E', say E, is a nonseparating disk in H.

Pushing E off away from S via a thin regular neighborhood of S produces a new disk, which we continue to call E, such that E is a nonseparating compression disk for the boundary of H and $(\mathcal{D} \setminus \{D_r\}) \cup \{E\}$ will be a new complete disk system for H. This complete disk system has fewer intersections with S since it is no longer intersecting γ, a contradiction with our choice of $\mathcal{D}$.

Therefore every arc of $D_r \cap S$ in S is essential in S. In general every arc of $\mathcal{D}^* \cap S$ in

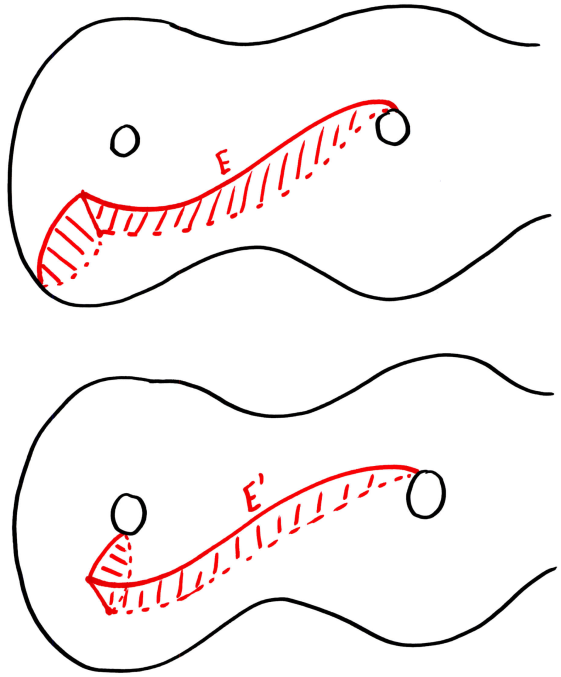

Figure 2.4: The disks E and E'.

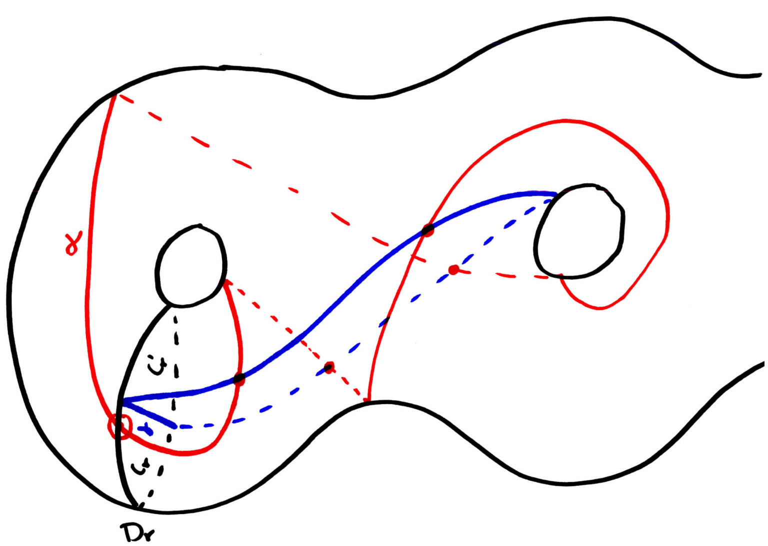

Figure 2.5: The circle α in ∂H.

S is nontrivial in S. As a result, S is boundary compressible in H. □

2.5 Circles in Handlebodies

Let H be a handlebody and α a circle in ∂H. We say that α is *primitive* if it represents a primitive word in $\pi_1(H)$ and that it is a *power* if it represents a nontrivial power of a nontrivial word in $\pi_1(H)$. The following result gives information on the structure of primitive and power circles in ∂H.

Lemma 2.5.1. *Let H be a handlebody and α a circle in ∂H.*

(a) If α is a power circle then, in $\pi_1(H)$, $\alpha = w^k$ where $|k| \geq 2$ and w is primitive in $\pi_1(H)$.

(b) If α is a primitive circle then, in $\pi_1(H) = \langle x, y \mid - \rangle$, the cyclic reduction of α is $x^{\pm 1}$, $y^{\pm 1}$, or of the form $x^{a_1}y^{b_1}x^{a_2}y^{b_2}\cdots x^{a_k}y^{b_k}$ where $k \geq 1$, the exponents $a_1, \ldots, a_k$ all have the same sign, and the exponents $b_1, \ldots, b_k$ all have the same sign; moreover we may assume all the exponents a_i are 1 or all are -1 while each exponent b_i is equal to n or $n+1$ for some integer $n \neq -1, 0$.

Proof. Part (a) follows from the work of Casson and Gordon [2], while (b) follows from the work of Gonzalez-Acuña and Ramirez [3]. □

2.6 Pairs

A *pair* (H, J) consists of a genus two handlebody and a circle $J \subset \partial H$ which separates ∂H into two once-punctured tori and is nontrivial in H. If one of these once-punctured tori is compressible then its compression would result in a disk whose boundary is J, contradicting the hypothesis that J is nontrivial in H; therefore these once-punctured tori are incompressible in H.

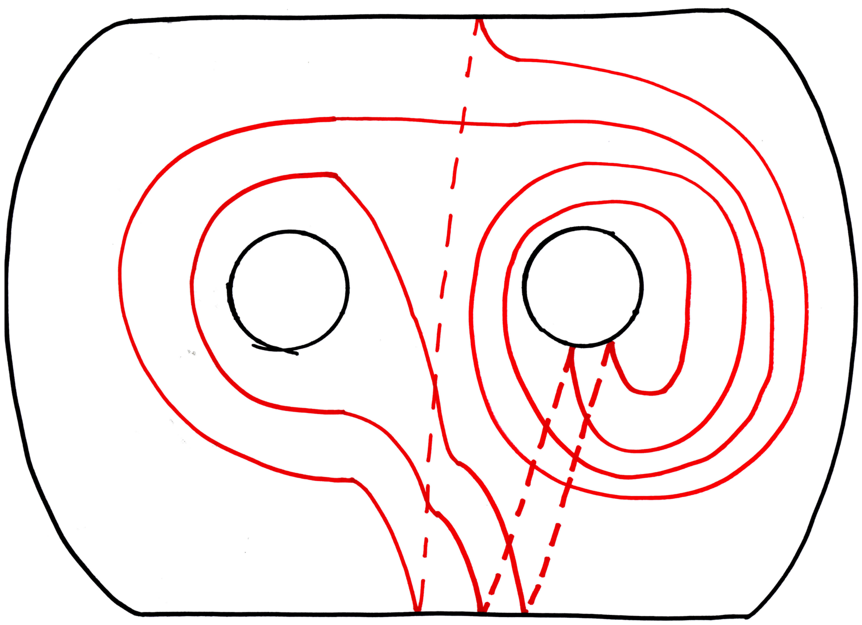

Figure 2.6: A pair of type $(0, 1; 1, 2)$ on a genus two handlebody

In [5] Tsutsumi defines a $(p, q; r, s)$ pair as follows. We see a genus two handlebody H along with a separating circle J embedded in ∂H similar to the one shown in Fig. 2.6. On the boundary of the handlebody, this circle can be seen as two pairs of arcs, labeled 1 and 2 in Fig. 2.7, disjoint from the two basic circles shown in ∂H. These arcs are then connected by taking a waist disk of ∂H and thickening it to create an annulus. In this annulus, we choose a matching between the ends of arcs 1 and 2 on one side with the ends of arcs 1 and 2 on the other side.

Now consider the genus two handlebody as two tori separated by a waist disk. On the boundary of the two tori, we can generalize the process above and take two torus knots of type (p, q) and (r, s), where p and r represent the standard meridional parameters of the knot and q and s the longitudinal parameters. We then similarly take two pairs of arcs labeled 1 and 2 disjoint from the torus knots and join them using the same process with the waist band. We will refer to these pairs as $(p, q; r, s)$. An example of a $(0, 1; r, s)$ pair is shown in Fig. 2.6 and will be called a *simple pair* for our purposes. Notice that for any p, q, r, s, the circle J is nontrivial in H, i.e. it does not bound a disk in H, since in the

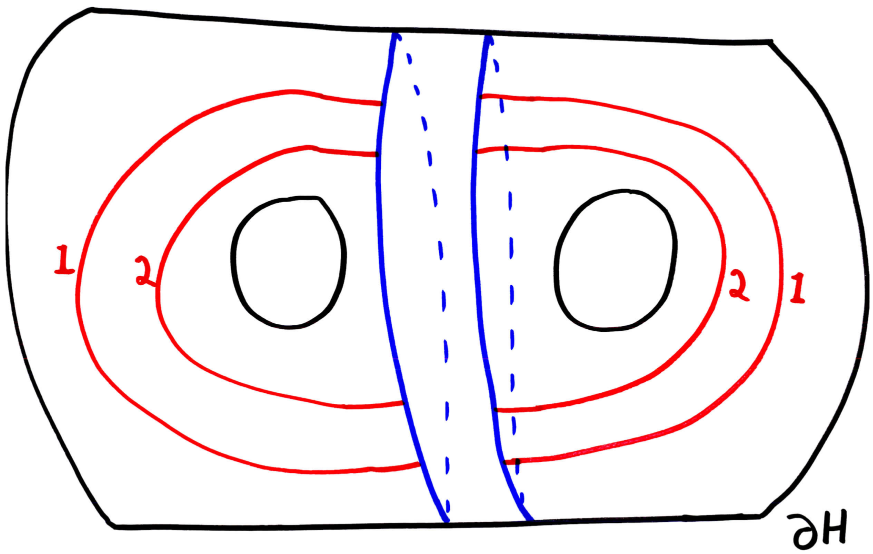

Figure 2.7: The boundary of a $(p, q; r, s)$ pair

fundamental group of H, the word generated by J is $x^q y^s \overline{x}^q \overline{y}^s$.

