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Finite Density QED at Strong Magnetic Field

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to my

MOTHER and FATHER

with love

FINITE DENSITY QED AT STRONG MAGNETIC FIELD

by

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Abstract

In this thesis we study the effect of a dense medium on the quantum properties of a relativistic electron-positron plasma under a magnetic field. From the calculation of the photon polarization operator depending on temperature, chemical potential, and the magnetic field; we study how the electric susceptibility of the strongly magnetized plasma is charged with temperature and density. In the same manner we study the dependence of the Debye mass with density. Then we investigated the phenomenon of magnetic catalysis of chiral symmetry breaking for a dense medium. We found that there exist a critical value of the chemical potential where a first-order phase transition takes place for the system to regain its chiral symmetry. Our results can be on interest for the astrophysics of compact objects and for planar condensed materials with Dirac Hamiltonians as graphene.

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Chapter 1

Introduction

1.1 General Remarks

At present there has been an intense activity in the scientific community to study the effects of magnetic fields on several physical scenarios.

In astrophysics pulsars have magnetic fields of order $10^{12} - 10^{13}$ G on their surface on average, with magnetars having magnetic fields of $10^{14}-10^{15}$G. Part of this higher magnetic field is due to the rotating plasma of a neutron star. When we apply the virial theorem to estimate the inner core magnetic field we find fields of the order of 10^{18} Gauss. There is a natural maximum of 10^{20}G for high-dense stars formed by quarks [4].

On the other hand, in relativistic heavy-ion collisions you can get massive magnetic fields from the charged gold nuclei with mass energies of about 200GeV. There can be magnetic fields of the order $2m_\pi^2$ from peripheral collisions, a value which is higher than the magnetic fields found on the surface of magnetars. Even higher magnetic fields of order $eB \sim 15_\pi^2 \sim 10^{19}$G can be reached in Pb-Pb collisions at CERN [15] [27].

The magnetic field affects the phase structure of QCD in a nontrivial fashion due to the behaivor of quarks and charged hadrons vs neutral hadrons in a magnetic field. The magnetic field can affect deconfinement and chiral symmetry restoration [21].

Originally, you would think that the vacuum should be empty in order to preserve its invariance under Lorentz transformations. However, since what is needed is the invariance of the action, and this is not the same as the invariance of the vacuum, you can have that vacuum be non-empty without violating Lorentz invariance of the action.

It is known that in 3+1 and 2+1 dimensions, a strong constant magnetic field causes dynamical symmetry breaking that leads to the generation of fermion dynamical mass. A very strong magnetic field results in the dimensional reduction D→D-2 so 3+1→1+1 and 2+1→0+1 dimensions. The dimensional reduction is due to the confinement of the charged fermions to the lowest Landau level (LLL), in the presence of the strong field. In the infrared region, even in the weak QED coupling there is spontaneous symmetry breaking in a magnetic field. This phenomenon is called in the literature magnetic catalysis of chiral symmetry breaking (MCχSB) [13] [27].

When there is a large external magnetic field interacting with charged particles there is a Landau quantization of the particle's transverse momentum. When that field is strong enough to confine the particles to the LLL it is easier to form a chiral condensate, because the fermions are right next to the antifermions of the Dirac sea. Then, the formation of the particle-antiparticle spin-0 condensate breaks the chiral symmetry a phenomenon referred as MCχSB. One of the main purposes of our investigation is to consider the MCχSB in a dense medium. This task will be carried out in the theoretical frame of QED in a strong magnetic field with finite density (i.e. with a non-zero chemical potential, [$\mu \neq 0$]).

There is another effect that plays a role in dynamical chiral symmetry breaking. It is also important to notice that the magnetic field also induces an anomalous magnetic moment (AMM) of the fermion-antifermion pair. This AMM is a consequence of the pairing of the particles with opposed charges and spins (i.e. with a net magnetic moment) [3] [2]. So the induced parameters (dynamical mass and AMM) change the vacuum properties in the presence of the magnetic field. The dynamical AMM leads to a nonperturbative Lande g-factor and Bohr magneton proportional to the inverse of the dynamical mass. The induction of the AMM leads to a nonperturbative Zeeman effect.

Unfortunately, it is very difficult to experimentally prove that an effect is due to the MCχSB because the dynamical parameter are extremely small. Also there is the added difficulty of ruling out the other competing mechanisms that can enter in a given field effect. In this regard, the finding in "Paraelectricity in Magnetized Massless QED" [6] is very significant. As it was shown in [6], there is a significant increase in the electrical susceptibility produced by magnetically catalyzed chiral pairs. If a system shows a sizable electrical polarization with a weak external electric field along the same direction as a strong external magnetic field, it will give plausible evidence of MCχSB.

The interesting thing about mass is that in any relativistic theory, as for example in Quantum Electrodynamics (QED), mass and energy can be interchangable. When using natural units the c in the famous equation of $E = mc^2$ is equal to 1. That means that mass and energy can be thought of as interchangeable or rather you can think of mass in terms of energy. Furthermore, there is the interesting concept that the very notion and concept of mass is that the interaction between quarks is the main factor responsible for the mass of an object. The contributions of these interactions to the mass can be calculated using nonperturbative Quantum Field Theory (QFT). Then, of course, there is also a small mass contribution that comes from the electroweak phase transition via the condensation of the Higgs field. That way, there is an explanation of why one proton will have a similar mass to another proton.

It has been discovered, on the other hand, that the origin of mass is associated with symmetries that have been broken [8]. Yoichiro Nambu was the first who presented the idea of a fermion pair generating a fermion mass and spontaneous broken symmetry in particle physics. He had the hypothesis that this pairing process contributed to the mass of the elementary particles. So the first models of dynamical symmetry breaking in QFT were constructed by Nambu and Jona-Lasinio. Their model, basically had dynamical symmetry breaking as one of its central concepts.

The dynamical mass of fermions that is generated in gauge theories requires a non-perturbative approach. It is usually calculated by using the Schwinger-Dyson equations. When doing the calculations an infinite series of loops arises in what is called the ladder approximation.

In this thesis we shall study the effects of a strong magnetic field on the chiral symmetry breaking of a dense system of electrons. In order to do this, we organize this thesis as follows:

In chapter 1, we give a review about some general concepts that allow for us to do the calculations in the following chapters.

In chapter 2, we calculate the photon polarization operator (PPO) at finite temperature and density. We also study the behavior of the electric susceptibility of a strong magnetized plasma. The electric susceptibility will tell us the properties that the medium imparts on the photon by breaking it into positron electron pair and then collapsing it back again. We can do this because the Heisenberg uncertainty principle allows for us to create large amounts of energy if we collapse them again in a short amount of time. It is important to consider that a positron and electron are charged particles that interact with an external magnetic field. This means that the magnetic field can have an effect on a photon indirectly through the vacuum, although it is a neutral particle.

In chapter 3, we study the electron self energy in order to find the dynamical mass that is created from quantum fluctuations by using the ladder approximation. It is also called the ladder approximation because you have the photon proagator over the fermion propagator. We continue to add photon propagators on top of the fermion propagator unitl it begins to look like a ladder. This calculation tells us the effect the interaction with the medium has on the mass of a particle.

Then we conclude and give some specific calculation in the Appendices.

1.2 Noether's Theorem

Every continuous symmetry of the Lagrangian implies a conservation law. The minimal action principle implies the conservation of the four current [3],

$$\delta S = 0 \quad \Rightarrow \quad \partial_\mu j^\mu = 0 \tag{1.1}$$

Where we used the well known Einstein convention. After we contract the indexes in Eq. (1.1) we have

$$\partial_0 j^0 + \partial_1 j^1 + \partial_2 j^2 + \partial_3 j^3 = 0 = \frac{\partial j^0}{\partial t} + \bigtriangledown \cdot j \tag{1.2}$$

The charge Q is defined as

$$Q = \int_{\mathcal{R}^3} d^3x j^0 \tag{1.3}$$

If we take the derivative with respect to time we have thanks to the conservation law Eq. (1.1)

$$\frac{\partial Q}{\partial t} = \int_{\mathcal{R}^3} d^3x \frac{\partial j^0}{\partial t} = -\int_{\mathcal{R}^3} d^3x \bigtriangledown \cdot \vec{j} \tag{1.4}$$

Since the integration of $\mathcal{R}^3$ can be over all space we can restrict it to a volume V. This would give us

$$\frac{\partial Q_V}{\partial t} = \int_V d^3V \frac{\partial j^0}{\partial t} = -\int_V d^3V \bigtriangledown \cdot \vec{j} = -\int_A j \cdot dS \tag{1.5}$$

Where we used Gauss's theorem to show that if we have a varying charge in a volume there is a current flux through the volume surface. Then, if the flux is zero the corresponding charge is conserved.

$$\frac{\partial Q}{\partial t} = 0 \tag{1.6}$$

1.3 Goldstone Fields

The Lagrangian density of a real scalar field is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi\right) - U(\phi) \tag{1.7}$$

where the potential is given by

$$U(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4; \quad \lambda > 0 \tag{1.8}$$

We do the transformation $\phi \to -\phi$, so

$$U(-\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 = U(\phi) \tag{1.9}$$

and then

$$\mathcal{L}' = \frac{1}{2}\left(\partial_\mu(-\phi)\partial^\mu(-\phi)\right) - U(\phi) = \frac{1}{2}(\partial_\mu(\phi)\partial^\mu(\phi)) - U(\phi) = \mathcal{L} \tag{1.10}$$

So the Lagrangian density is invariant under the discrete transformation $\phi \to -\phi$. Let's assume that there is a constant field ϕ that minimizes the action. Then the minimal action principle reduces to

$$\frac{dU}{d\phi} = \mu^2\phi + \frac{\lambda}{3!}\phi^3 = 0 \tag{1.11}$$

That is,

$$\phi\left(\mu^2 + \frac{\lambda}{6}\phi^2\right) = 0 \tag{1.12}$$

The roots would then depend on whether $\mu^2 > 0$ or $\mu^2 < 0$. If $\mu^2 > 0$ then

$$\phi = 0 \tag{1.13}$$

However if $\mu^2 < 0$, then

$$\phi = 0, \quad \phi = \pm\sqrt{\frac{-6\mu^2}{\lambda}} = \pm\nu \tag{1.14}$$

It is also good to note that when μ^2 >0 the solution is invariant under $\phi \to -\phi$, however when $\mu^2 < 0$ we note that the roots are not invariant under this transformation.

Now we use the model of a complex scalar field with Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - U(\phi^\dagger\phi) \tag{1.15}$$

where

$$U = \mu^2\phi^\dagger\phi + \frac{\lambda}{3!}(\phi^\dagger\phi)^3 \tag{1.16}$$

We use the U(1) global transformation with ω constant

$$\phi \to e^{i\omega}\phi, \;\; \phi^\dagger \to e^{-i\omega}\phi^\dagger \tag{1.17}$$

Under this transformation our Lagrangian density is invariant

$$\begin{aligned}\mathcal{L}' &= \partial_\mu e^{-\omega}\phi^\dagger\partial^\mu e^{i\omega}\phi \\ &- \left(\mu^2 e^{-i\omega}\phi^\dagger e^{i\omega}\phi + \frac{\lambda}{3!}\left(e^{-i\omega}\phi^\dagger e^{i\omega}\phi\right)^3\right) \\ &= \partial_\mu\phi^\dagger\partial^\mu\phi - (\mu^2\phi^\dagger\phi + \frac{\lambda}{3!}(\phi^\dagger\phi)^3)(\phi^\dagger\phi) = \mathcal{L} \end{aligned} \tag{1.18}$$

For ϕ= const the minimum solution for the Lagrangian density in Eq(1.15) is obtained from

$$\frac{\partial U}{\partial(\phi^\dagger\phi)} = \mu^2 + \frac{\lambda}{3!}(2\phi^\dagger\phi) = 0 \tag{1.19}$$

For $\mu^2 < 0$ the solution is given by

$$|\phi_0|^2 = -\frac{3\mu^2}{\lambda} \tag{1.20}$$

Where $\phi_0 =< \phi >$ is the vev.

Of course in this case there is a circle of degenerate minima , because the minimum solution fixes the magnitude of the complex field ϕ, but does not fix its phase. Then, we can write

$$\phi_0 = \frac{1}{\sqrt{2}}\nu e^{i\delta} \tag{1.21}$$

where δ is a constant phase, and $\nu = +(-\frac{6\mu^2}{\lambda})^{-1/2}$

For the field we can also write

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)e^{i\delta} \tag{1.22}$$

and the vev is the given by

$$< \phi >= \frac{1}{\sqrt{2}}(< \phi_1 > +i < \phi_2 >)e^{i\delta} = \phi_0 \tag{1.23}$$

Thus

$$< \phi_1 >= \nu, \quad < \phi_2 >= 0 \tag{1.24}$$

is a good choice for the vev of the component fields.

