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Griselda Acosta
University of Texas at El Paso, gvacosta@miners.utep.edu

Eric Smith
University of Texas at El Paso, esmith2@utep.edu

Olga Kosheleva
University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
University of Texas at El Paso, vladik@utep.edu

Epicycles Are Almost as Good as Trigonometric Series: General System-Based Analysis

Griselda Acosta[1], Eric Smith[2],
Olga Kosheleva[3], and Vladik Kreinovich[4]
[1]Department of Electrical and Computer Engineering
[2]Department of Industrial, Manufacturing, and
Systems Engineering
[3]Department of Teacher Education
[4]Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
gvacosta@miners.utep.edu, esmith2@utep.edu,
olgak@utep.edu, vladik@utep.edu

Abstract

To adequately describe the planets' motion, ancient astronomers used epicycles, when a planet makes a circular motion not around the Earth, but around a special auxiliary point which, in turn, performs a circular motion around the Earth – or around a second auxiliary point which, in turns, rotates around the Earth, etc. Standard textbooks malign this approach by calling it bad science, but in reality, this is, in effect, trigonometric (Fourier) series – an extremely useful tool in science and engineering. It should be mentioned, however, that the epicycles are *almost* as good as trigonometric series – in the sense that in some cases, they need twice as many parameters to achieve the same accuracy.

1 Epicycles: Bad Science or Genius Idea

Epicycles: what they are. For purposes of navigation, since the ancient times, astronomers have studied the visible motion of the planets and the stars. In the first crude approximation, their trajectories form a circle.

To provide a more accurate description, astronomers proposed the following idea – called *epicycles*. First step is to assume that while a point corresponding to a planet follows a circular motion around the Earth, the planet itself performs a circular motion around this point. To get an even more accurate description, we assume that it is not the planet itself, but rather the second point associated

with the planet that performs a circular motion around the first point. The planet itself performs a circular motion around the second point.

An even more accurate description is that for each planet, there is a third point that performs a circular motion around the second point, and the planet itself moves in a circle around this third point, etc.

This idea was originally proposed by Apollonius of Perga (late 3rd — early 2nd centuries BCE), developed by several others, and finalized by Claudius Ptolemy ($\approx$100–$\approx$170).

Epicycles: traditional textbook-based negative coverage. Traditional textbooks on history of science treated epicycles as bad science, a bad idea that was overcome by the genius of Nicolaus Copernicus (1473–1543).

Epicycles as Fourier (trigonometric) series. From the mathematical viewpoint, a circular rotation around the origin of the coordinate system can be described in simple trigonometric terms:

$$x(t) = r \cdot \cos(\omega \cdot t + \varphi), \quad y(t) = r \cdot \sin(\omega \cdot t + \varphi),$$

i.e., equivalently, as

$$x(t) = x_0 \cdot \cos(\omega \cdot t) - y_0 \cdot \sin(\omega \cdot t);$$

$$y(t) = x_0 \cdot \sin(\omega \cdot t) + y_0 \cdot \cos(\omega \cdot t).$$

In these terms, Ptolemy's description means that we represent the motion of a planet as a sum of such motions – i.e., as a linear combination of sines and cosines, a combination that we now call *trigonometric series* or *Fourier series*, after Joseph Fourier (1768–1830), a researcher who, in effect, reinvented them for modern times.

Fourier series is exactly how we now describe the visible motion of the planets (see, e.g., [1]). Taking into account the ubiquity of Fourier series and their importance for science and engineering (see, e.g., [2]), the epicycles idea sounds more like a stroke of genius than the bad science as described by traditional textbooks.

What is better: Fourier series or epicycles? Epicycles are, in effect, Fourier series. So, any trajectory that can be represented by an epicycle can also be represented by the corresponding Fourier series.

Natural question are:

- are these two representations truly equivalent, i.e., can we represent each Fourier series motion in terms of epicycles?

- if yes, which representation is better? is there any computational advantage in using Fourier series in contrast to epicycles?

These are the questions that we deal with in this paper.

2

2 Analysis of the Problem and the Resulting Conclusions

Can any Fourier series be represented in epicycle terms? In the above text, we showed that every epicycle-based motion can be represented in terms of Fourier series. A natural question is: is the opposite true? Can any Fourier-series motion be represented in epicycle terms?

When the dependence of both coordinates x and y on time is described by Fourier series, the question is whether we can separately represent the x-motion and the y-motion by epicycles; if we can, then by adding these two representations, we will be able to represent any Fourier motion this way. Thus, the question is: can epicycles represent the purely x-motion in which $y = 0$ or the purely y-motion in which $x = 0$?

The answer to this natural question was provided, for the first time, by the Arabic astronomer Nasir al-Din at-Tusi (1201–1274); see, e.g., [3]. One of the solutions that he proposed to represent an x-only motion is to have a circle of half-radius rotate inside a circle of the original radius. In this case, a point on a small circle moves only along the x-axis. An even simpler solution would be to represent an x-only motion

$$x(t) = r \cdot \cos(\omega \cdot t + \varphi), \quad y(t) = 0$$

as the sum

$$(x(t), y(t)) = (x_1(t), y_1(t)) + (x_2(t), y_2(t)),$$

where

$$x_1(t) = \frac{r}{2} \cdot \cos(\omega \cdot t + \varphi), \quad y_1(t) = \frac{r}{2} \cdot \sin(\omega \cdot t + \varphi)$$

and

$$x_2(t) = \frac{r}{2} \cdot \cos((-\omega) \cdot t + \varphi) = \frac{r}{2} \cdot \cos(\omega \cdot t + \varphi);$$

$$y_2(t) = \frac{r}{2} \cdot \sin((-\omega) \cdot t + \varphi) = -\frac{r}{2} \cdot \sin(\omega \cdot t + \varphi).$$

Similarly, we can represent an y-only motion.

Comment. It is worth mentioning that, as shown in [3], Copernicus used al-Tusi results in his analysis of celestial mechanics – to the extent that he even borrowed some illustrative pictures from a translation of al-Tusi's book.

So which representation is computationally more efficient? In both cases – of using epicycles and of using trigonometric series – we approximate the observed motion $x(t), y(t)$ by a linear combination of several standard motions $(x_i(t), y_i(t))$:

$$x(t) \approx \sum_i a_i \cdot x_i(t), \quad y(t) \approx \sum_i a_i \cdot y_i(t);$$

see, e.g., [4].

In general, there is a small advantage in using trigonometric series. This advantage is related to the fact that in many cases, the motion is close to linear. In this case, e.g., for the x-only motion:

- in trigonometric series, we need two terms for each frequency ω, namely, we need a linear combination of the terms $(x(t), y(t)) = (\cos(\omega \cdot t), 0)$ and $(x(t), y(t)) = (\sin(\omega \cdot t), 0)$ while

- to represent this particular frequency component via epicycles, we need two at-Tusi-type pairs – i.e., we need four terms and thus, four (twice as many) coefficients.

This advantage is not that big for the regular planetary motions, since such motions are indeed close to circular, but it is important for other motions – and for higher-frequency (i.e., higher-order) terms describing this motion.

Acknowledgments

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