The following lemma shows the role that $(p, q; r, s)$ pairs play in the context of pairs.

Lemma 2.6.1. *Let (H, J) be a pair and let T be a once-punctured torus properly embedded in H with $\partial T = J$. Then T separates H into two components, one of which is a genus two handlebody H' with $J \subset \partial H'$ such that the pair (H', J) is of type $(0, 1; r, s)$ for some r, s.*

Proof. Suppose J separates H into once-punctured tori T_1 and T_2. If T does not separate H then, after embedding H in $\mathbb{R}^3$, we obtain the closed surface $T \cup T_1$ which does not separate $\mathbb{R}^3$, an impossibility. Hence the once-punctured torus T separates H.

Since J is nontrivial in H it follows that T is incompressible and hence boundary compressible by Lemma 3.2.2. A boundary compression disk D of T must lie between T and one of the once-punctured tori, say T_1, as shown in Fig 2.8. Therefore the manifold H' on the side of T that contains D has boundary $\partial H' = T_1 \cup_J T$, or a genus two surface.

In Fig 2.9 we can see that the surface obtained in the first step of boundary compressing

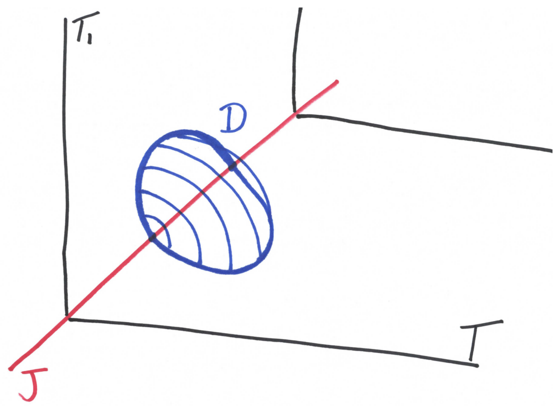

Figure 2.8: The union of T and T_1 along the boundary compression disk D.

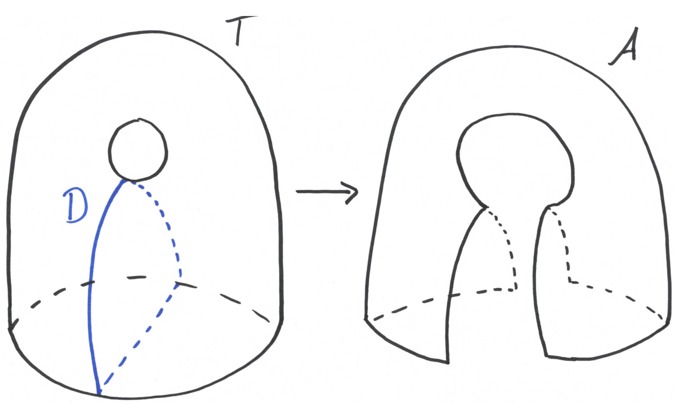

Figure 2.9: First step in the boundary compression of T along D.

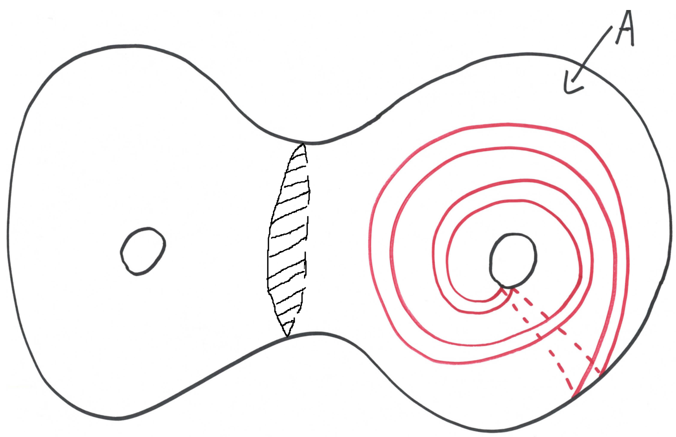

Figure 2.10: The annulus A resulting from the boundary compression on T.

T along the disk D is an annulus A. Since T was separating in H, we also have that A is separating in H, so that after capping off ∂A with copies of D we obtain a separating waist disk of H as full boundary compression of T. In Fig 2.10 we can see A as the annulus complementary to a torus knot in $\partial H'$. By [6] it follows that A cobounds a solid torus V with an annulus $A' \subset \partial H'$, so that undoing the boundary compression along D will be equivalent to adding a one-handle along the solid torus V and therefore resulting in a genus two handlebody. In this genus two handlebody we can see that J is the union of A', a torus knot of type (r, s) in V, and a rectangle, a $(0, 1)$ knot. Thus (H', J) forms a simple pair. $\square$

Lemma 2.6.2. *A simple pair contains no properly embedded once-punctured tori.*

Proof. Let (H, J) be a simple pair and T_1, T_2 the two once-punctured tori in ∂H separated by J. Assuming there is a properly embedded once-punctured torus T in the simple pair (H, J) then from Lemma 2.6.1 we know that T separates H into two components.

In Lemma 2.6.1 we saw that the boundary compression disk D or a properly embedded once-punctured torus was the bigon of the $(0, 1; r, s)$ pair located on one side of the once-punctured torus.

Consider a complete disk system $\{D_1, D_2\}$ of H as shown in Fig 2.11. Since (H, J)

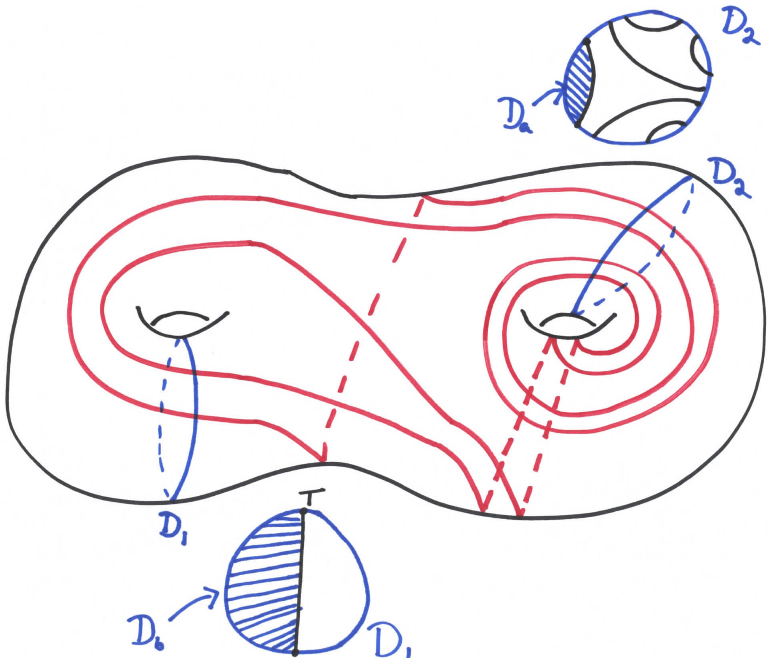

Figure 2.11: A $(p, q; r, s)$ pair with two compression disks D_1 and D_2 and their intersections with T.

is a simple pair, T intersects D_1 only once, resulting in the arc shown in figure Fig 2.11. However, T may intersect D_2 in a number of arcs from which we can choose an outermost arc α as shown in Fig 2.11. The arc α separates T. We will call the disk $D_a \subseteq D_2$ as the disk with no other intersections with T.

Since $T \cap D_1$ only has two sides, we may choose the disk $D_b \subseteq D_1$ as the disk that is on the same side of T as D_a.