The Lagrangian density in terms of the fluctuation field with zero vev

$$\tilde{\phi} = \phi - < \phi > \tag{1.25}$$

is given as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}[(\partial_\mu \tilde{\phi}_1)(\partial^\mu \tilde{\phi}_1) + (\partial_\mu \tilde{\phi}_2)(\partial^\mu \tilde{\phi}_2) + \frac{3}{2}\mu^2 \tilde{\phi_1}^2] \\ &- \frac{1}{16}\lambda[(\tilde{\phi_1}^2) + (\tilde{\phi}_2)]^2 - \frac{1}{4}\lambda\nu\tilde{\phi}_1[(\tilde{\phi}_1)^2 + (\tilde{\phi}_2)^2] - 3\mu^2\nu^2 \end{aligned} \tag{1.26}$$

We see that the field $\tilde{\phi}_1$ has a positive mass squared $-\frac{3}{4}\mu^2$, while $\tilde{\phi}_2$ is massless. This massless mode, which arises from the degeneracy of the ground state after spontaneous symmetry breaking, are the so called "Goldstone bosons." Thus, we have that the Goldstone bosons are a consequence of the spontaneous breaking of a continuous global symmetry [11, 10].

1.4 Higgs Mechanism

Now we will consider the situation where a local gauge invariance is broken. We start with the Lagrangian of scalar electrodynamics.

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi^*) - \mu^2 \phi\phi^* - \frac{\lambda}{4}(\phi\phi^*)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.27}$$

where

$$\begin{aligned} D_\mu \phi &\equiv (\partial_\mu + iqA_\mu)\phi \\ D_\mu \phi^* &\equiv (\partial_\mu - iqA_\mu)\phi^* \end{aligned} \tag{1.28}$$

are the U(1) gauge covariant derivatives acting on the complex scalar field ϕ, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.29}$$

is the gauge invariant field tensor. When $\mu^2 > 0$ the U(1) symmetry is unbroken as we know from the analysis of the complex scalar field case. This doesn't apply to $\mu^2 < 0$ though. In this case the symmetry is broken and the vacuum expectation value becomes

$$\phi_c \quad = \quad < 0|\phi(x)|0 >= \frac{1}{\sqrt{2}}\nu e^{i\delta}, \quad \nu = +(-\frac{4\mu^2}{\lambda})^{1/2} \tag{1.30}$$

Defining the fluctuation field

$$\tilde{\phi} \quad = \quad \phi - \phi_c \tag{1.31}$$

and using $\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)e^{i\delta}$ we have that the covariant derivative is

$$\begin{aligned} D_\mu(\tilde{\phi} + \phi_c) &= D_\mu(\frac{e^{i\delta}}{\sqrt{2}}(\tilde{\phi}_1 + i\tilde{\phi}_2) + \frac{e^{i\delta}}{\sqrt{2}}\nu) \\ &= \frac{e^{i\delta}}{\sqrt{2}}(\partial_\mu\tilde{\phi}_1 + i(\partial_\mu\tilde{\phi}_2 + qA_\mu\nu) + iqA_\mu(\tilde{\phi}_1 + i\tilde{\phi}_2)) \end{aligned} \tag{1.32}$$

Aside from interaction terms, $\tilde{\phi}_2$ and A_μ happen to only enter in the Lagrangian in the combination

$$A'_\mu \equiv A_\mu + \frac{1}{q\nu}\partial_\mu\tilde{\phi}_2 \tag{1.33}$$

If we eliminate A_μ in terms of A'_μ in (1.19), we find that A'_μ is massive, with mass given by

$$M_{A'} = q\nu \tag{1.34}$$

Hence, the gauge field becomes massive as a consequence of spontaneous symmetry breaking ($\nu \neq 0$) and its interaction with the Goldstone field ϕ_2

We note that A'_μ may be obtained from A_μ in Eq. (1.27) by the gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda(x)$ with

$$\Lambda(x) = \frac{1}{q\nu}\tilde{\phi}_2(x) \tag{1.35}$$

Since in terms of the fluctuation fields

$$\phi = \frac{1}{\sqrt{2}}(\nu + \tilde{\phi}_1 + i\tilde{\phi}_2)e^{i\delta} \tag{1.36}$$

If we choose the gauge parameter as

$$q\Lambda = arctan\frac{\tilde{\phi}_2}{\nu + \tilde{\phi}_1} \tag{1.37}$$

Through the gauge transformation $\phi \rightarrow \phi' = e^{-iq\Lambda}\phi \equiv \frac{1}{\sqrt{2}}(\nu + \tilde{\phi}_1{}' + i\tilde{\phi}_2)e^{i\delta}$ we obtain

$$\tilde{\phi}_2{}' = 0 \tag{1.38}$$

Then, the gauge transformation can be written as

$$\phi(x) \rightarrow \phi'(x) = \frac{1}{\sqrt{2}}(\nu + H(x)) \tag{1.39}$$

with H = $\tilde{\phi}_1$. This causes for U(1) covariant derivative to be transformed as

$$D_\mu\phi \rightarrow (D_\mu\phi)' = \frac{e^{i\delta}}{\sqrt{2}}(\partial_\mu H + iq\nu A'_\mu + iqA'_\mu H) \tag{1.40}$$

In this gauge the Lagrangian density is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu H + iqA'_\mu(\nu + H))(\partial^\mu H - iqA'^\mu(\nu + H)) - \mu^2\frac{1}{2}(\nu + H)^2 \\
&- \frac{\lambda}{16}(\nu + H)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \frac{1}{2}(\partial_\mu H\partial^\mu H + q^2A'_\mu A'^\mu(\nu + H)^2) - \frac{\mu^2}{2}(\nu + H)^2 - \frac{\lambda}{16}(\nu + H)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
\end{aligned} \tag{1.41}$$

where $\mu^2 = -\frac{\nu^2\lambda}{4}$ as stated in Eq. (1.30) After we simply we obtain

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[(\partial_\mu H)(\partial^\mu H) + 3\mu^2H^2\right] - \frac{1}{4}\mu^2\nu^2 - \frac{\lambda}{16}(H^4 + 4\nu H^3) \\
&- \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}q^2A'_\mu A'^\mu(\nu^2 + 2\nu H + H^2)
\end{aligned} \tag{1.42}$$

The Goldstone model has been completely consumed by the transformed gauge boson A'_μ. The only remaining scalar field is the Higgs field, H_1, with a mass of $(-2\mu)^{\frac{1}{2}}$. We can check that the total number of degrees of freedom remains the same. As a consequence of the Higgs mechanism the gauge symmetry is broken because the gauge field A'_μ is massive and the scalar field H is real.

1.5 Chiral Symmetry Breaking

Let us discuss another important symmetry, which is present in many models of QFT. This is the so called chiral symmetry that appears in massless theories. Let's start from the Dirac Lagrangian for a fermion with mass m.

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{1.43}$$

We then set m=0 for the massless case the Lagrangian density Eq. (1.43) is invariant under the chiral transformation

$$\Psi' = e^{-i\alpha\gamma_5}\Psi \tag{1.44}$$

with α being a constant. That is, denoting $\hat{T} = e^{-i\alpha\gamma_5}$ we have

$$\mathcal{L}_{||} = \bar{\Psi}(i\partial\!\!\!/)\Psi \rightarrow \mathcal{L}'_{||} = \bar{\Psi}'(i\partial\!\!\!/)\Psi' = (\hat{T}\Psi)^{\dagger}\gamma^0(i\partial\!\!\!/)\hat{T}\Psi = \Psi^{\dagger}\hat{T}^{\dagger}\gamma^0(i\partial\!\!\!/)\hat{T}\Psi \tag{1.45}$$

At this point we make note of the relation.

$$\hat{T}\gamma^{\mu} = \gamma^{\mu}\hat{T}^{\dagger} \tag{1.46}$$

Going back to the Lagrangian we have

$$\mathcal{L}'_{||} = \Psi^{\dagger}\gamma^0\hat{T}(i\partial^{\mu}\gamma_{\mu})\hat{T}\Psi = \Psi^{\dagger}\gamma^0(i\partial\!\!\!/)\hat{T}^{\dagger}\hat{T}\Psi = \Psi^{\dagger}\gamma^0(i\partial\!\!\!/)\Psi = \mathcal{L} \tag{1.47}$$

Thus, $\mathcal{L}$ is invariant under chiral symmetry. But if we include a mass term in Eq. (1.43) we find that the chiral symmetry is broken because

$$\begin{aligned} \mathcal{L}_2 &= \bar{\Psi}(m)\Psi \rightarrow \mathcal{L}'_2 = \bar{\Psi}'(m)\Psi' = (\hat{T}\Psi)^{\dagger}m\hat{T}\Psi = \Psi^{\dagger}\hat{T}^{\dagger}\gamma^0 m\hat{T}\Psi \\ &= \Psi^{\dagger}\gamma^0\hat{T}m\hat{T}\Psi = m\bar{\Psi}\hat{T}\hat{T}\Psi \neq m\bar{\Psi}\Psi \end{aligned} \tag{1.48}$$

Therefore, there is an explicit chiral symmetry breaking due to the mass.

1.6 Feynman Propagator

First find the expectation value of the vacuum

$$\begin{aligned} < 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0 > &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(\not{p}+m)_{\alpha\beta}e^{ip\cdot(x-y)} \\ < 0|\bar{\psi}_\beta(x)\psi_\alpha(y)|0 > &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(\not{p}-m)_{\alpha\beta}e^{ip\cdot(x-y)} \end{aligned} \tag{1.49}$$

Now we define the Feynman propagator $S_F(x-y)$

$$S_F(x-y) =< 0|T(\psi(x)\bar{\psi}(y))|0 >\equiv \begin{cases} < 0|\psi(x)\bar{\psi}(y)|0 >, & x^0 > y^0 \\ < 0|-\bar{\psi}(y)\psi(x)|0 >, & y^0 > x^0 \end{cases} \tag{1.50}$$

First the symbol $T(x)$ means time ordered. The minus sign is necessary to make the two definitions agree outside the lightcone ($\{\psi(x), \bar{\psi}(y)\} = 0$)

The 4 momentum integral representation for the Feynman propagator is

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{\gamma\cdot p+m}{p^2-m^2+i\epsilon}, \tag{1.51}$$

which statisfies $(i\not{\partial}-m)S_F(x-y) = i\delta^4(x-y)$, showing that S_F is a Green's function for the Dirac operator.

We see in Eq. (1.50) that bosonic operators commute ($[\psi_1, \psi_2]$=0) and the fermionic operators anti-commute ($\{\psi_1, \psi_2\} = 0$)

1.6.1 Feynman Rules for Fermions

The Feynman Rules are a simplified way to calculate the amplitude of scattering that can also be illustrated in Feynman diagrams. These Feynman Diagrams in turn are a visual way to describe the behaivor of a particle or particles.

A fermion of spin r and momentum p is represented as $u^r(\vec{p})$ for incoming and $\bar{u}^r(\vec{p})$ for outgoing fermions

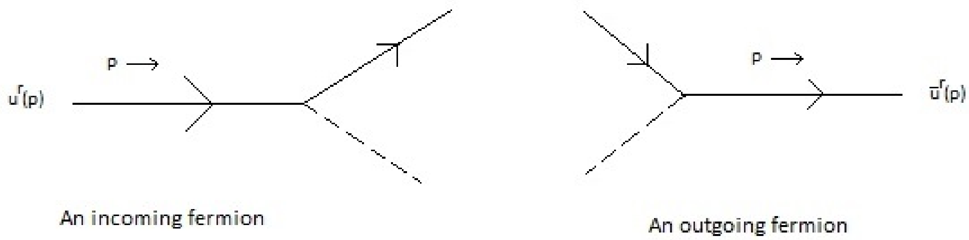

Figure 1.1: Image of incoming fermion (left) and outgoing fermion (right)

The anti-fermion with momentum p and spin r is represented as $\bar{v}^r(\vec{p})$ for incoming and $v^r(\vec{p})$ for outgoing anti-fermions

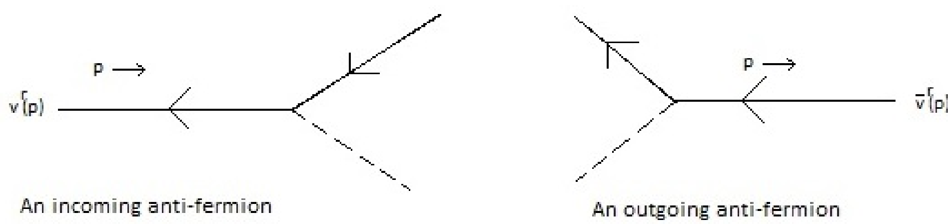

Figure 1.2: Image of incoming anti-fermion (left) and outgoing anti-fermion (right)

In Fig.1.1 and 1.2 we are representing the Feynman diagrams for some veritas there. There the dashed lines are for scalars and the solid lines are for fermions. The arrows of the fermion lines must flow consistently through the diagram. The matrix indices of the propagators are contracted at each vertex either with further propagators or external spinors.

1.6.2 Feyman Rules in QED

For QED the Feynman rules are that we write a vertex and internal lines as

$$\begin{aligned} Vertex &\rightarrow -ie\gamma^{\mu} \\ Photon\ propagator &\rightarrow -\frac{ig^{\mu\nu}}{p^2+i\epsilon} \\ Fermion\ propagator &\rightarrow \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} \end{aligned} \tag{1.52}$$

Internal lines are the propagators that are present inside of a closed loop in a Feynman diagram. External lines are the propagator lines that are on the outside of these loops.

For photons: a polarization vector ϵ^{μ}_{in} for incoming photons and ϵ^{μ}_{out} for outgoing photons is introduced,

For fermions: a spinor of $u^r(\vec{p})$ for incoming fermions an $\bar{u}^r(\vec{p})$ for outgoing fermions. Conversely we write a $\bar{v}^r(\vec{p})$ for incoming and $v^r(\vec{p})$ for outgoing anti-fermions.

$$\begin{aligned} iG_{\alpha\beta}(x,y) &= \sum_{\nu=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{\nu}\frac{1}{\nu!}\int_{-\infty}^{\infty}dt_1...\int_{-\infty}^{\infty}dt_{\nu} \\ &\times \frac{<\Phi_0|T[\hat{H}_1(t_1)...\hat{H}(t_{\nu})\hat{\psi}_{\alpha}(x)\hat{\psi}^{\dagger}_{\beta}(y)]|\Phi_0>}{<\Phi_0|\hat{S}|\Phi_0>} \end{aligned} \tag{1.53}$$

1.6.3 Fermion Propagator in the Presence of a Magnetic Field

We begin with the Green's equation for an electron in a constant magnetic field

$$[\gamma_{\mu}\Pi^{\mu}-\Sigma(x,x')]\,G(x,x')=\delta^4(x-x') \tag{1.54}$$

where $\Pi = i\partial_{\mu} - ieA_{\mu}$ and $\Sigma(x,x')$ is the fermion self-energy operator.

The self-energy operator $\Sigma(x,x')$ is a function of the operators $\gamma_{\mu}\Pi^{\mu}$, $\sigma_{\mu\nu}F^{\mu\nu}$, $(F^{\mu\nu}\Pi_{\mu})^2$ and $(\gamma_{\mu}\Pi^{\mu})^2$. $(\gamma_{\mu}\Pi^{\mu})^2$ commutes with all of the operators mentioned. So the eigenfunction of $(\gamma\cdot\pi)^2$ is a Gamma eigenfunction.

$$(\gamma\cdot\Pi)^2\mathbb{E}_p = \bar{p}^2\mathbb{E}_p \tag{1.55}$$

Where $\mathbb{E}_p$ is the eigenfunction matrices of $(\gamma_\mu \Pi^\mu)^2$. We also have that $(\gamma \cdot \pi)^2$ commute with γ_5 and $\Sigma_3 = i\gamma_1\gamma_2$.

When γ_5 and Σ_3 are diagonal with eigenvalues of ± 1, the eigenfunction $\mathbb{E}_p$ is given by

$$\mathbb{E}_p = \sum_{\sigma=\pm 1} E_{p\sigma}(x)\Delta(\sigma) \tag{1.56}$$

with the spin projectors given by

$$\Delta(\sigma) = \frac{I + i\sigma\gamma^1\gamma^2}{2} \tag{1.57}$$

and $E_{p\sigma}(x)$ is

$$E_{p\sigma}(x) = N_n e^{i(p_0 x^0 + p_2 x^2 + p_3 x^3)} D_n(\rho), \quad \rho = \sqrt{2|eH|}(x_1 - \frac{p_2}{eH}) \tag{1.58}$$

in an asymmetric gauge. $D_n(\rho)$ are the parabolic cylinder functions with the normalization factor

$$N_n = \frac{(4\pi eH)^{1/4}}{\sqrt{n!}}, \quad n = n(l,\sigma) \equiv l + \frac{\sigma + 1}{2} \tag{1.59}$$

The $\mathbb{E}_p$ functions form a complete set of functions,

$$\sum\!\!\!\!\!\!\int \frac{d^4p}{(2\pi)^4} \mathbb{E}_p(x)\bar{\mathbb{E}}_p(x') = \delta(x - x') \tag{1.60}$$

and satisfy the generalized orthogonal condition

$$\int d^4x \bar{\mathbb{E}}_p(x)\mathbb{E}_{p'}(x) = (2\pi)^4 \hat{\delta}^4(p - p')\sqcap_l \tag{1.61}$$

where

$$\begin{aligned} \bar{\mathbb{E}}_p(x) &= \gamma^0 \mathbb{E}_p^\dagger \gamma^0, \quad \sqcap_l = \delta^{0l}\Delta(+) + I(1 - \delta^{0l}) \\ \hat{\delta}^4(p - p') &= \delta^{ll'}\delta(p_0 - p_0')\delta(p_1 - p_1')\delta(p_3 - p_3') \end{aligned} \tag{1.62}$$

and

$$\sum\!\!\!\!\!\!\int \frac{d^4p}{(2\pi)^4} \equiv \sum_l \int \frac{dp_0 dp_1 dp_2}{(2\pi)^4} \tag{1.63}$$

From the spin structure of $\mathbb{E}_p$ we can say that

$$(\gamma \cdot \Pi)\mathbb{E}_p = \mathbb{E}_p(\gamma \cdot \bar{p}), \quad (Z_{||}\not{\Pi}_{||} + Z_{\perp}\not{\Pi}_{\perp})\mathbb{E}_p(x) = \mathbb{E}_p(\bar{\not{p}}_{||} + \bar{\not{p}}_{\perp}) \tag{1.64}$$

with

$$\bar{p}^{\mu} = \bar{p}^{\mu}_{||} + \bar{p}^{\mu}_{\perp}, \quad \bar{p}^{\mu}_{||} = (p_0, 0, 0, p_3), \quad \bar{p}^{\mu}_{\perp} = (0, 0, -\sqrt{2eHl}, 0) \tag{1.65}$$

$Z_{||}$ and $Z_{\perp}$ are the coefficients of the wave functions after renormalization.

The $\mathbb{E}_p$ functions diagonalize $\Sigma(p, p')$ giving us the equation

$$\Sigma(p, p') \equiv (2\pi)^4 \hat{\delta}^4(p - p') \sqcap_l \Sigma^l(\bar{p}) \tag{1.66}$$

Since we are using the full fermion propagator, which includes the anomalous magnetic moment (AMM) [2], we have

$$\sqcap_l \Sigma^l(\bar{p}) = Z^l_{||}\bar{\not{p}}_{||} + Z^l_{\perp}\bar{\not{p}}_{\perp} + (M^l + T^l)\Delta(+) + (M^l - T^l)\Delta(-) \tag{1.67}$$

with M^l being the fermion mass and T^l being the AMM. From Eq. (1.61) , Eq. (1.64), Eq. (1.65), and Eq. (1.67) we can get the fermion full propagator which takes into account AMM effects [2],

$$\tilde{G}^l(\bar{p}) = \sum_{\sigma,\bar{\sigma}} \frac{N^l(\sigma T, \bar{\sigma} V_{||}) - iV^l_{\perp}(\Lambda^+_{\perp} - \Lambda^-_{\perp})}{D^l(\sigma\bar{\sigma}T)} \Delta(\sigma)\Lambda^{\bar{\sigma}}_{||} \tag{1.68}$$

where

$$\begin{aligned}
\Lambda^{\sigma}_{||} = \frac{1}{2}\left(1 + \sigma\frac{\bar{\not{p}}_{||}}{|\bar{p}_{||}|}\right), &\qquad \Lambda^{\sigma}_{\perp} = \frac{1}{2}(1 + i\sigma\gamma^2), \\
V^l_{||} = (1 - Z^l_{||})|\bar{p}_{||}| &\qquad V^l_{\perp} = (1 - Z^l_{\perp})|\bar{p}_{\perp}| \\
N^l(\sigma T, \bar{\sigma} V_{||}) &= \sigma T^l - M^l - \bar{\sigma} V^l_{||} \\
D^l(\sigma T) &= (M^l)^2 - (V^l_{||} - \sigma T^l)^2 + (V^l_{\perp})^2
\end{aligned} \tag{1.69}$$

Since it will be used later, we should calculate the fermion propagator without AMM. This is done by setting $T^l = 0$ and performing the summations over σ and $\bar{\sigma}$. This means that

$$
\begin{aligned}
N^l(\sigma T, \bar{\sigma} V_{||}) &= -M^l - \bar{\sigma} V^l_{||} \\
D^l(\sigma T) &= (M^l)^2 - (V^l_{||})^2 + (V^l_{\perp})^2
\end{aligned} \tag{1.70}
$$

With this we write the Fermion propagator as

$$
\bar{G}^l(\bar{p}) = \sum_{\sigma,\bar{\sigma}=\pm 1} \frac{-M^l - \bar{\sigma}((1 - z^l_{||})|\bar{p}_{||}|) - i(1 - z^l_{\perp})|\bar{p}_{\perp}|(i\gamma^2)}{(M^l)^2 - (1 - z^l_{||})^2 \bar{p}^2_{||} + (1 - z^l_{\perp})^2} \Lambda^{\bar{\sigma}}_{||} \Delta(\sigma) \tag{1.71}
$$

After we perform the sum in σ and $\bar{\sigma}$ we get

$$
\bar{G}^l(\bar{p}) = \frac{(1 - z^l_{\perp})|\bar{p}_{\perp}|\gamma^2 - M^l - \frac{\bar{p}\!\!/_{||}}{|\bar{p}_{||}|}(1 - z^l_{||})|\bar{p}_{||}|}{(M^l)^2 - (1 - z^l_{||})|\bar{p}_{||}|^2 + (1 - z^l_{\perp})|\bar{p}_{\perp}|^2} \tag{1.72}
$$

Chapter 2

Photon Polarization Operator of a Charged e^--e^+ Plasma in a Strong Magnetic Field

In this chapter we calculate the Photon Polarization Operator (PPO) in the presence of a strong magnetic field for different media such as the QFT vacuum, the charged positron-electron plasma, and the neutral positron-electron plasma. Graphically the PPO is represented by the Feynmann diagram of Fig 2.1 where the wavy line represents the propagating photon and the straight line the virtual positron-electron pair. So, the first initial photon would be the one on the left. Then the photon annihilates to create a positron-electron pair. This process is allowed because of the Heisenberg Uncertainty Principle which allows for us to have an infinite energy in an infinitesimal amount of time. This virtual process is represented by the circular loop in the middle of the diagram. The straight lines in the loop represent fermion propagators. Afterward the positron-electron pair annihilates to create a photon again.

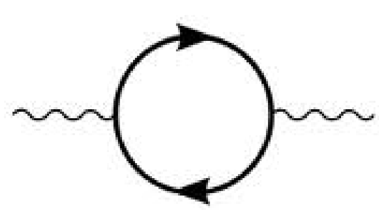

Figure 2.1: The photon polarization operator in the one-loop approximation

The process represented by this diagram accounts for the quantum corrections to the photon propagator due to its interaction with the medium, or even with the vacuum.

The PPO is important because it also can tell us about the electromagnetic properties of a medium. In this case the photon can be used as a probe to activate the medium response. For example we will use the PPO to calculate the electric susceptibility of the magnetized medium and the Debye screening. Thus, we can find the degree of polarization of such a medium (the fact that electrons and positrons are charged particles and can interact with the applied magnetic field may affect the electromagnetic properties of the medium). Also the Debye screening will give us information about the modification of the Coulomb interaction in the medium under consideration.