Both D_a and D_b are boundary compression disks of T. Therefore we have that $T \cup_J T_1$ is a simple pair with two bigons D_1 and D_2, so $T \cup_J T_1$ must be of the form $(0, 1; 0, 1)$. This genus two handlebody can also be written as $T_1 \times I$, which implies T is parallel to the boundary of H, a contradiction to our original hypothesis. $\square$

Lemma 2.6.3. *Let (H, J) be a pair. Then J bounds at most two mutually disjoint, non-parallel, once-punctured tori in H which are not boundary parallel.*

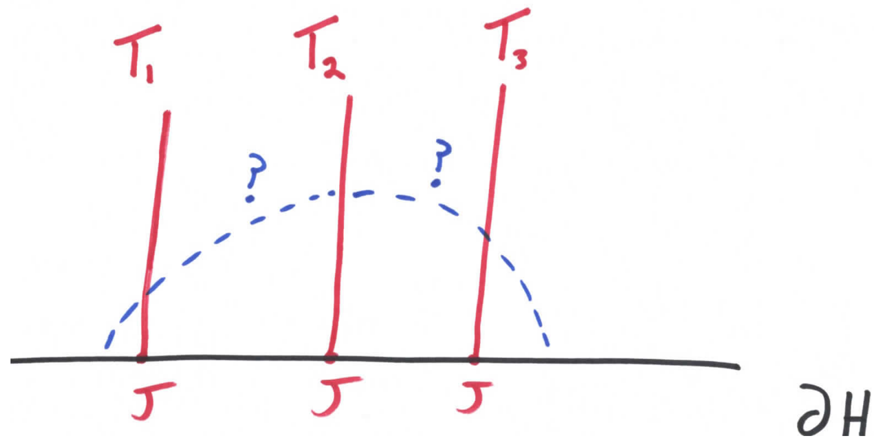

Figure 2.12: Three mutually disjoint, non-parallel, once-punctured tori in H

Proof. Assume there are three mutually disjoint, non-parallel, once-punctured tori T_1, T_2, and T_3 in H which are not boundary parallel, as shown in Fig 2.12. By our Lemma 2.6.1 we can see that T_2 bounds a simple pair on one side. But from Lemma 2.6.2 we know that a simple pair does not contain any non-parallel once-punctured tori. Thus if T_2 bounds a simple pair on the side of T_1, T_1 cannot exist, and if T_2 bounds a simple pair on the side of T_3, T_3 cannot exist. Hence T_2 cannot exist, contradicting our hypothesis. $\square$

Chapter 3

$2n, p$ pairs

3.1 Construction

Let H be a genus two handlebody with a waist disk D_w separating two basic primitive circles x and y as shown in Fig. 3.1 (up to homeomorphism). The disk D_w separates H into two solid tori V_1, V_2 which contain meridian disks D_1, D_2, respectively, disjoint from D_w. In H, D_1 and D_2 form a complete disk system. Cut ∂H along ∂D_w to produce two once-punctured tori T_x and T_y containing x and y, respectively, so that $\partial H = T_x \cup_\partial A \cup_\partial T_y$ for some regular neighborhood A of ∂D_w in ∂H. For each positive integer $n \geq 2$ let $\mathcal{A}_x$ and $\mathcal{A}_y$ be families of $2n$ mutually disjoint parallel arcs in T_x and T_y disjoint from x and y, respectively.

We will label the arcs of $\mathcal{A}_x$ and $\mathcal{A}_y$ consecutively with the labels 1 through $2n$ as shown in Fig. 3.1, and their endpoints with the labels 1 through $4n$ as shown in the same figure.

We then construct a collection of circles $\mathcal{C}_{n,\alpha}$ in ∂H by connecting the endpoint with label 1 in $\mathcal{A}_x$ to some endpoint in $\mathcal{A}_y$ with label p using a spanning arc α in A which goes from left to right and adding arcs in A parallel to α connecting the remaining endpoints of $\mathcal{A}_x$ to $\mathcal{A}_y$, as shown in Fig. 3.2. Notice that each endpoint with label t in $\mathcal{A}_x$ will be connected to the endpoint with label $t + p$ in $\mathcal{A}_y$ for some $0 \leq p \leq 4n - 1$, and we say that $\mathcal{C}_{n,\alpha}$ has been constructed by giving p *clicks* to the endpoints of the arcs in $\mathcal{A}_x$ before identifying with the endpoints of the arcs in $\mathcal{A}_y$.

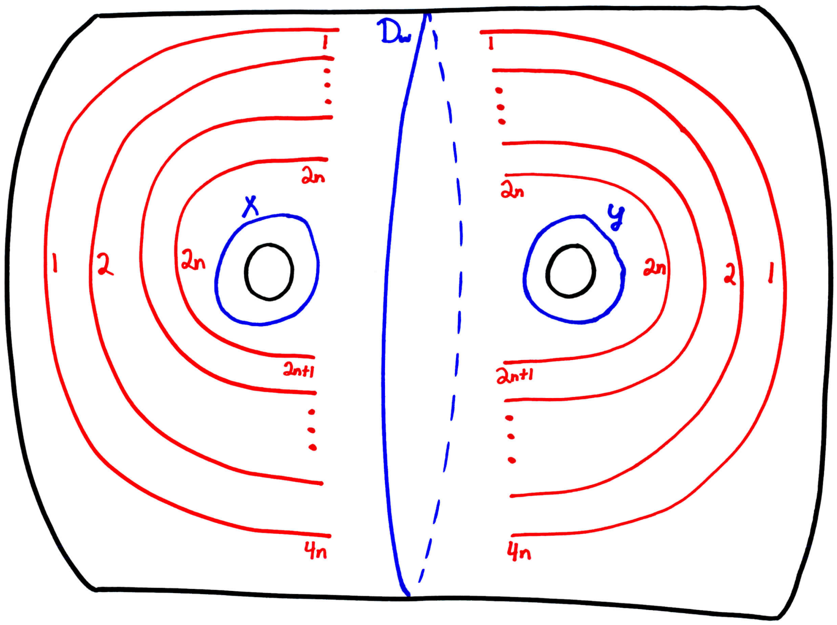

Figure 3.1: Construction of a $2n, p$ pair.

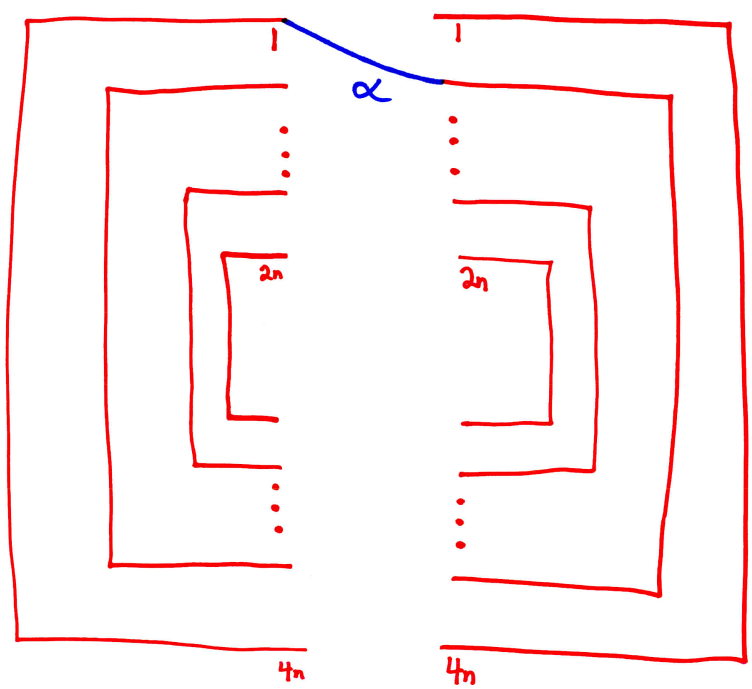

Figure 3.2: The endpoints of $2n$ arcs with the arc α

3.2 Properties of $\mathcal{C}_{n,\alpha}$

In $\mathcal{C}_{n,\alpha}$ we may get more than one component depending on the number p of clicks used in its construction. Our main interest lies in the case where $\mathcal{C}_{n,\alpha}$ has only one component which we now characterize in the next proposition, which is the main result of this section.

Proposition 3.2.1. *The genus two handlebody H along with the collection of circles $\mathcal{C}_{n,\alpha}$ constructed with p clicks forms a pair iff $\gcd(2n, p) = 1$, in which case we may assume that $0 \leq p \leq 2n - 1$.*

The proof follows from the next 3 lemmas.

Lemma 3.2.2. *If $\mathcal{C}_{n,\alpha}$ is constructed using p clicks then the number of components in $\mathcal{C}_{n,\alpha}$ is $\gcd(2n, p)$. In particular, $(H, \mathcal{C}_{n,\alpha})$ is a pair iff $\gcd(2n, p) = 1$.*

Proof. Take a $2n, p$ construction with endpoints labeled as in Fig. 3.3. Observe that the two labels at the endpoints of each arc add to 1 modulo $4n$. Let p be an integer. Starting from the left side, we say a *revolution* is taken when, after connecting a left-sided arc with its corresponding right-sided arc via p number of clicks, we arrive once more to a left-sided arc ready to connect to a right-sided arc. As seen in Fig. 3.3, one revolution consists of four assignments given by the mapping: $x \to x + p \to 1 - x - p \to 1 - x - 2p \to x + 2p$. Hence after r revolutions, the endpoint x is mapped onto the endpoint $x + 2rp$.

Consider a component of $\mathcal{C}_{n,\alpha}$ which contains the endpoint with label x on the left-hand side. From the discussion above, tracing the component once starting at x will require a number of revolutions $r \geq 1$ which is the smallest positive integer satisfying $x + r \cdot 2p \equiv x \mod 4n$, or equivalently r is the smallest positive integer such that $4n | 2pr$, whence $r = 2n/\gcd(p, 2n)$. Since each revolution consists of two arcs, it follows that each component of $\mathcal{C}_{n,\alpha}$ is a union of $2r$ arcs and therefore that there must be a total of $4n/2r = \gcd(p, 2n)$ components in $\mathcal{C}_{n,\alpha}$. $\square$

The following two lemmas give restrictions to the arc α and the number of clicks p.