2.1 The One-loop PPO in a Strong Magnetic Field

In order to do the calculation of the PPO in the one-loop approximation, we must first use the Feynman rules to take the magnitude of the fermion full propagator. G(x,y) is the fermion full propagator magnitude in the presence of a constant magnetic field and depending on the anomalous magnetic moment and with the bare vertices represented by $(ie\gamma^{\mu})$. This gives us the equation

$$\Pi_{\mu\nu}(x,y) = -4\pi\alpha Tr\left[\gamma_{\mu} G(x,y)\gamma_{\nu} G(y,x)\right] \tag{2.1}$$

When we take Eq. (2.1) and do the transformation to momentum space we obtain [2],

$$G(x,x') = \sum\!\!\!\!\!\!\!\int \frac{d^4p}{(2\pi)^4}\mathbb{E}_p(x)\sqcap_l \tilde{G}^l(\bar{p})\bar{\mathbb{E}}_p(x'). \tag{2.2}$$

From Eq. (2.2) and the properties of the trace in Eq. (2.1) we have

$$\Pi_{\mu\nu}(x,y) = -4\pi\alpha \sum\!\!\!\!\!\!\!\int \frac{d^4d^4p'}{(2\pi)^8} Tr\left[\bar{\mathbb{E}}_{p'}(x)\gamma_{\mu}\mathbb{E}_p(x)\sqcap_l \tilde{G}^l(\bar{p})\bar{\mathbb{E}}_p(y)\gamma_{\nu}\mathbb{E}_{p'}(y)\sqcap_{l'} \tilde{G}^{l'}(\bar{p}')\right] \tag{2.3}$$

From [17] we know that

$$\int dye^{-iq'\cdot y}\bar{\mathbb{E}}_p(y)\gamma_{\nu}\mathbb{E}_{p'}(y) = (2\pi)^4\hat{\delta}^3(p'+q'-p)e^{-\frac{\hat{q}'^2}{2}}\sum_{\sigma,\sigma'}\frac{J_{nn'}(\hat{q}'_{\perp})e^{i(n-n')\phi}}{\sqrt{n!n'!}}\Delta(\sigma)\gamma_{\nu}\Delta(\sigma') \tag{2.4}$$

and that

$$\int dx e^{-iq'\cdot x}\bar{\mathbb{E}}_p(x)\gamma_\mu \mathbb{E}_{p'}(x) = (2\pi)^4\hat{\delta}^3(p'+q'-p)e^{-\frac{\hat{q}'^2}{2}}\sum_{\bar{\sigma},\bar{\sigma}'}\frac{J_{\bar{n}'\bar{n}}(\hat{q}'_\perp)e^{i(\bar{n}'-\bar{n})\phi}}{\sqrt{\bar{n}!\bar{n}'!}}\Delta(\bar{\sigma}')\gamma_\mu\Delta(\bar{\sigma}) \quad (2.5)$$

where $n = n(l,\sigma)$, $n' = n(l',\sigma')$, $\bar{n} = n(l,\bar{\sigma})$ and $\bar{n}' = n(l',\bar{\sigma}')$ with $n \equiv l + \frac{\sigma+1}{2}$ and

$$J_{nn'}(\hat{q}_\perp) \equiv \sum_{m=0}^{min(n,n')}\frac{n!n'!|i\hat{q}_\perp|^{n+n'-2m}}{m!(n-m)!(n'-m')!} \quad (2.6)$$

where $\hat{q}_\perp \equiv \frac{q_\perp}{2eH}$.

Since the photon has no interaction with the magnetic field we can do a Fourier transformation so that

$$\int d^4xd^4y\Pi_{\mu\nu}(x,y)e^{iq\cdot x-iq'\cdot y} \equiv (2\pi)^4\delta^4(q-q')\Pi_{\mu\nu}(q) \quad (2.7)$$

From Eq. (2.4), Eq. (2.5), and Eq. (2.7) we get for the PPO

$$\Pi_{\mu\nu}(q) = -2\alpha eHe^{-\hat{q}_\perp^2}\sum_{l,l'}\int\frac{d^2p}{(2\pi)^2}\sum_{\sigma,\sigma',\bar{\sigma},\bar{\sigma}'}\frac{e^{i(n-n'+\bar{n}'-\bar{n})\phi}}{\sqrt{n!n'!\bar{n}!\bar{n}'!}}J_{nn'}(\hat{q}_\perp)J_{\bar{n}'\bar{n}}(\hat{q}_\perp)$$
$$\times Tr\left[\Delta(\bar{\sigma}')\gamma_\mu\Delta(\bar{\sigma})\sqcap_l\tilde{G}^l(\bar{p})\Delta(\sigma)\gamma_\nu\Delta(\sigma')\sqcap_{l'}\tilde{G}^{l'}(p\bar{-}q)\right] \quad (2.8)$$

When we consider that in the strong-field limit only the terms of $J_{nn'}$ and $J_{\bar{n}'\bar{n}}$ with the smallest contribution in $\hat{q}$ contribute to the PPO because of the factor $e^{-\hat{q}_\perp}$ we have

$$J_{nn'}(\hat{q}'_\perp) \to n!\delta_{nn'} \quad \text{and} \quad J_{\overline{n}'\overline{n}} \to \overline{n}!\delta_{\overline{n}'\overline{n}}\,. \quad (2.9)$$

Using Eq. (2.9), Eq. (2.8) becomes

$$\Pi_{\mu\nu}(q) = -2\alpha eHe^{-\hat{q}_\perp^2}\sum_{l,l'}\int\frac{d^2p}{(2\pi)^2}\sum_{\sigma,\sigma',\bar{\sigma},\bar{\sigma}'}\delta_{nn'}\delta_{\overline{n}'\overline{n}}$$
$$\times\ Tr\left[\Delta(\overline{\sigma}')\gamma_\mu\Delta(\overline{\sigma})\sqcap_l\tilde{G}^l(\overline{p})\Delta(\sigma)\gamma_\nu\Delta(\sigma')\sqcap_{l'}\tilde{G}^{l'}(\overline{p-q})\right]. \quad (2.10)$$

After we perform the sums in $\sigma,\sigma',\bar{\sigma},\bar{\sigma}'$ we get

$$\Pi_{\mu\nu}(q) = -2\alpha eHe^{-\hat{q}_\perp^2}\sum_{ll'}\int\frac{d^2p}{(2\pi)^2}\left\{\delta_{ll'}Tr\left[\gamma_\mu^{||}\sqcap_l\tilde{G}^l(\overline{p})\gamma_\nu^{||}\sqcap_l\tilde{G}^l(\overline{p-q})\right]\right.$$
$$+\delta_{l'l+1}Tr\left[\gamma_\mu^\perp\Delta(+)\sqcap_l\tilde{G}^l(\overline{p})\gamma_\nu^\perp\Delta(-)\sqcap_{l'}\tilde{G}^{l'}(\overline{p-q})\right]$$
$$\left.+\ \delta_{l'+1l}Tr\left[\gamma_\mu^\perp\Delta(-)\sqcap_l\tilde{G}^l(\overline{p})\gamma_\nu^\perp\Delta(+)\sqcap_{l'}\tilde{G}^{l'}(\overline{p-q})\right]\right\}, \quad (2.11)$$

When we use the fact that in the low energy region, $\hat{q} \ll 1$, the only contributions come from the LLL, we have

$$\Pi_{\mu\nu} = -2i\alpha|eH|e^{-\frac{q_\perp^2}{2|eH|}} \int \frac{d^2p}{(2\pi)^2} Tr[\gamma_\mu^{||}\Delta(+)\tilde{G}^0(\overline{p})\gamma_\nu^{||}\Delta(+)\tilde{G}^0(\overline{p-q})r] \tag{2.12}$$

The first step in order to calculate Eq. (2.12) is to calculate the trace. The trace always appears when there is a closed line of fermions in a Feynman diagram.

From Eq. (1.72) and [2] we know that

$$\tilde{G}^0(\overline{p}) = \frac{\not{p}^{||} + m}{p_{||}^2 + m^2} \tag{2.13}$$

The trace in Eq. (2.12) is given by

$$Tr[\gamma_\mu^{||}\Delta(+)\tilde{G}^0(\overline{p})\gamma_\nu^{||}\Delta(+)\tilde{G}^0(\overline{p-q})r] = 2\frac{p_\mu^{||}(p-q)_\nu^{||} + p_\mu^{||}(p-q)_\nu^{||} - g_{\mu\nu}^{||}(p_{||}\cdot(p-q)_{||} - m^2)}{(p_{||}^2 - m^2)((p-q)_{||}^2 - m^2)} \tag{2.14}$$

Finally, by substituting Eq. (2.14) in Eq. (2.12), we find that the PPO in the strong-field limit is

$$\Pi_{\mu\nu} = -4i\alpha eHe^{-\hat{q}^2} \int \frac{d^2p}{(2\pi)^2} \frac{p_\mu^{||}(p-q)_\nu^{||} + p_\mu^{||}(p-q)_\nu^{||} - g_{\mu\nu}^{||}(p_{||}\cdot(p-q)_{||} - m^2)}{(p_{||}^2 - m^2)((p-q)_{||}^2 - m^2)} \tag{2.15}$$

To perform the Wick rotation we on Eq. (2.15) we get

$$p_0 \to ip_4 \Rightarrow \begin{cases} g_{\mu\nu} & \to & -\delta_{\mu\nu} \\ X_\mu Y^\mu = x_0y_0 - x_3y_3 & \to & -X_\mu Y_\mu = -(x_0y_o + x_3y_3) \end{cases} \tag{2.16}$$

This is introduced through the replacement

$$\int \frac{dp_4}{2\pi} \to \frac{1}{\beta}\sum_{p_4} p_4 = \frac{(2n+1)\pi}{\beta}; \quad n = 0, \pm 1, \pm 2... \tag{2.17}$$

with $\beta = \frac{1}{T}$ were T is the absolute temperature. The photon's external fourth momentum is quantized as $q_4 = \frac{2m\pi}{\beta}, m = 0, \pm 1, \pm 2.$

$$-\int \frac{dp_3}{p_4^2 - \vec{p}^2 - M^2} \to \int \frac{dp_4}{p_4^2 + \vec{p}^2 + M^2} \to 2\pi \sum_n \frac{1}{i\omega_n^2 + \vec{p} + M^2} \tag{2.18}$$

This allows for us to perform a summation over an infinite discrete set of frequencies instead of over a continuous spacial component.[30] [16]

2.2 The PPO in a Strongly Magnetized Positron-Electron Plasma

In order to take into account the effect of the medium, we introduce the Matsubara sum for the discrete fourth component of the fermion Euclidean momentum.

$$\Pi_{\mu\nu} = 4\alpha e H e^{-\hat{q}^2} \frac{1}{\beta} \sum_{n=\infty}^{\infty} \int \frac{dp_3}{(2\pi)^2} \frac{p_\mu^{||}(p-q)_\nu^{||} + p_\mu^{||}(p-q)_\nu^{||} - \delta_{\mu\nu}^{||}(p_{||}\cdot(p-q)_{||} + m^2)}{(p_{||}^2 + m^2)((p-q)_{||}^2 + m^2)} \quad (2.19)$$

The density of the medium is introduced by the chemical potential μ, which enters as a shift in the fermion momentum fourth component

$$p_\mu^{||} = (p_4 - i\mu, 0, 0, p_3), \qquad (p-q)_\mu^{||} = (p_4 - q_4 - i\mu, 0, 0, p_3) \quad (2.20)$$

Now, we perform the Matsubara summation in Eq. (2.19) as shown in Appendix A.

The standard procedure at finite temperature suggests that we might perform first the summation over the Matsubara frequencies and then the integration over the spatial momentum components.

To perform the Matsubara sum we work with the different Lorentz components of the PPO. The $\mu = \nu = 0$ component of Eq. (2.19) is given by

$$\Pi_{00} = \frac{4\alpha e H e^{-\hat{q}^2}}{\beta} \int \frac{dp_3}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \frac{(\omega_n - i\mu)(\omega_n - \omega_m - i\mu) - (p_3(p_3 - q_3) + m^2)}{(p_{||}^2 + m^2)((p-q)_{||}^2 + m^2)} \quad (2.21)$$

where we introduced the Matsubara frequencies, which for fermion is given by $p_4 = \omega_n = \frac{(2n+1)\pi}{\beta}, n = 0, \pm 1, \pm 2, ...$

This can be written as a combination of the sums Sum_1 and Sum_4 shown in Appendix A,

$$\Pi_{00} = \frac{4\alpha e H e^{-\hat{q}^2}}{\beta} \int \frac{dp_3}{(2\pi)^3} \left\{ Sum_4 - [p_3(p_3 - q_3) + m^2] Sum_1 \right\} \quad (2.22)$$

and it is explicitly given by

$$\Pi_{00} = \alpha e H e^{-\hat{q}_\perp^2} \int \frac{dp_3}{(2\pi)^3} \sum_{\sigma,\sigma'=\pm 1} \frac{[\sigma n_f(E_1+\sigma\mu) - \sigma' n_f(E_2+\sigma'\mu) - \frac{1}{2}(\sigma-\sigma')]}{-i\omega_m + \sigma' E_2 + \sigma E_1} \times$$
$$\times (1 + \sigma\sigma' \frac{p_3(p_3-q_3)+m^2}{E_1 E_2}) \tag{2.23}$$

where

$$E_1^2 = p_3^2 + m^2, \quad E_2^2 \equiv (p_3-q_3)^2 + m^2 \tag{2.24}$$

In Eq. (2.23) the terms proportional to n_f correspond to the medium contribution, while the others correspond to the vacuum contribution. By taking the analytic continuation $i\omega_m \to (1+i\eta)q_0 \equiv q_0$, we go back to the Minkowski space and Π_{00} becomes

$$\Pi_{00} = -\alpha e H e^{-\hat{q}_\perp^2} \int \frac{dp_3}{(2\pi)^3} \sum_{\sigma,\sigma'=\pm 1} \frac{[\sigma n_f(E_1+\sigma\mu) - \sigma' n_f(E_2+\sigma'\mu) - \frac{1}{2}(\sigma-\sigma')]}{-q_0 + \sigma' E_2 + \sigma E_1} \times$$
$$\times (1 + \sigma\sigma' \frac{p_3(p_3+q_3)+m^2}{E_1 E_2}) \tag{2.25}$$

The $\mu = \nu = 3$ component of the PPO is given from Eq. (2.19) by

$$\Pi_{33} = \frac{-4\alpha e H e^{-\hat{q}^2}}{\beta} \int \frac{dp_3}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(\omega_n - i\mu)(\omega_n - \omega_m - i\mu) - [p_3(p_3-q_3) - m^2]}{[(\omega_n - i\mu)^2 + E_1^2][(\omega_n - \omega_m - i\mu)^2 + E_2^2]} \tag{2.26}$$

Comparing Eq. (2.26) with Eq. (2.21) we note that Π_{33} becomes similar to Π_{00} as long as we multiply it by -1 and replace m^2 with $-m^2$. Therefore we obtain

$$\Pi_{00} = -\alpha e H e^{-\hat{q}_\perp^2} \int \frac{dp_3}{(2\pi)^3} \sum_{\sigma,\sigma'=\pm 1} \frac{[\sigma n_f(E_1+\sigma\mu) - \sigma' n_f(E_2+\sigma'\mu) - \frac{1}{2}(\sigma-\sigma')]}{-q_0 + \sigma' E_2 + \sigma E_1} \times$$
$$\times (1 + \sigma\sigma' \frac{p_3(p_3+q_3)-m^2}{E_1 E_2}) \tag{2.27}$$

where we have performed the analytic continuation $i\omega_n \to q_0$.