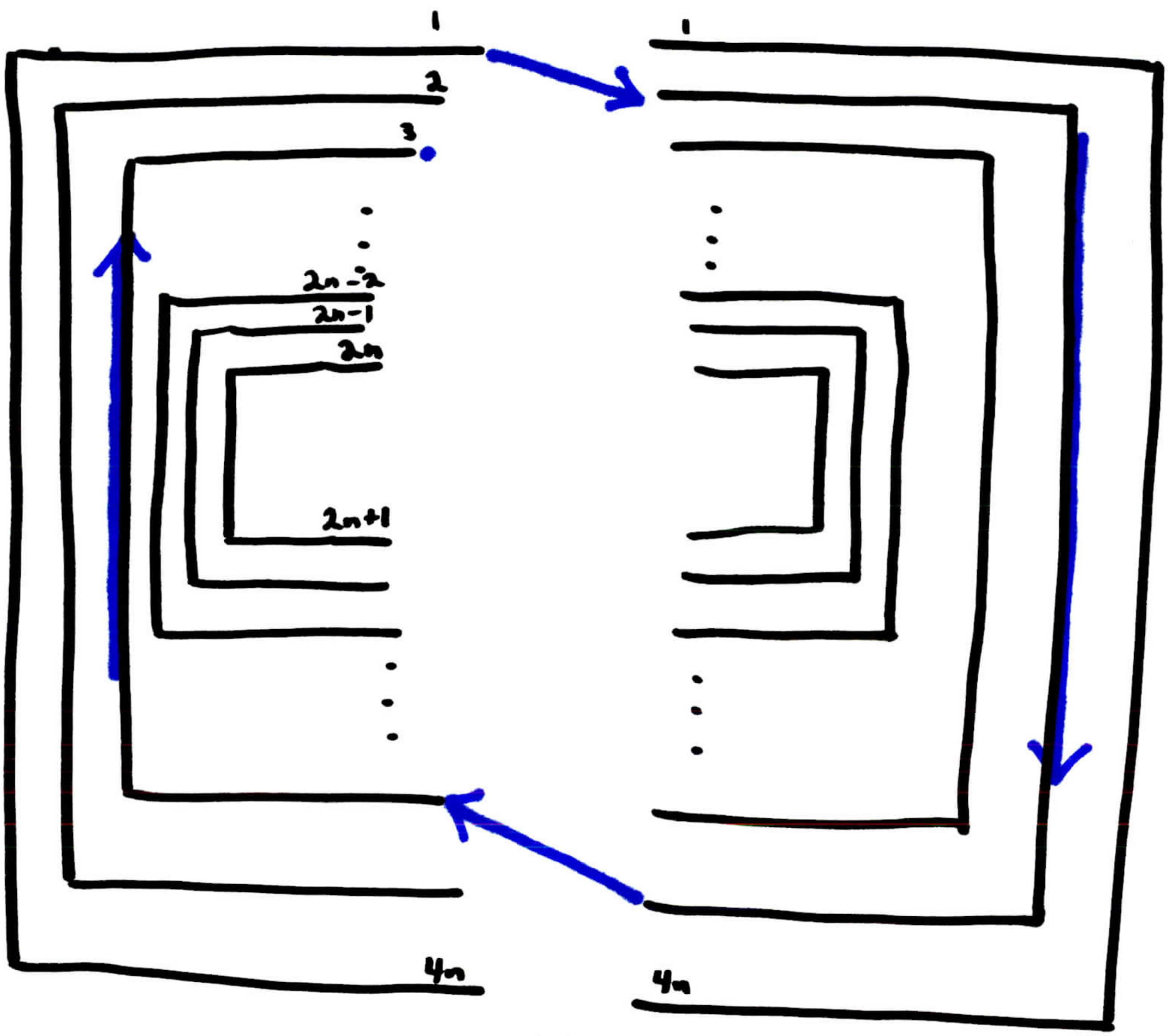

Figure 3.3: A pair of $2n$ arcs with 1 click.

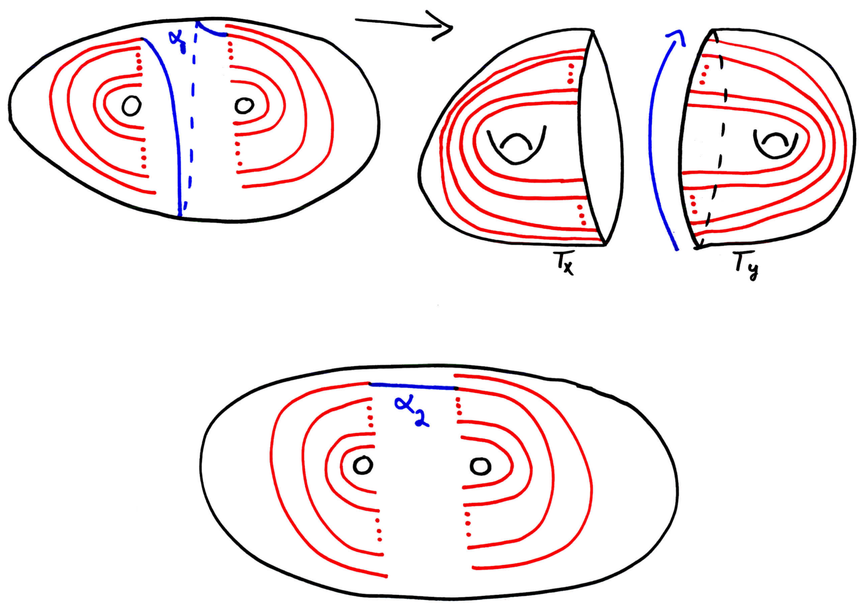

Figure 3.4: A Dehn twist.

Lemma 3.2.3. *There is a homeomorphism $f : H \to H$ that maps the family $\mathcal{C}_{n,\alpha_1}$ onto $\mathcal{C}_{n,\alpha_2}$ iff the endpoints of α_1 and α_2 in $\mathcal{A}_y$ are the same.*

Proof. We construct the homeomorphism f by giving H an appropriate number of Dehn twists along the core of the annulus A as shown in Fig. 3.4. □

Lemma 3.2.4. *There is a homeomorphism $f : H \to H$ that maps the family of circles $\mathcal{C}_{n,\alpha_1}$ constructed with p clicks onto the family of circles $\mathcal{C}_{n,\alpha_2}$ constructed with $4n - p + 1$ clicks.*

Proof. We construct the homeomorphism f by reflecting the pair $(H, \mathcal{C}_{n,\alpha})$ with p clicks along the plane shown in Fig 3.5 (where the case $p = 1$ is represented). □

From now on we may assume that in the family $\mathcal{C}_{n,\alpha}$ the endpoints of α are labeled 1 and $p+1$ where $0 \leq p \leq 2n-1$ and refer to $\mathcal{C}_{n,\alpha}$ by $\mathcal{C}_{n,p}$. For every positive integer n with $\gcd(2n, p) = 1$, we will call $(H, \mathcal{C}_{n,p})$ a $2n, p$ pair.

We may further narrow our choice of p with the following lemma.

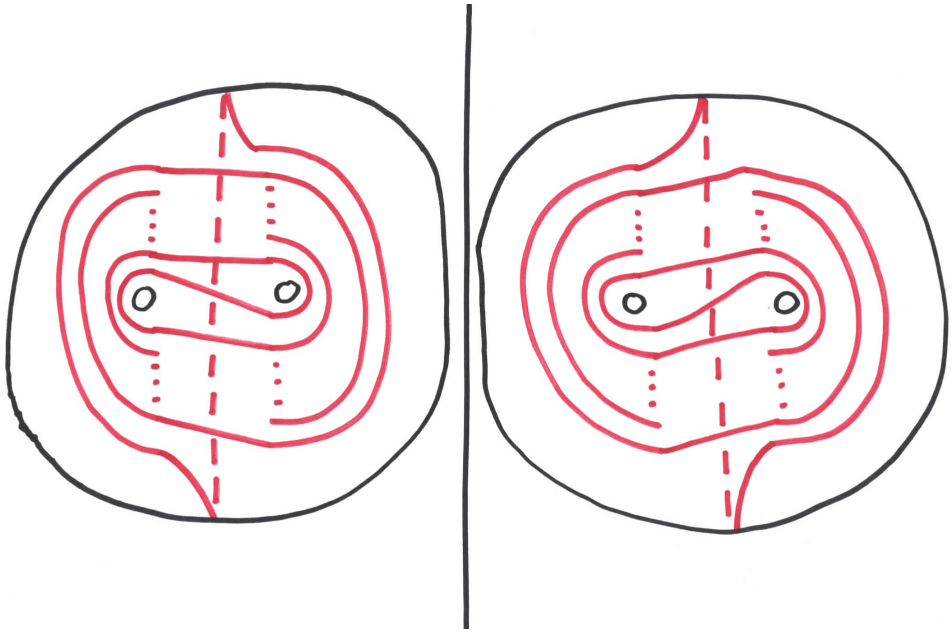

Figure 3.5: The pair $(H, \mathcal{C}_{n,\alpha})$ with 1 click (left) and its reflection.