The case when $\mu = 0$ and $\nu = 3$ (and μ=3, ν=0) gives

$$\Pi_{03} = \Pi_{30} = \frac{4\alpha e H e^{-\hat{q}_\perp^2}}{\beta} \int \frac{dp_3}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(\omega_n - i\mu)(p_3-q_3) + p_3(\omega_n - \omega_m - i\mu)}{[(\omega_n - i\mu)^2 + E_1^2][(\omega_n - \omega_m - i\mu)^2 + E_2^2]} \tag{2.28}$$

Using the notation introduced in Appendix A, this can be written in terms of Sum2 and Sum3 as

$$\Pi_{03} = \frac{4\alpha eHe^{-\hat{q}^2}}{\beta}\int\frac{dp_3}{2\pi}[(p_3-q_3)Sum_2+p_3Sum_3] \tag{2.29}$$

After performing the Matsubara sum (see Appendix A for the details) we obtain

$$\begin{aligned}\Pi_{03} &= -i\alpha eHe^{-\hat{q}_\perp^2}\int\frac{dp_3}{2\pi}\sum_{\sigma,\sigma'=\pm1}\frac{1}{-q_0+\sigma' E_2-\sigma E_1}\\ &\times\left\{(\sigma\sigma' n_f(E_1+\sigma\mu)-n_f(E_2+\sigma'\mu))(p_3(1+\sigma\sigma')-q_3)+\frac{q_3}{2}(\sigma\sigma'-1)\right\}\end{aligned} \tag{2.30}$$

where we have performed the analytical continuation $i\omega_m \to q_0$.

Summarizing, the PPO components in vacuum, $\Pi^{vac}_{\mu,\nu}$, and in the medium, $\Pi^{med}_{\mu,\nu}$, are given respectively by

$$\Pi^{vac}_{00} = -2\alpha eHe^{-\hat{q}^2}\int\frac{dp_3}{2\pi}\left(1-\frac{p_3(p_3-q_3)+m^2}{E_1E_2}\right)\left(\frac{E_1+E_2}{q_0^2-(E_1+E_2)^2}\right) \tag{2.31}$$

$$\Pi^{vac}_{33} = 2\alpha eHe^{-\hat{q}^2}\int\frac{dp_3}{2\pi}\left(1-\frac{p_3(p_3-q_3)-m^2}{E_1E_2}\right)\left(\frac{E_1+E_2}{q_0^2-(E_1+E_2)^2}\right) \tag{2.32}$$

$$\Pi^{vac}_{30} = \Pi^{vac}_{03} = 2\alpha ieHe^{-\hat{q}^2}\int\frac{dp_3}{2\pi}\left(\frac{(p_3-q_3)E_1-p_3E_2}{E_1E_2}\right)\left(\frac{q_0}{q_0^2-(E_1+E_2)^2}\right) \tag{2.33}$$

$$\begin{aligned}\Pi^{med}_{00} &= 4\alpha eHe^{-\hat{q}^2}\int\frac{dp_3}{2\pi}\\ \times &\left\{\frac{1}{4}\left[\frac{n_f(E_1+\mu)-n_f(E_2+\mu)}{-q_0+E_2-E_1}-\frac{n_f(E_1-\mu)-n_f(E_2-\mu)}{-q_0-E_2+E_1}\right]\left(1+\frac{p_3(p_3-q_3)+m^2}{E_1E_2}\right)\right.\\ + &\left.\frac{1}{4}\left[\frac{n_f(E_1+\mu)+n_f(E_2-\mu)}{-q_0-E_2-E_1}-\frac{n_f(E_1-\mu)+n_f(E_2+\mu)}{-q_0+E_2+E_1}\right]\left(1-\frac{p_3(p_3-q_3)+m^2}{E_1E_2}\right)\right\}\end{aligned} \tag{2.34}$$

$$\begin{aligned}\Pi^{med}_{33} &= 4\alpha eHe^{-\hat{q}^2}\int\frac{dp_3}{2\pi}\\ \times &\left\{-\frac{1}{4}\left[\frac{n_f(E_1+\mu)-n_f(E_2+\mu)}{-q_0+E_2-E_1}-\frac{n_f(E_1-\mu)-n_f(E_2-\mu)}{-q_0-E_2+E_1}\right]\left(1+\frac{p_3(p_3-q_3)-m^2}{E_1E_2}\right)\right.\\ - &\left.\frac{1}{4}\left[\frac{n_f(E_1+\mu)+n_f(E_2-\mu)}{-q_0-E_2-E_1}-\frac{n_f(E_1-\mu)+n_f(E_2+\mu)}{-q_0+E_2+E_1}\right]\left(1-\frac{p_3(p_3-q_3)-m^2}{E_1E_2}\right)\right\}\end{aligned} \tag{2.35}$$

$$\begin{aligned}
\Pi_{30}^{med} &= \Pi_{03}^{med} = 4\alpha eHe^{-\hat{q}^2}\int\frac{dp_3}{2\pi} \\
\times &\left\{\frac{1}{4i}\left[\frac{n_f(E_1+\mu)-n_f(E_2+\mu)}{-q_0+E_2-E_1}+\frac{n_f(E_1-\mu)-n_f(E_2-\mu)}{-q_0-E_2+E_1}\right]\left(\frac{(p_3-q_3)E_1+p_3E_2}{E_1E_2}\right)\right. \\
+ &\left.\frac{1}{4i}\left[\frac{n_f(E_1+\mu)+n_f(E_2-\mu)}{-q_0-E_2-E_1}+\frac{n_f(E_1-\mu)+n_f(E_2+\mu)}{-q_0+E_2+E_1}\right]\left(\frac{-(p_3-q_3)E_1+p_3E_2}{E_1E_2}\right)\right\}
\end{aligned} \tag{2.36}$$

2.3 Vacuum Polarization Tensor in the Infrared Limit

We will now compare the results that we calculated for the PPO in vacuum with those that are already known [29][1].

First we note that the only combinations of μ and ν in the PPO are $\Pi_{00},\ \Pi_{03} = \Pi_{30}$, and Π_{33}.

Starting from Π_{00} of Eq. (2.31), we explore its infrared limit ($q_0 = 0, q_3 \to 0$). The first step in line with this goal is that we set $q_0 = 0$ in the above equation to get

$$\Pi_{00}(q_0 = 0, q_3) = -2\alpha eHe^{\hat{q}^2}\int\frac{dp_3}{2\pi}\left(\frac{q_3^2-(E_1-E_2)^2}{2E_1E_2}\right)\left(\frac{1}{E_1+E_2}\right) \tag{2.37}$$

where we have used $2p_3q_3 = E_1^2 - E_2^2 + q_3^2$. Note that the integration over p_3 can be performed analytically and the result is

$$\Pi_{00}(q_0 = 0, q_3) = 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}}\frac{1}{4\pi}\left[2-\frac{4m^2}{q_3\sqrt{q_3^2+4m^2}}\ln\left(\frac{\sqrt{q_3^2+4m^2}+q_3}{\sqrt{q_3^2+4m^2}-q_3}\right)\right] \tag{2.38}$$

Next, we take the limit $q_3 \to 0$, and perform a Taylor expansion up to $\mathcal{O}(q_3^2)$ in Eq. (2.38). So, Π_{00} in the static limit has the form

$$\Pi_{00}(q_0 = 0, q_3 \to 0) = 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}}\frac{1}{4\pi}\left(\frac{q_3^2}{3m^2}\right) \tag{2.39}$$

The use of dimensional regularization preserves the transversality of $\Pi_{\mu\nu}$ and we have that Π_{33}=0 when q_0=0 [6].

2.4 Infrared Polarization Tensor for the Neutral e^--e^+ Plasma

To help us move forward let us consider the PPO for a neutral plasma. This will provide us with an idea of what we should expect for the more complicated case of a charged plasma.

For a neutral plasma we should take $\mu = 0$ in the medium contribution to the PPO Lorentz components. Then, taking $\mu = 0$ in Eq. (2.34) and static limit ($q_0 = 0$)

$$\begin{aligned}\Pi_{00}^{med}(q_0 = 0) \;\; &= \;\; 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}} \int_{-\infty}^{\infty} \frac{dp_3}{4\pi} \left\{ -\left(1 + \frac{p_3(p_3 - q_3) + m^2}{E_1 E_2}\right) \left[\frac{n_f(E_2) - n_f(E_1)}{E_2 - E_1}\right] \right. \\ &\left. - \left(1 - \frac{p_3(p_3 - q_3) + m^2}{E_1 E_2}\right) \left[\frac{n_f(E_1) + n_f(E_2)}{E_2 + E_1}\right] \right\} \end{aligned} \tag{2.40}$$

Now, we take the limit $q_3 \to 0$ in Eq. (2.40), we are going to perform a Taylor expansion for E_2 and for the Fermi-Dirac distribution $n_f(E_2)$ up to $\mathcal{O}(q_3^2)$ to obtain

$$\begin{aligned} E_2 \;\; &\approx \;\; E_1 - \frac{p_3 q_3}{E_1} + \frac{m^2 q_3^2}{E_1^3 2!} + \frac{m^2 p_3 q_3^3}{E_1^5 2!} \\ n_f(E_2) \;\; &\approx \;\; n_f(E_1) - \frac{p_3 q_3}{E_1} \frac{\partial n_f(E_1)}{\partial E_1} + \frac{1}{2!} \frac{m^2}{E_1^2} \left[-\frac{1}{E_1} \frac{\partial n_f(E_1)}{\partial E_1} + \frac{p_3^2}{m^2} \frac{\partial^2 n_f(E_1)}{\partial E_1^2} \right] q_3^2 \\ &+ \frac{1}{3!} \frac{p_3}{E_1^3} \left[\frac{3m^2}{E_1^2} \frac{\partial n_f(E_1)}{\partial E_1} - \frac{3m^2}{E_1} \frac{\partial^2 n_f(E_1)}{\partial E_1^2} - p_3^2 \frac{\partial^3 n_f(E_1)}{\partial E_1^3} \right] q_3^3 \end{aligned} \tag{2.41}$$

Note that there is no infrared divergence in Eq. (2.40). Once we take into account the previous approximations in Eq. (2.41), we expand Π_{00}^{med} up to $\mathcal{O}(q_3^2)$ and obtain

$$\begin{aligned} \Pi_{00}^{med}(q_0 = 0) \;\; &\approx \;\; 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}} \times \int_{-\infty}^{\infty} \frac{dp_3}{4\pi} \left\{ -2 \frac{\partial n_f(E_1)}{\partial E_1} \right. \\ &+ \left. \frac{1}{2} \frac{q_3^2}{E_1^2} \left[-\frac{m^2}{E_1^2} \frac{n_f(E_1)}{E_1} + \frac{m^2}{E_1^2} \frac{\partial n_f(E_1)}{\partial E_1} - \frac{m^2}{E_1} \frac{\partial^2 n_f(E_1)}{\partial E_1^2} - \frac{2p_3^2}{3} \frac{\partial^3 n_f(E_1)}{\partial E_1^3} \right] \right\} \end{aligned} \tag{2.42}$$

where we have dropped out the linear term in p_3 because the integrand is odd and the integral has symmetric limits, so it vanishes. Note that the second term in the above equation can be related to the susceptibility [6].

For π_{33}, we apply the same procedure, in order to analyze the static limit. We first set $q_0 = 0$ in Eq. (2.35)

$$\Pi_{33}^{med}(q_0 = 0) = 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}} \int_{-\infty}^{\infty} \frac{dp_3}{4\pi} \left\{ \left(1 + \frac{p_3(p_3 - q_3) - m^2}{E_1 E_2}\right) \frac{n_f(E_2) - n_f(E_1)}{E_2 - E_1} + \left(1 - \frac{p_3(p_3 - q_3) - m^2}{E_1 E_2}\right) \frac{n_f(E_1) + n_f(E_2)}{E_2 + E_1} \right\} \quad (2.43)$$

Therefore, performing the change of variable $p_3 = -(p_3' - q_3) \Rightarrow (E_1 \leftrightarrow E_2)$ in Eq. (2.43), only for those terms which involve the Fermi-Dirac distribution $n_f(E_2)$, gives us the equation

$$\begin{aligned} \Pi_{33}^{med}(q_0 = 0) &= 4\alpha eHe^{-\hat{q}_\perp^2} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} n_f(E_1) \left[\frac{2p_3}{q_3 E_1}\right] \\ &= 0, \end{aligned} \quad (2.44)$$

where in the last line we have used the following formula

$$E_1^2 - E_2^2 = -q_3^2 + 2p_3 q_3 = q_3(2p_3 - q_3) \quad (2.45)$$

Thus Π_{33} in the static (q_0=0) limit is zero.

Finally, it is easy to see that in the static limit the components 30 and 03 become null

$$\Pi_{03}^{med} = \Pi_{30}^{med} = 0 \quad (2.46)$$

2.4.1 Electric Susceptibility

From the term proportional to q_3^2 in Eq. (2.42) we get the electric susceptibility and from the other term we obtain the Debye mass. Note that when we take $q_3 = 0$ only the first term in the integrand remains. This term is precisely the Debye mass.

From Eq. (2.39) and from Eq. (2.42), the electric susceptibility is found as

$$\frac{\chi}{\chi_0} = 1 + \int_{-\infty}^{\infty} dp_3 \frac{3m^2}{4E_1^2} \left[-\frac{m^2}{E_1^2} \frac{n_f(E_1)}{E_1} + \frac{m^2}{E_1^2} \frac{\partial n_f(E_1)}{\partial E_1} - \frac{m^2}{E_1^2} \frac{\partial^2 n_f(E_1)}{(\partial E_1)^2} - \frac{2p_3^2}{3} \frac{\partial^3 n_f(E_1)}{(\partial E_1)^3} \right] \quad (2.47)$$

where

$$\chi_0 = \frac{\alpha eH}{3\pi m^2} e^{-\frac{q_\perp^2}{2|eH|}} \quad (2.48)$$

with $\beta = \frac{1}{T}$ and μ is the chemical potential. We plot χ/χ_0 as a function of β at zero chemical potential $\mu = 0$.

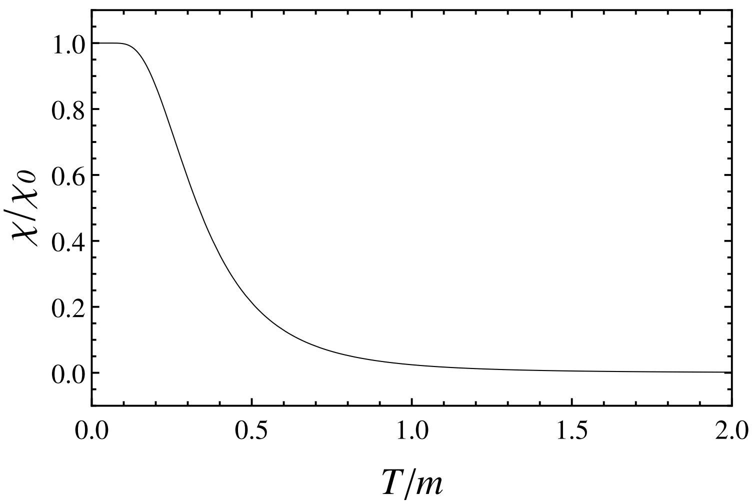

Figure 2.2: Electric Susceptibility versus Temperature

Note that increasing the temperature the electric susceptibility decreases from its zero value, χ_0, to zero at T≥M.

2.4.2 High Temperature Limit

The effect of the magnetic field on the vacuum is to orient the particle's dipole moment along its direction. In vacuum we have two physical scales: the fermion mass m and the magnetic field $\frac{1}{\sqrt{eH}}$, which in the strong magnetic field limit has the hierarchy of $m^2 \ll eH$. The medium provides us with the additional scale T, with T being the medium temperature. Since we declared that these calculations are in a strong magnetic field we must assume that ($m^2 \ll eH$ and $T^2 \ll eH$).

That leaves us with two possible regions $m \ll T$ and $T \ll m$. With the former being the high-temperature limit and the later being the low temperature limit of the strongly magnetized system. So we analyze both of those possibilities here starting with the high-temperature limit. The zero temperature limit being somewhat trivial as the all medium contributions becomes zero, in the neutral case.

Let us start this subsection by noticing that in Eq. (2.42) the derivatives with respect to the energy can be replaced by derivatives with respect to β, in the form

$$\frac{\partial n_f(E_1)}{\partial E_1} = \frac{\beta}{E_1}\frac{\partial n_f(E_1)}{\partial \beta} \tag{2.49}$$

and in general

$$\frac{\partial^k n_f(E_1)}{\partial E_1^k} = \left(\frac{\beta}{E_1}\right)^k \frac{\partial^k n_f(E_1)}{\partial \beta^k} \tag{2.50}$$

So, using Eq. (2.50) we rewrite Eq. (2.42), as

$$\begin{aligned}\Pi_{00}^{med}(q_0=0) &= 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}}\int_{-\infty}^{\infty}\frac{dp_3}{4\pi}\left\{-2\beta\frac{\partial}{\partial\beta}\frac{n_f(E_1)}{E_1}+\frac{1}{2}\frac{q_3^2}{E_1^2}\left[\left(-\frac{2\beta^3}{3}\frac{\partial^3}{\partial\beta^3}\right)\frac{n_f(E_1)}{E_1}\right.\right.\\ &+ \left.\left.\left(-\frac{1}{2}+\frac{1}{2}\beta\frac{\partial}{\partial\beta}-\frac{1}{2}\beta^2\frac{\partial^2}{\partial\beta^2}+\frac{\beta^3}{3}\frac{\partial^3}{\partial\beta^3}\right)\frac{2m^2 n_f(E_1)}{E_1^3}\right]\right\}\end{aligned} \tag{2.51}$$

Note that there are several integrals with the same structure, therefore we introducing the notation

$$I_\alpha = \int_{-\infty}^{\infty} dp_3 \frac{n_f(E)}{E^{\alpha+1}} \tag{2.52}$$

we rewrite Eq. (2.51) as follows

$$\begin{aligned}\Pi_{00}^{med}(q_0=0,q_3\to 0) &= 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}}\frac{1}{4\pi}\left\{-2\beta\frac{\partial}{\partial\beta}I_0+\frac{q_3^2}{2}\left[\left(-\frac{2\beta^3}{3}\frac{\partial^3}{\partial\beta^3}\right)I_2\right.\right.\\ &+ \left.\left.\left(-\frac{1}{2}+\frac{1}{2}\beta\frac{\partial}{\partial\beta}-\frac{1}{2}\beta^2\frac{\partial^2}{\partial\beta^2}+\frac{\beta^3}{3}\frac{\partial^3}{\partial\beta^3}\right)2m^2 I_4\right]\right\}\end{aligned} \tag{2.53}$$

We calculate each I_α in the high temperature limit. Thus, Eq. (2.53) in the high temperature limit looks

$$\begin{aligned}\Pi_{00}^{med}(q_0=0,q_3\to 0,\beta\to 0) = 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}}\frac{1}{4\pi}\times\\ \times\ \left\{2-\frac{2m^2\beta^2}{(2\pi)^2}\zeta\left(3,\frac{1}{2}\right)+\frac{q_3^2}{2m^2}\left[-\frac{2}{3}+\frac{\beta^4m^4}{(2\pi)^4}\zeta\left(5,\frac{1}{2}\right)\right]\right\}\end{aligned} \tag{2.54}$$

where we kept terms up to $\mathcal{O}((\beta m)^4)$ in the term proportional to q_3^2, and ζ(a,b) is the generalized Reimann zeta function.

When we combine this result with the one from the vacuum we obtain

$$\Pi_{00}(q_0 = 0, q_3 \to 0, \beta \to 0) = 4\alpha e H e^{-\frac{q_\perp^2}{2|eH|}} \frac{1}{4\pi} \times$$
$$\times \quad \left\{ \left(\frac{q_3^2}{3m^2} \right) + 2 - \frac{2m^2\beta^2}{(2\pi)^2} \zeta\left(3, \frac{1}{2}\right) + \frac{q_3^2}{2m^2} \left[-\frac{2}{3} + \frac{\beta^4 m^4}{(2\pi)^4} \zeta\left(5, \frac{1}{2}\right) \right] \right\} \tag{2.55}$$

From where the electric susceptibility is found as

$$\frac{\chi}{\chi_0} = \frac{3\beta^4 m^4}{2(2\pi)^4} \zeta\left(5, \frac{1}{2}\right) \tag{2.56}$$

We see that the electric susceptibility of the system decreases as the tempearture increases. This is in agreement with the numerical result reported in Fig 2.2. This means that the temperature acts against the electric susceptibility of the system. The temperature increases the system entropy destroying the polarization of the electric dipoles.

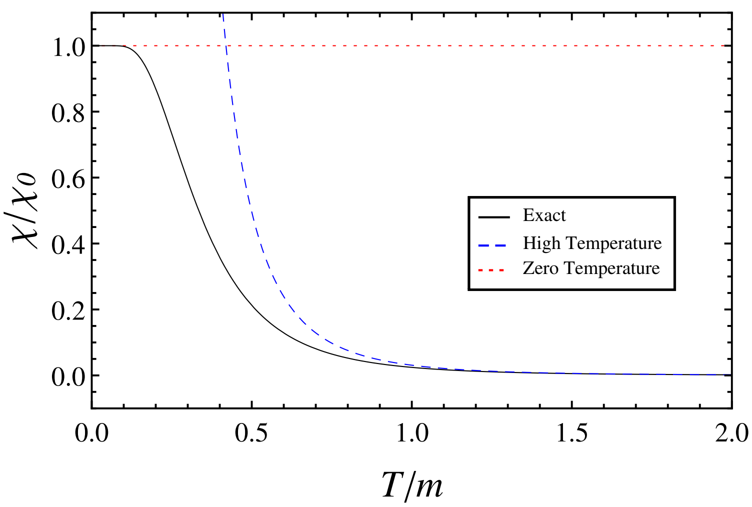

Figure 2.3: Graph of the Electrical Susceptibility as a function of temperature

2.5 Polarization operator of the Charged e^--e^+ Plasma

2.5.1 Infrared Limit

As in the case of a neutral plasma, in this section we shall analyze each component of the PPO in the infrared limit q_0=0, and $\vec{q_3} \to 0$ and keeping $\mu \neq 0$.

We start this analysis by setting $q_0 = 0$ in the PPO given in Eq. (2.34)

$$\begin{aligned}\Pi_{00}(q_0 = 0) &= 4\alpha e H e^{-\hat{q}_\perp^2} \int_{-\infty}^{\infty} \frac{dp_3}{8\pi} \left\{ -\left(1 + \frac{p_3(p_3 - q_3) + m^2}{E_1 E_2}\right) \left[\frac{N_f(E_2) - N_f(E_1)}{E_2 - E_1}\right] \right. \\ &\left. - \left(1 - \frac{p_3(p_3 - q_3) + m^2}{E_1 E_2}\right) \left[\frac{N_f(E_2) + N_f(E_1)}{E_2 + E_1}\right] \right\} \end{aligned} \quad (2.57)$$

where to simplify the equation we introduced the notation

$$N_f(x) \equiv n_f(x + \mu) + n_f(x - \mu) \quad (2.58)$$

Expanding q_3 up to $\mathcal{O}(q_3^2)$ and using the expansion Eq. (2.41), we get

$$\begin{aligned}\Pi_{00}^{med}(q_0 = 0) &\approx 4\alpha e H e^{-\frac{q_\perp^2}{2|eH|}} \int_{-\infty}^{\infty} \frac{dp_3}{8\pi} \left\{ -2\frac{\partial N_f(E_1)}{\partial E_1} \right. \\ &+ \left. \frac{1}{2}\frac{q_3^2}{E_1^2} \left[-\frac{m^2}{E_1^2}\frac{N_f(E_1)}{E_1} + \frac{m^2}{E_1^2}\frac{\partial N_f(E_1)}{\partial E_1} - \frac{m^2}{E_1}\frac{\partial^2 N_f(E_1)}{\partial E_1^2} - \frac{2p_3^2}{3}\frac{\partial^3 N_f(E_1)}{\partial E_1^3} \right] \right\} \end{aligned} \quad (2.59)$$

Now, for the $\mu = \nu = 3$ component of the PPO given in Eq. (2.34) after the $q_0 = 0$ limit, we obtain

$$\begin{aligned}\Pi_{33}^{med}(q_0 = 0) &= 4\alpha e H e^{-\hat{q}_\perp^2} \int_{-\infty}^{\infty} \frac{dp_3}{8\pi} \left\{ \left(1 + \frac{p_3(p_3 - q_3) - m^2}{E_1 E_2}\right) \left[\frac{N_f(E_2) - N_f(E_1)}{E_2 - E_1}\right] \right. \\ &\left. + \left(1 - \frac{p_3(p_3 - q_3) - m^2}{E_1 E_2}\right) \left[\frac{N_f(E_1) + N_f(E_2)}{E_1 + E_2}\right] \right\} \end{aligned} \quad (2.60)$$

Notice that this last equation has the same structure as Eq. (2.43). We can make the change of variable $p_3 = -(p_3' - q_3)$ ($\Rightarrow E_1 \leftrightarrow E_2$) in those terms which involve E_2 and expand in powers of q_3 up to $\mathcal{O}(q_3^2)$ to obtain

$$\begin{aligned}\Pi_{33}^{med}(q_0 = 0) &= 4\alpha e H e^{-\hat{q}_\perp^2} \int_{-\infty}^{\infty} \frac{dp_3}{4\pi} N_f(E_1) \left[\frac{2p_3}{q_3 E_1}\right] \\ &= 0 \end{aligned} \quad (2.61)$$

where we followed the steps that we show in Eq. (2.44).