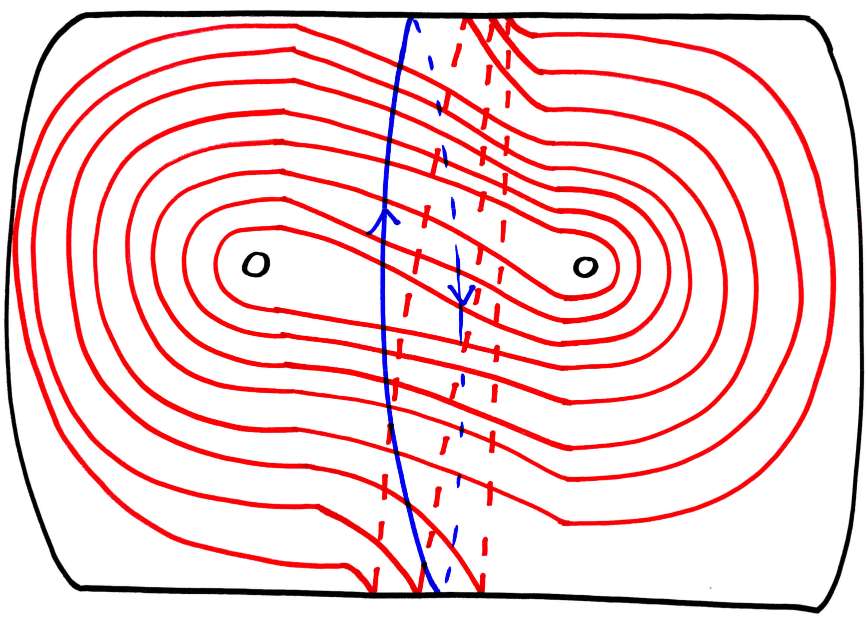

Figure 3.6: An $8, 3$ construction

Lemma 3.2.5. *There is a homeomorphism $f : H \to H$ that maps the $2n, p$ pair to the $2n, 2n - p$.*

Proof. Consider a $2n, p$ pair. We construct the homeomorphism f by giving H a half-Dehn twist and isotoping the arcs such that they are all on the top of the handlebody. As seen in Fig 3.7 relabeling the arcs on $\mathcal{A}_y$ results in the endpoint with label 1 in $\mathcal{A}_x$ being mapped onto the endpoint with label $4n - 2n - p + 1$ in the homeomorphic image of $\mathcal{A}_y$. Now from Lemma 3.2.4 we see this is homeomorphic to a $2n, 2n - p$ pair. $\square$

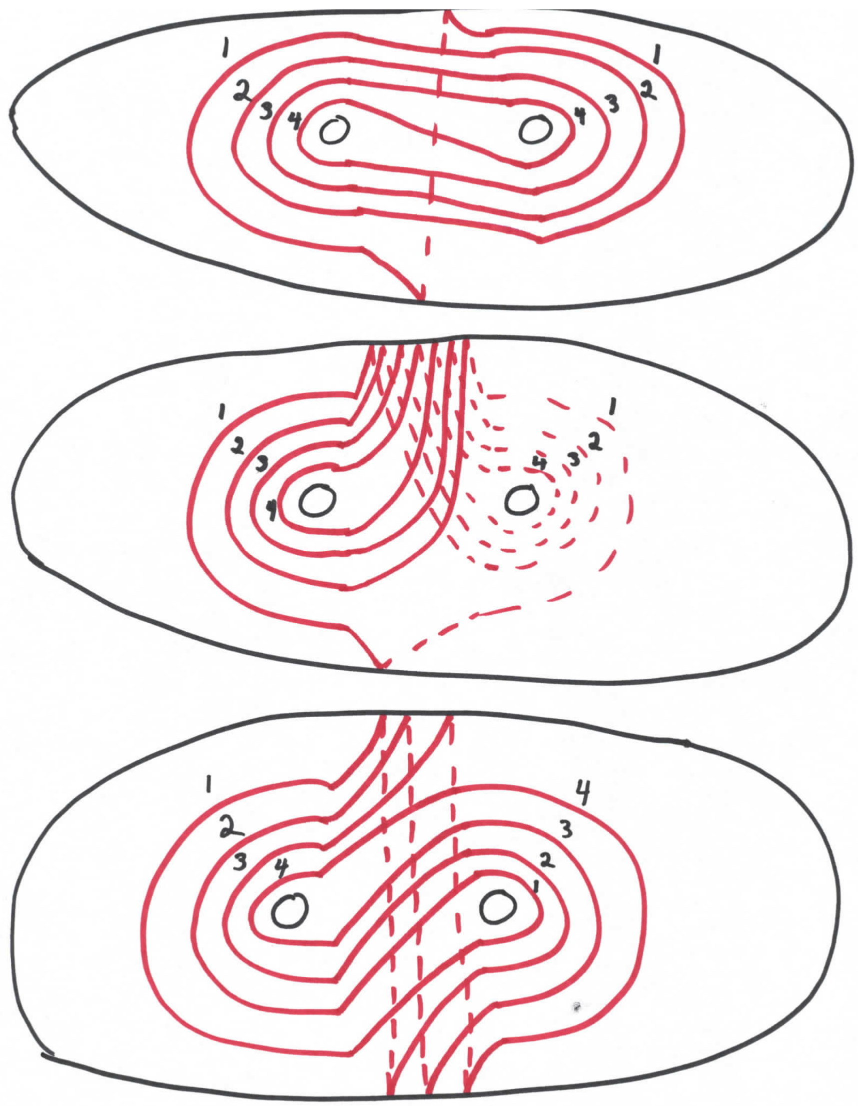

Figure 3.7: Performing a half-Dehn twist and bringing the arcs to the top of H.

Chapter 4

Simple Pairs and $2n, p$ Pairs

4.1 Nonsimple $2n, p$ Pairs

By Proposition 3.2.1 the family of circles $\mathcal{C}_{n,p}$ in ∂H has exactly one component iff $\gcd(2n, p) = 1$, and we may further assume that $1 \leq p \leq 2n - 1$. In this chapter we will assume that $\mathcal{C}_{n,p}$ consists of a single component J and that p satisfies the above restrictions, and refer to (H, J) as a $2n, p$ pair. In our first result we show that such pairs are generically different from simple pairs.

Theorem 4.1.1. *A $2n, p$ pair with $p \neq 1$ is not homeomorphic to any simple pair.*

Proof. Let $G = (H, J)$ be a $2n, p$ pair and let c_1 and c_2 be the boundary circles of the complete disk system of H used in the construction of G. The circle J then cuts each circle c_1 and c_2 into collections of arcs properly embedded in T_1 and T_2; since there is only one arc in c_1 or c_2 that intersects the basic circle x and only one arc that intersects the basic circle y, it follows that the arcs of c_1 and c_2 must lie in ∂H as shown in 4.3; that is, on either side of J there are $2n - 1$ parallel arcs and 1 arc which is not parallel to the others.

We label the arcs of c_1 and c_2 cut by J consecutively with $1, 2, \ldots, 2n$ in a fashion similar to the one shown for the $8, 3$ pair in Fig. 4.1, so that the arc intersecting the basic circle in T_1 or T_2 gets the label $2n$.

It is not hard to see that the arcs of c_1 with labels p and $2n - p$ cut T_2 into two components: an annulus B containing y and a rectangle R. We construct a circle $L_y \subset T_2$ as the union of a core arc of the rectangle R and an spanning arc of B which is disjoint from the arcs of c_2, as shown in Fig 4.2.

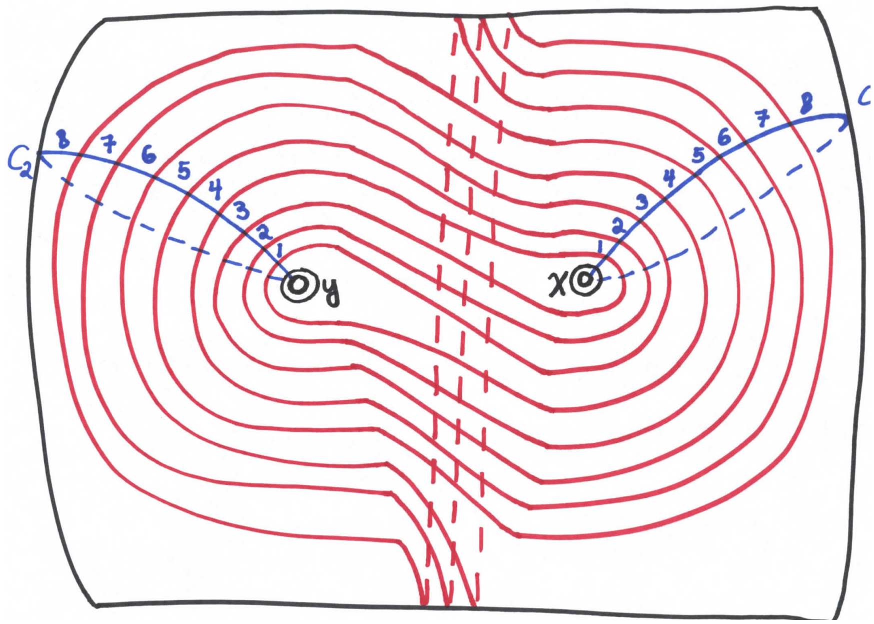

Figure 4.1: An $8, 3$ pair with two nontrivial circles c_1 and c_2.

We define the *arc order* of the arcs of $c_1 \cup c_2$ in T_2 as the sequence of labels read while traversing the circle L_y starting at the point $y \cap L_y$, so that the first label read is p. For example, the arcs in T_2 in Fig. 4.2 have arc order $3_1, 6_2, 7_1, 4_2, 1_1, 2_2, 5_1$ where the subindices 1 and 2 represent arcs from c_1 and c_2 respectively. Notice that the subindices 1 and 2 alternate in the arc order induced by L_y; this property can be traced back to the constructions of J and the rectangle used in the construction of L_y.