Finally for the $\mu = 3$, $\nu = 0$ component of the PPO we get from Eq. (2.34) after setting $q_0 = 0$

$$\Pi_{30}^{med}(q_0 = 0) = 0 \tag{2.62}$$

In a similar fashion, we also have that $\Pi_{03}^{med}(q_0 = 0, q_3 \to 0) = 0$.

2.5.2 Zero-temperature Limit

We should mention that this limit is meaningful for astrophysical applications. Neutron stars have a very large density compared to their temperature. This means that effectively we could use a zero-temperature limit approximation for this case.

In order to simplify the analysis in the zero temperature limit, we rewrite dp_3 in terms of dE_1 as $dp_3 = \frac{E_1 dE_1}{\sqrt{E_1^2 - m^2}}$ in Eq. (2.42), Then we write

$$\begin{aligned}\Pi_{00}^{med} &= 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}} \int_m^\infty \frac{dE_1}{4\pi} \frac{E_1}{\sqrt{E1^2 - m^2}} \left\{ -2\frac{\partial N_f(E_1)}{\partial E_1} + \frac{1}{2}\frac{q_3^2}{E_1^2}\left[-\frac{m^2}{E_1^2}\frac{N_f(E_1)}{E_1} \right.\right. \\ &+ \left.\left. \frac{m^2}{E_1^2}\frac{\partial N_f(E_1)}{\partial E_1} - \frac{m^2}{E_1}\frac{\partial^2 N_f(E_1)}{\partial E_1^2} - \frac{2(E_1^2 - m^2)}{3}\frac{\partial^3 N_f(E_1)}{\partial E_1^3} \right]\right\} \end{aligned} \tag{2.63}$$

Now, taking into account that in the zero temperature limit, ($\beta \to \infty$), the function $N_f(E_1)$ becomes

$$N_f(E_1) = \Theta(\mu - E_1) \tag{2.64}$$

with $\Theta(x)$ the Heaviside step function, we have that Eq. (2.63) in the zero-temperature limit turns out to be of the form

$$\begin{aligned}\Pi_{00}^{med}(q_0 = 0, \beta \to \infty) &= 4\alpha eHe^{-\frac{q_\perp^2}{2|eH|}} \int_m^\mu \frac{dE_1}{4\pi} \frac{E_1}{\sqrt{E_1^2 - m^2}} \left\{ 2\delta(\mu - E_1) \right. \\ &\left. + \frac{1}{2}\frac{q_3^2}{E_1^2}\left[-\frac{m^2}{E_1^3}\Theta(\mu - E_1) - \frac{m^2}{E_1^2}\delta(\mu - E_1) + \frac{m^2}{E_1}\frac{\partial\delta(\mu - E_1)}{\partial E_1} + \frac{2(E_1^2 - m^2)}{3}\frac{\partial^2\delta(\mu - E_1)}{\partial E_1^2} \right]\right\} \end{aligned} \tag{2.65}$$

Next, using the relation

$$\int dx f(x)\frac{\partial^n \delta(x)}{\partial x^n} = (-1)^n \frac{\partial^n f(x)}{\partial x^n}\bigg|_{x=0} \tag{2.66}$$

and bearing in mind that the dirac delta function($\delta(x)$) is an even function, Eq. (2.65) becomes

$$\begin{aligned}
&\Pi_{00}^{med}(q_0 = 0, \beta \to \infty) \\
&\quad = \frac{\alpha e H}{\pi} e^{-\frac{q_\perp^2}{2|eH|}} \left\{ \frac{2\mu}{\sqrt{\mu^2 - m^2}} + \frac{q_3^2}{2}\left[\frac{3\mu m^2 - 2\mu^3}{3m^2(\mu^2 - m^2)^{\frac{3}{2}}}\right]\right\}\Theta(\mu - m) \\
&\quad = \frac{\alpha e H}{\pi} e^{-\hat{q}^2} \left\{ \frac{2\mu}{\sqrt{\mu^2 - m^2}} + \frac{q_3^2}{2}\left[\frac{3\mu m^2}{3m^2(\mu^2 - m^2)^{\frac{3}{2}}} - \frac{2\mu^3}{3m^2(\mu^2 - m^2)^{\frac{3}{2}}}\right]\right\}\Theta(\mu - m)
\end{aligned} \tag{2.67}$$

This result is an exact result in which we only used the assumption $\mu > m$.

If we consider the extreme scenario $\mu \gg m$, then Eq. (2.67) reduces. Therefore we need take the taylor expansion of the first term to the second order. Then we take the taylor expansion of the second term to the second order. After this is done we recieve the following

$$\Pi_{00}^{med} \approx \frac{\alpha e H}{\pi} e^{-\frac{q_\perp^2}{2|eH|}} \left\{ 2 + \frac{2m}{\mu} + + \frac{q_3^2}{2}\left[-\frac{2}{3m^2} + \frac{3m^2}{2\mu^4} + \frac{1}{\mu^2}\right]\right\} \tag{2.68}$$

We then combine Eq. (2.68) with the vacuum term in Eq. (2.39) giving us

$$\begin{aligned}
\Pi_{00} &= \Pi_{00}^{vac} + \Pi_{00}^{med} \\
\Pi_{00} &= \frac{\alpha e H}{\pi} e^{-\hat{q}_\perp^2} \left\{ 2 + \frac{2m}{\mu} + \frac{q_3^2}{2}\left[\frac{1}{\mu^2} + \frac{3m^2}{2\mu^4}\right]\right\}
\end{aligned} \tag{2.69}$$

The medium contribution to the electrical susceptibility is obtained from the terms proportional to q_3^2 in Eq. (2.67). Thus, the total susceptibility normalized with respect to the vacuum susceptibility reads.

$$\begin{aligned}
\frac{\chi}{\chi_0} &= 1 + \frac{3\mu m^2 - 2\mu^3}{2(\mu^2 - m^2)^{\frac{3}{2}}}\Theta(\mu - m), \\
or &= \frac{3m^2}{2}\left[\frac{3m^2 + \mu^2}{2\mu^4}\right]
\end{aligned} \tag{2.70}$$

in which the high-density limit ($\mu \gg m$) gives

$$\frac{\chi}{\chi_0} = \frac{3}{8}\frac{m^4}{\mu^4} \tag{2.71}$$

The graphical representation of the electric susceptibilities in the zero-temperature limit of Eq. (2.70) and Eq. (2.71) are given in Fig 2.4 as a function of the chemical potential.

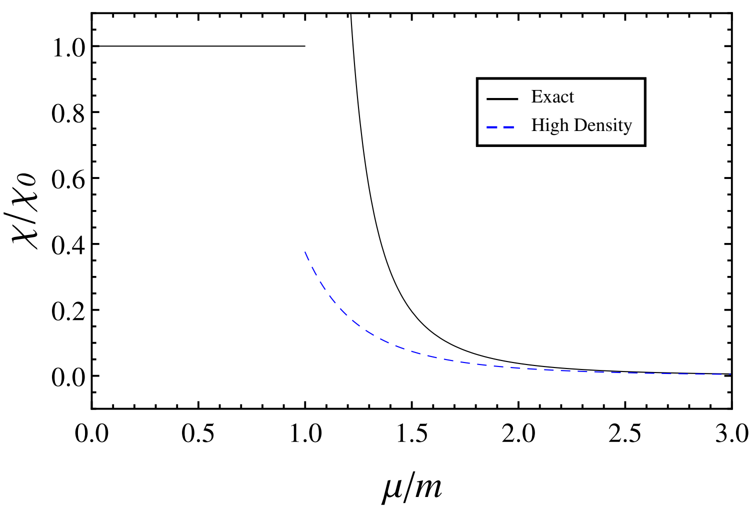

Figure 2.4: Graph of the Electrical Susceptibility as a function of chemical potential

We note that for $\mu > m$, the electric susceptibility decreases with the chemical potential. This means that the increase of the density of $e^- - e^+$ pairs screens the medium polarizability.

Also note that there is a singularity when μ=m in Fig. 2.4, that can be associated with a passable phase transition.

Likewise from the term independent of q_3 appearing in Eq. (2.68), we obtain the contribution of the medium to the Debye Mass

$$M^2_{Debye} = \frac{\alpha e H}{3\pi}\frac{6\mu}{(\mu^2 - m^2)^{1/2}}\Theta(\mu - m) \tag{2.72}$$

In Fig. 2.5 we represent the Debye mass as a function of chemical potential.

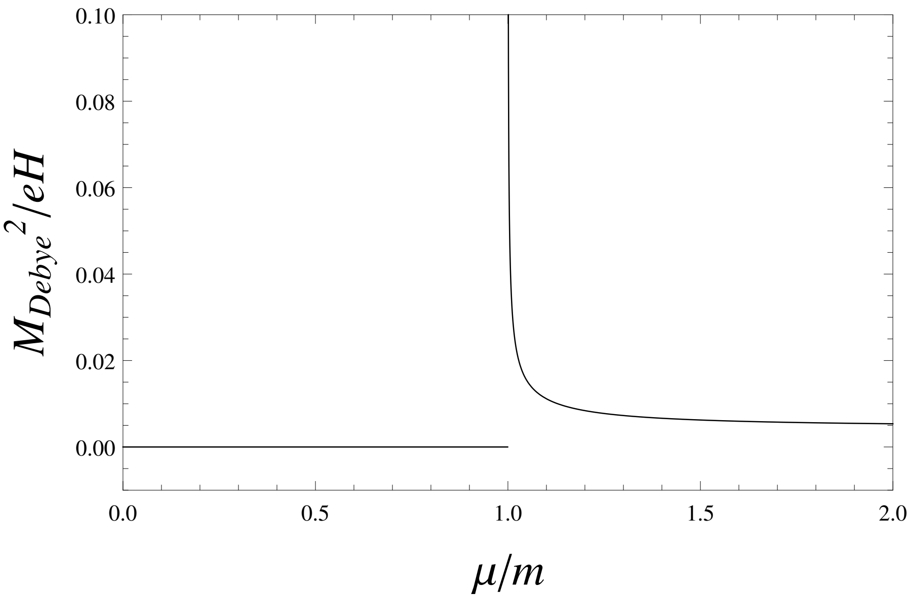

Figure 2.5: Debye Mass vs Density

We see that as in the electric susceptibility case the chemical potential decreases the Debye mass of the strongly magnetized medium and at μ=m there is also a singularity.

Chapter 3

Magnetic Catalysis of Chiral Symmetry Breaking at Finite Density in the Ladder Approximation

3.1 General Goal

It is known that because of the effect of quantum contributions on the mass, there is a dynamical mass. To find this dynamical generated effect, a nonperturbative approach is required. This is usually worked out in the ladder approximation, where the full fermion Green's function contributes to its self-energy. Then, we will investigate the effect of the dynamically generated parameters on the electric susceptibility and Debye mass.