Similarly we construct a circle $L_x \subset T_1$ which induces the arc order $3_2, 6_1, 7_2, 4_1, 1_2, 2_1, 5_2$ in the arcs of $c_1 \cup c_2$ in T_1.

We construct a homeomorphism of the $4-$tuple (H, J, c_1, c_2) by representing J as a 'waist' circle of ∂H separating the circles x and y as shown in Fig. 4.3, where the arcs of c_1 and c_2 on either side of J are labeled following the arc order induced by the circles L_x and L_y. In this homeomorphic image, the endpoint with label 1 may go to the endpoint with label 2 or the endpoint with label $2n$. Since there are four such endpoints on the right side, we have four possible matchings. Fig 4.4 shows two such matchings. In fact, these two matchings are the only ones which will correctly match all other arcs. We can see that,

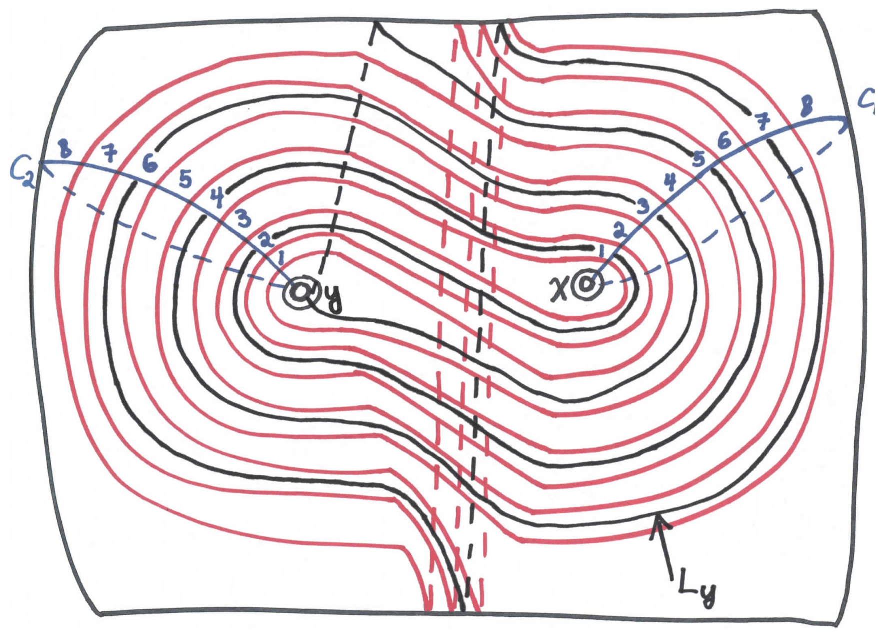

Figure 4.2: The curve L_y going along an $8,3$ pair with two nontrivial circles c_1 and c_2.

on the right side, the order of the parallel arcs may be reversed. We keep the notation of c_1 and c_2 for the two distinct circles in the homeomorphism as seen in Fig 4.3.

From the work of Valdez-Sánchez [pre-print] we can say the following about simple pairs.

Lemma 4.1.2. *In any $(0,1;c,d)$ pair, a circle α is a power circle iff α intersects a primitive circle of the handlebody in one point.*

Suppose now that $p \not\equiv \pm 1 \mod 2n$. To see that the pair G is not homeomorphic to any $(0,1;c,d)$ pair by Lemma 4.1.2 it suffices to show that any circle a in T_1 that intersects x in one point does not represent a power in $\pi_1(H)$.

Since the circle a intersects x in one point, it must be a homological sum of the form $L_y + qy$; Fig 4.5 shows the circle a for $q = 1$.

Now, $\pi_1(H)$ is the free group on the generators x and y which are dual to the complete disk system $\{D_1, D_2\}$ of H. Since $c_1 = \partial D_1$ and $c_2 = \partial D_2$, the word represented by a in $\pi_1(H)$ can be found by simply reading the intersections of a with c_1 and c_2.

Observe that, under suitable orientations for c_1, c_2, L_y, the first and last letters in the

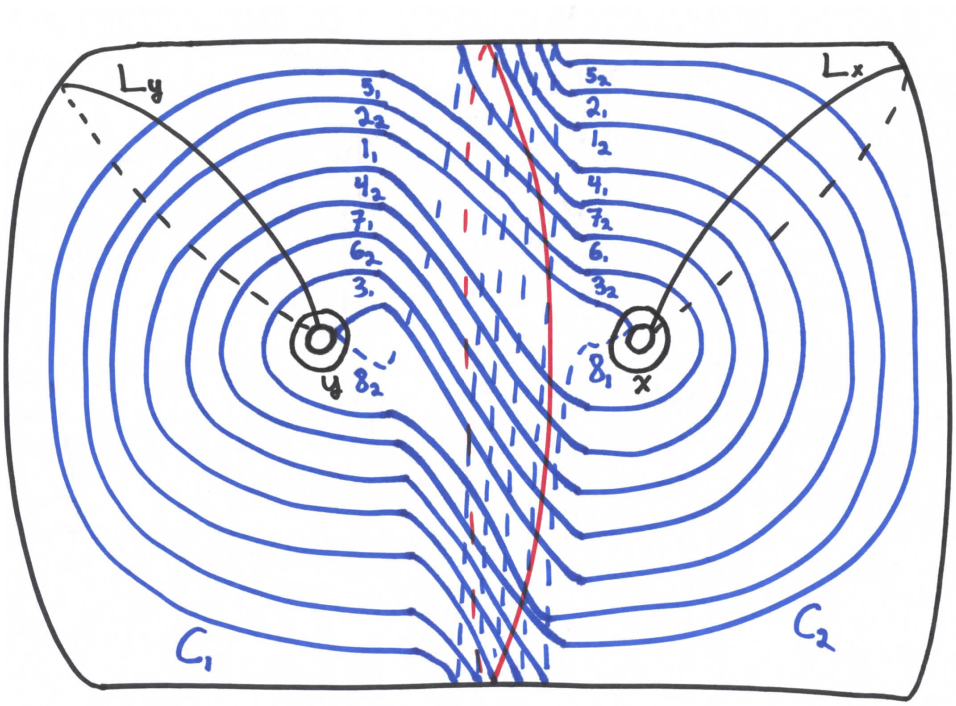

Figure 4.3: The circles c_1, c_2, x, y, L_x, L_y in the homeomorphic image of the 8, 3 pair.

word of L_y will always be x because, from Fig. 4.2, we can see, using the part of L_y in the annulus B, that the arcs with labels $(p)_1$ and $(2n-p)_1$ are intersected by L_y in the same direction, and thus in the circles c_1 and c_2, the first and last arc components have the same orientation. Since the subindices 1 and 2 in the arc order induced by L_y alternate, it follows that the word for L_y will have the form $L_y = x \cdot y^{k_2} x^{k_3} \ldots y^{k_{2n-2}} \cdot x$, where $k_i = \pm 1$. Therefore we have $a = L_y \cdot y^q = xy^{k_2}x^{k_3} \ldots y^{k_{2n-2}} x \cdot y^q$.

Recall that $L_y = x \cdot w \cdot x$ where $w = y^{k_2} x^{k_3} \ldots y^{k_{2n-2}}$ with $k_i = \pm 1$; since L_y intersects c_1 and c_2 alternatingly, it follows that a cyclic reduction of L_y is $x^2 \cdot w$. We now claim that w contains a pair $x, \overline{x}$ or $y, \overline{y}$; these pairs are necessarily nonadjacent since the word $x^2 \cdot w$ is cyclically reduced.

From Fig 4.4 we see that the labels of the endpoints on the right side may be in reverse order. We consider both cases in Fig 4.6 by letting the first endpoint on the right side have the label $\epsilon \cdot p \mod 2n$ where $\epsilon = \pm 1$. This endpoint necessarily connects to the endpoint

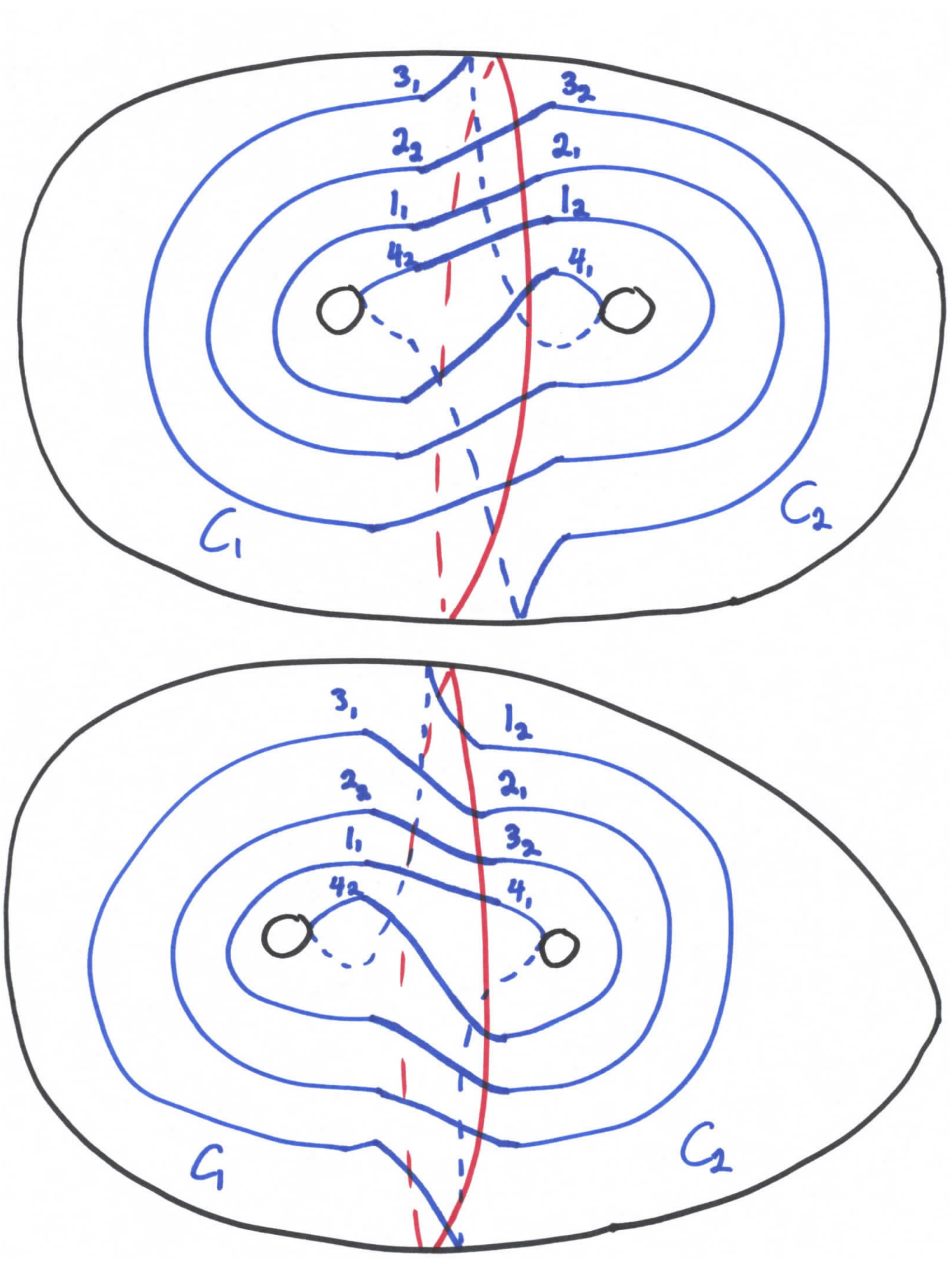

Figure 4.4: The two possible matchings in the homeomorphic image of the 4, 1 pair.

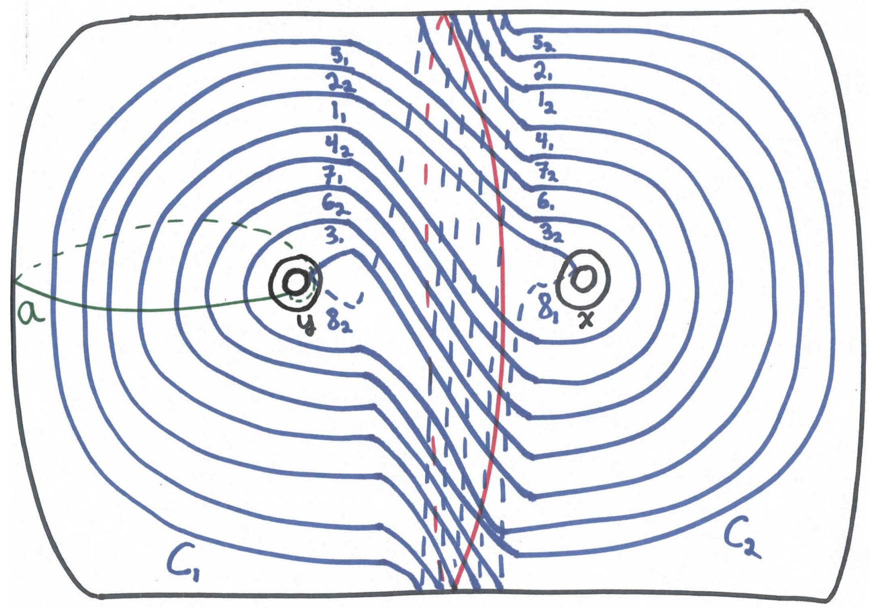

Figure 4.5: The circle $a = L_y + qy$ with $q = \pm 1$.

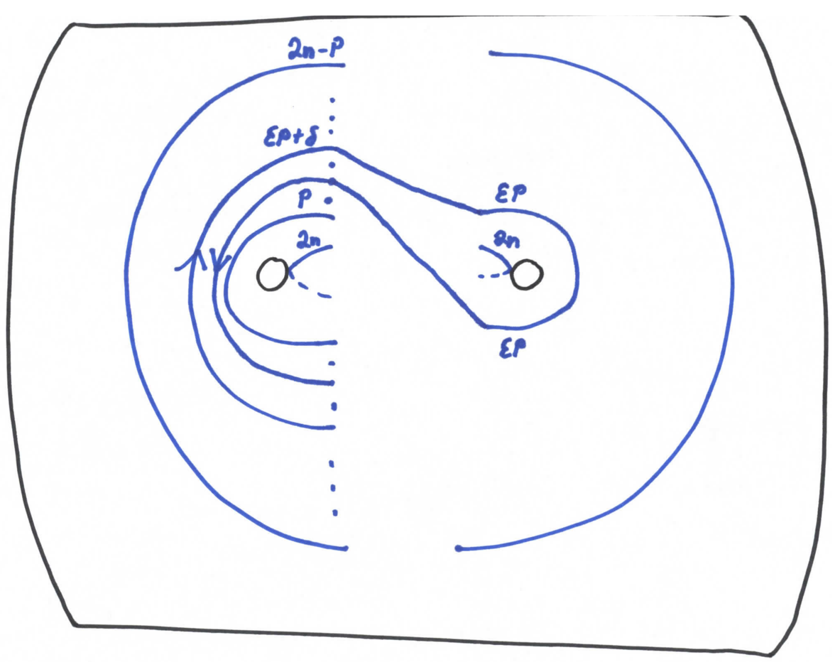

Figure 4.6: The endpoints of the homeomorphic image shown in Fig. 4.3.

$\epsilon \cdot p + \delta$ on the left side for $\delta = \pm 1$ and we may assume this endpoint is on the top portion of $2n$ arcs. Since $p \not\equiv \pm 1 \mod 2n$ we can say $p \neq \epsilon \cdot p + \delta \neq 2n - p$. That is the endpoint with label $\epsilon \cdot p$ connects to an endpoint between the two endpoints with labels p and $2n - p$ on the left side. Since the endpoints with labels p and $2n$ remain on the top portion of the arcs, we now see that there must exist an endpoint between $\epsilon \cdot p + \delta$ and p that connects to the endpoint with label $\epsilon \cdot p$ on the bottom portion of the arcs. From Fig 4.6 we can see that, since $\epsilon \cdot p$ was an 'innermost' arc and both endpoints connect to the top portion of the arcs, we have a switch in direction for the circle c_1 or c_2. This results in a nonadjacent pair $x, \overline{x}$ or $y, \overline{y}$ in the word w.

It now follows that for any q the cyclic reduction of $a = L_y \cdot y^q$ also contains a nonadjacent pair $x, \overline{x}$ or $y, \overline{y}$, and so by Lemma 2.5.1 the circle a is not a power in $\pi_1(H)$. Since from Lemma 4.1.2 we have that each side of J in any simple pair contains a power circle, it follows that a $2n, p$ pair with $p \not\equiv \pm 1 \mod 2n$ cannot be homeomorphic to a $2n, p$ pair. $\square$

4.2 Simple $2n, p$ Pairs

We have now narrowed down the number of $2n, p$ pairs which can be simple to only two cases. The next result shows that indeed in these remaining cases the $2n, p$ pairs are simple.

Theorem 4.2.1. *A $2n, p$ pair is homeomorphic to a simple pair iff $p =\not\equiv \pm 1 \mod 2n$.*

Proof. Theorem 4.1.1 shows that if a $2n, p$ pair is homeomorphic to a simple pair then $p \equiv \pm 1 \mod 2n$.

From Lemma 3.2.5 we see that we only need to consider the case for $p = 1$.

Consider a $2n, 1$ pair and a regular neighborhood A of a waist disk of H shown in Fig. 4.7. In this neighborhood A, we can see the existence of a bigon disk $D \subset H$ with the two intersections with J shown in Fig. 4.7. We can also see two circles c_1 and c_2 which are both disjoint from J and D as seen in Fig. 4.7.

By cutting H along the bigon D, we get a solid torus as shown in Fig. 4.8 where c_1 and c_2 are torus knots and the arcs of J are between these two circles. By reattaching the

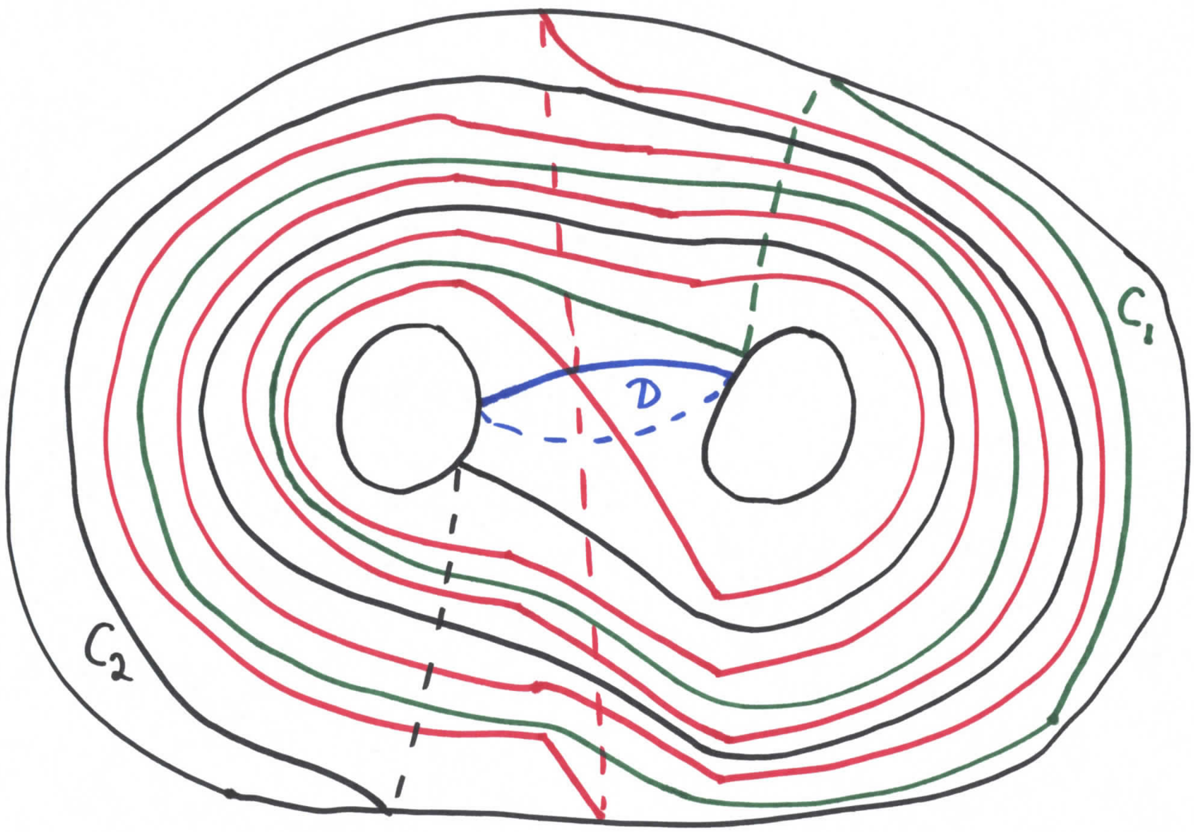

Figure 4.7: The two circles c_1 and c_2 disjoint from both J and the bigon.

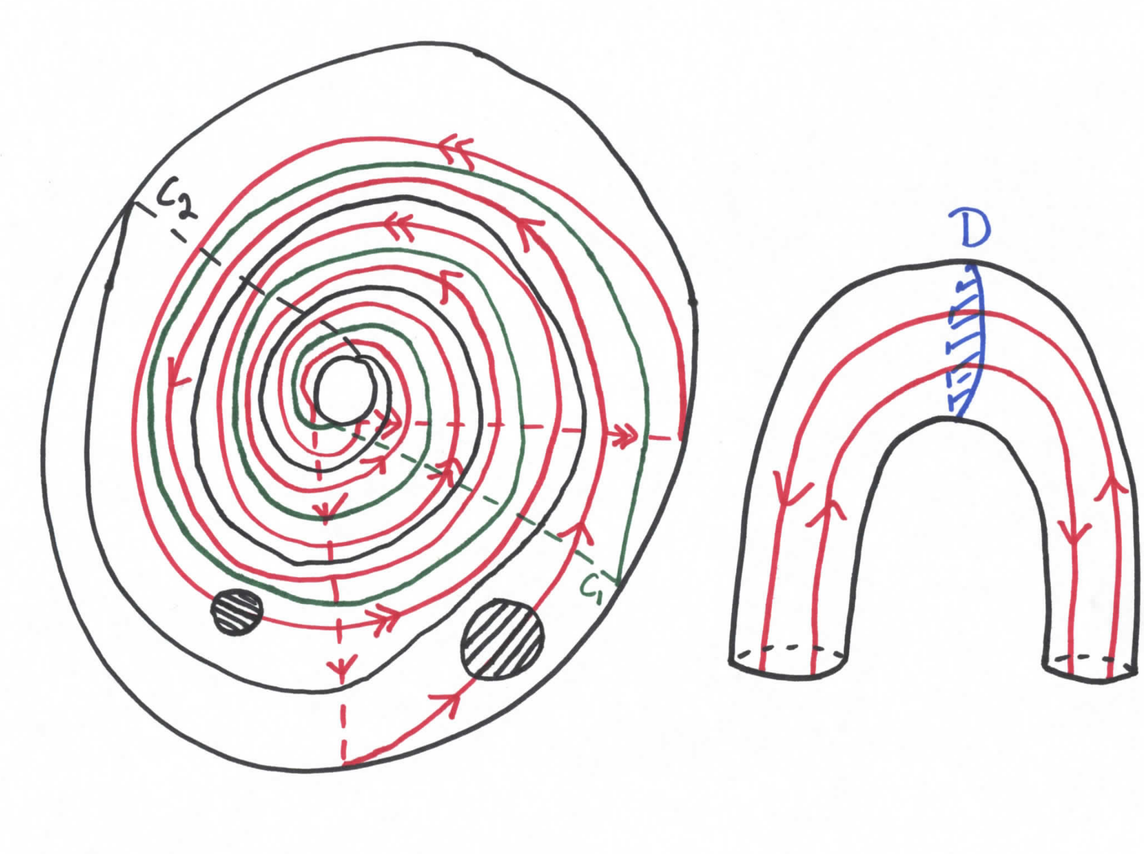

Figure 4.8: The result of cutting H along D

1-handle shown in Fig. 4.8 we can see we have a simple pair $(1, 0; r, s)$ where r and s are determined by the torus knots separated by arcs of J and the feet of the 1-handle $N(D)$ as shown in Fig. 4.8. Thus we have that a $2n, 1$ pair is homeomorphic to a simple pair.

$\square$

Chapter 5

Applications

By identifying this homeomorphism between $2n, 1$ pairs and simple pairs we can answer questions about more general separating circles in a genus two handlebody. For instance, any pair which does not contain a bigon is not homeomorphic to a $2n, 1$ pair and therefore is not homeomorphic to any simple pair. Rather than compute the fundamental group of a pair to establish a mapping between two handlebodies, we can take a geometric approach and consider constructions which may be easier in one homeomorphic version of the pair.

In section 3.1 we saw a that a nontrivial separating curve in any genus two handlebody bounds at most two mutually disjoint, nonparallel, once-punctured tori. However, it may often be the case that only one once-punctured torus is present in an (H, J) pair which is not boundary parallel. We expect that all $(p, q; r, s)$ pairs will only contain one once-punctured torus not parallel to the boundary of H. On the other hand, we expect two mutually disjoint, non-parallel once-punctured tori to be present in $2n, p$ constructions with $\gcd(2n, p) = 1$ which use power circles in place of basic circles; in fact a stronger claim is that such pairs are the only ones which contain two mutually disjoint, non-parallel once-punctured tori not parallel to the boundary of H. Proofs of these claims are left as future work. Intuition leads one to believe the basic circles in $2n, p$ pairs which are disjoint from J are heavily involved in these proofs.

References

[1] Colin Adams. Hyperbolic knots. In *Handbook of knot theory*, pages 1–18. Elsevier B. V., Amsterdam, 2005.

[2] A. J. Casson and C. McA. Gordon. Reducing Heegaard splittings. *Topology Appl.*, 27(3):275–283, 1987.

[3] Francisco González-Acuña and Arturo Ramírez. A composition formula in the rank two free group. *Proc. Amer. Math. Soc.*, 127(9):2779–2782, 1999.

[4] Dale Rolfsen. *Knots and links.* Publish or Perish, Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.

[5] Yukihiro Tsutsumi. Universal bounds for genus one Seifert surfaces for hyperbolic knots and surgeries with non-trivial JSJT-decompositions. In *Proceedings of the Winter Workshop of Topology/Workshop of Topology and Computer (Sendai, 2002/Nara, 2001)*, volume 9, pages 53–60, 2003.

[6] Luis G. Valdez-Sánchez. Seifert Klein bottles for knots with common boundary slopes. In *Proceedings of the Casson Fest*, volume 7 of *Geom. Topol. Monogr.*, pages 27–68 (electronic). Geom. Topol. Publ., Coventry, 2004.

Curriculum Vitae

Daniel Orlando Vazquez was born on April 7, 1992. The second son of Alejandro Vazquez and Alicia Vazquez, he graduated from Mission Early College High School, El Paso, Texas, in the spring of 2010. He entered El Paso Community College in spring of 2006 and, in the fall of 2009, The University of Texas at El Paso. He received his bachelor's degree in Mathematics in the spring of 2012 along with a certification to teach secondary education on the state of Texas.

In the fall of 2012, he entered the Graduate School of The University of Texas at El Paso. While pursuing a master's degree in Mathematics he worked as a Teaching Assistant. The area in which he specialized was low-dimensional topology with Dr. Luis Valdez-Sánchez.

Permanent address: 1333 Clay Basket Circle
El Paso, Texas 79936-6846