3.2 Shwinger-Dyson Equation in the One-loop Approximation at B=0

In this section we will use the Shwinger Dyson equation to determine the quantum correction to the fermion mass in the one-loop approximation. The Schwinger-Dyson equation can be written as

$$S^{-1}(p) = S_0^{-1}(p) + \Sigma, \tag{3.1}$$

where $S^{-1}(p)$ is the full fermion propogator, $S_0^{-1}(p)$ is the bare fermion propagator, and Σ is the full correction provided by the self-energy operator. This equation is equivalent to

$$\not{p} - M = \not{p} - M_0 - e^2 \int \frac{d^4q}{(2\pi)^4} D_{\mu\nu} \gamma^\mu S(q) \gamma^\nu \tag{3.2}$$

Where $D_{\mu\nu}$ is the photon propagator, and S(q) is the free electron propagator in a magnetic field.

$$S(q) = \frac{1}{(\not{p} - \not{q}) - M} \tag{3.3}$$

To get the correction to the fermion mass due to Σ, we take the trace of Eq. (3.2) to obtain

$$M = M_0 + e^2 \int \frac{d^4q}{(2\pi)^4} D_\mu{}^\mu \frac{M}{(p-q)^2 - M^2} \tag{3.4}$$

3.3 Electon's Self Energy in Vacuum

First we start off with the Feynmann diagram for self-energy

Figure 3.1: Self Energy

The diagram consists of an electron breaking apart into an electron and a photon and then recombining back again into an electron. So the diagram has two vertices that are ieγ_μ and ieγ_ν.

The fermion propagator is

$$S(p-q) = \frac{i}{(\not{p} - \not{q}) - m} = i \frac{(\not{p} - \not{q}) + m}{(p-q)^2 - m^2} \tag{3.5}$$

The photon propagator

$$D_{\mu\nu} = -i\left(\frac{g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}}{q^2}\right) \tag{3.6}$$

Where ξ is the gauge fixing parameter. So the fermion self energy is

$$\Sigma(p) = \int \frac{d^4q}{(2\pi)^4}(ie\gamma^\nu)(\frac{i}{(\not{p}-\not{q})-M})(ie\gamma^\mu)\left(-i\frac{g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}}{q^2}\right) \tag{3.7}$$

with the Schwinger-Dyson equation in mind we replace m by m in Σ to obtain.

$$\Sigma(p) = -ie^2\int \frac{d^4q}{(2\pi)^4}\gamma^\nu(\frac{(\not{p}-\not{q})+M}{(p-q)^2-M^2})\gamma^\mu\left(-i\frac{g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}}{q^2}\right), \tag{3.8}$$

Which can be rearranged as

$$\Sigma(p) = -e^2\int \frac{d^4q}{(2\pi)^4}\frac{-2(\not{p}-\not{q}) + (3+\xi)M - (1-\xi)[-(\not{p}+\not{q}) + 2\frac{p\cdot q}{q^2}\not{q}]}{[(p-q)^2-M^2]q^2} \tag{3.9}$$

Expressions Eq. (3.8) and Eq. (3.9) in the Feynman guage (ξ=1) reduce respectively to

$$\Sigma(p) = -e^2\int \frac{d^4q}{(2\pi)^4}\frac{-2(\not{p}-\not{q}) + 4M}{[(p-q)^2-M^2]q^2} \tag{3.10}$$

3.4 Electron Self-Energy at Finite Temperature

The self energy operator at finite temperature is given by

$$\Sigma(T) = e^2T\sum_{n=-\infty}^{\infty}\int \frac{d^3q}{(2\pi)^3}\frac{2(\not{p}-\not{q}) + 4M}{[(p-q)^2+M^2]q^2} \tag{3.11}$$

Where the momenta are defined as

$$\begin{aligned} q_\mu &= (\omega_n, q_1, q_2, q_3) \\ (p-q)_\mu &= (\omega_m - \omega_n, p_1 - q_1, p_2 - q_2, p_3 - q_3) \end{aligned} \tag{3.12}$$

with the Matsubara's frequencies for fermions given by

$$\omega_n = \frac{(2n+1)\pi}{\beta}; \quad n = 0, \pm 1, \pm 2, ... \tag{3.13}$$

To make the Matsubara sum in Eq. (3.11), we consider first the following partial summations

$$\begin{aligned} S_1 &= T\sum_n \frac{1}{[(\omega_m-\omega_n)^2+E_1^2](\omega_n^2+E_2^2)} \\ &= \frac{1}{4E_1E_2}\sum_{r,s=\pm 1}\frac{rn_b(E_2)-sn_f(E_1)+\frac{r-s}{2}}{i\omega_m-sE_2-rE_1} \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} E_1^2 &= (\vec{p}-\vec{q}\,)^2+M^2 \\ E_2^2 &= \vec{q}\,^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} n_b(E,\mu) &= \frac{1}{e^{\beta(E-\mu)}-1} \\ n_f(E,\mu) &= \frac{1}{e^{\beta(E-\mu)}+1} \end{aligned} \tag{3.16}$$

where $n_f(E,\mu)$ and $n_b(E,\mu)$ are the fermion and boson distribution functions respectively.

The last step is covered in Appendix B. Similar to Eq. (3.14) we have for S_2

$$\begin{aligned} S_2 &= T\sum_n \frac{\omega_m-\omega_n}{[(\omega_m-\omega_n)^2+E_1^2](\omega_n^2+E_2^2)} \\ &= \frac{1}{4iE_2}\sum_{r,s=\pm 1}\frac{sn_b(E_2)-rn_f(E_1)+\frac{s-r}{2}}{i\omega_m+rE_1+sE_2} \end{aligned} \tag{3.17}$$

Substituting Eq. (3.14) and Eq. (3.17) in Eq. (3.11) we obtain

$$\begin{aligned} \Sigma(T) &= 2e^2\int\frac{d^3q}{(2\pi)^3}\left\{[(\not{p}-\not{q})+2M]S_1+\gamma^0 S_2\right\} \\ &= 2e^2\int\frac{d^3q}{(2\pi)^3}\left(\frac{\vec{\not{p}}-\vec{\not{q}}+2M}{4E_1E_2}\sum_{r,s=\pm 1}\frac{rn_b(E_2)-sn_f(E_1)+\frac{r-s}{2}}{i\omega_m-sE_2-rE_1}\right. \\ &+ \left.\frac{\gamma^0}{4iE_2}\sum_{r,s=\pm 1}\frac{sn_b(E_2)-rn_f(E_1)+\frac{s-r}{2}}{i\omega_m+rE_1+sE_2}\right) \end{aligned} \tag{3.18}$$

Since for the variables s and r are dummy variables, we can just switch them in the second summation. Then, what we end up with is a simplified integral.

$$
\begin{aligned}
\Sigma(T) &= e^2 \int \frac{d^3q}{2E_2E_1(2\pi)^3} \times \\
&\times \sum_{r,s=\pm 1} \frac{(\not{p} - \not{q} + 2M - \gamma^0 iE_1 rs)\left(rn_b(E_2) - sn_f(E_1) + \frac{r-s}{2}\right)}{i\omega_m - rE_1 - sE_2}
\end{aligned} \tag{3.19}
$$

3.5 Electron Self-Energy and Dynamical Mass at Finite Temperature and Density at B=0

In order to incorporate the chemical potential and temperature, let us start with Eq. (3.10). Now, we perform the Wick rotation to go from Minkowski to Euclidean space. To introduce the chemical potential, μ, we perform a shift on the electron four momentum,

$$
\begin{aligned}
q_\mu &= (\omega_n, q_1, q_2, q_3) \\
(p-q)_\mu &= (\omega_m - \omega_n - i\mu, p_1 - q_1, p_2 - q_2, p_3 - q_3)
\end{aligned} \tag{3.20}
$$

Then, introducing the Matsubara sum and the chemical potential, Eq. (3.10) is transformed into

$$
\Sigma(\mu, T) = e^2 \int \frac{d^3q}{(2\pi)^3} \sum_n \frac{2((\omega_n - \omega_m - i\mu)\gamma_4 + (\vec{p} - \vec{q}) \cdot \vec{\gamma}) + 4M}{[(\omega_n - \omega_m - i\mu)^2 + (p \vec{-} q)^2 + M^2](\omega_n + \vec{q})^2} \tag{3.21}
$$

So in a similar fashion as before let us begin by introducing some basic forms to help us with calculating the Matsubara sums,

$$
\begin{aligned}
S_1(\mu) &= T \sum_n \frac{1}{((\omega_m - \omega_n - i\mu)^2 + E_1^2)(\omega_n^2 + E_2^2)} \\
&= \frac{1}{4E_1E_2} \sum_{r,s=\pm 1} \left(\frac{sn_f(E_1 - r\mu) - rn_b(E_2) - \frac{r+s}{2}}{i\omega_m + \mu - rE_1 - sE_2} \right)
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
S_2(\mu) &= T \sum_n \frac{\omega_m - \omega_n - i\mu}{((\omega_m - \omega_n - i\mu)^2 + E_1^2)(\omega_n^2 + E_2^2)} \\
&= \frac{i}{4E_2} \sum_{s,r=\pm 1} s \frac{sn_b(E_2) - rn_f(E_1 - r\mu) + \frac{s+r}{2}}{i\omega_m + \mu - rE_1 - sE_2}
\end{aligned} \tag{3.23}
$$

where it was assumed that $E_1 - \mu > 0$.

Then, substituting Eq. (3.22) and Eq. (3.24) into Eq. (3.21) we obtain

$$\begin{aligned}\Sigma(\mu, T) &= 2e^2 \int \frac{d^3q}{(2\pi)^3} \left\{[(\not{p} - \not{q}) + 2M]S_1(\mu) + \gamma^0 S_2(\mu)\right\} \\ &= e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_1E_2} \times \\ &\times \left(\sum_{r,s=\pm1} \frac{(sn_f(E_1 - r\mu) - rn_b(E_2) - \frac{s+r}{2})(\not{p} - \not{q} + 2M - \gamma^0 iE_1 r)}{i\omega_m + \mu - rE_1 - sE_2}\right)\end{aligned} \quad (3.24)$$

3.5.1 Zero-Temperature Limit

Taking the $T \to 0$ limit of Eq. (3.24) we have that the Bose-Einstein distribution functions vanish as E_2 is always positive, and the Fermi distribution functions give rise to the corresponding step functions,

$$\begin{aligned}\Sigma(P, T = 0, \mu) &= 2e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{4E_1E_2} \left[\sum_{s=\pm1} \frac{s(\Theta(\mu - E_1) - \frac{1+s}{2})((\not{p} - \not{q} + 2M) - i\gamma^0 E_1)}{i\omega_m + \mu - E_1 - sE_2}\right. \\ &+ \left.\frac{(\not{p} - \not{q} + 2M) + i\gamma^0 E_1}{i\omega_m + \mu + E_1 + E_2}\right] \\ &= 2e^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{4E_1E_2} \left[\sum_{s=\pm1} \frac{s\Theta(\mu - E_1)((\not{p} - \not{q} + 2M) - i\gamma^0 E_1)}{i\omega_m + \mu - E_1 - sE_2}\right. \\ &- \left.\frac{((\not{p} - \not{q} + 2M) - i\gamma^0 E_1)}{i\omega_m + \mu - E_1 - E_2} + \frac{((\not{p} - \not{q} + 2M) + i\gamma^0 E_1)}{i\omega_m + \mu + E_1 + E_2}\right]\end{aligned} \quad (3.25)$$

In Eq. (3.25) we performed the summation over r, also we moved the $\frac{1+s}{2}$ of the first term of the Eq. (3.25) to get to the second equality of the Eq. (3.25).

Finally we take the analytical continuation $i\omega_m \to p_0$ to obtain

$$\begin{aligned}\Sigma(T = 0, \mu) &= e^2 \int \frac{d^3q}{(2\pi)^3} \left(\frac{1}{2E_1E_2}\right) \left[\sum_{s=\pm1} \frac{s\Theta(\mu - E_1)((\not{p} - \not{q} + 2M) - i\gamma^0 E_1)}{p_0 + \mu - E_1 - sE_2}\right. \\ &+ \left. 2\frac{(p_0 + \mu)(i\gamma^0 E_1) - (E_1 + E_2)(\not{p} - \not{q} + 2M)}{(p_0 + \mu)^2 - (E_1 + E_2)^2}\right]\end{aligned} \quad (3.26)$$

As it was stated in the beginning of the chapter the dynamical mass can be obtained by computing the trace of Eq. (3.26) to get

$$M' = \int \frac{d^3q}{(2\pi)^3} \frac{e^2 M}{E_1 E_2} \left[2 \frac{E_1 + E_2}{(p_0 + \mu)^2 - (E_1 + E_2)^2} - \sum_{s=\pm 1} \frac{s\Theta(\mu - E_1)}{p_0 + \mu - E_1 - sE_2} \right] \tag{3.27}$$

3.6 Electron Dynamical Mass at Strong Magnetic Field in the Ladder Approximation

Let's consider the Schwinger-Dyson equation for massless QED ($M_0 = 0$) in the ladder approximation at finite density and in the presence of a strong magnetic field ($eH \gg T^2$,$eH \gg M^2$). In this approximation we need to consider bare vertices and a bare photon propagator but a full fermion propagator in the strong magnetic field limit.

In the strong-field limit the electrons are confined to the (LLL) so only longitudinal modes are present. This implies that the electron momenta and vertices reduce to

$$\begin{aligned} \not{\vec{p}} - \not{\vec{q}} \quad & \rightarrow \not{\vec{p}}_{||} - \not{\vec{q}}_{||} \\ \gamma^\mu \quad & \rightarrow \gamma^\mu_{||} \Delta(t) \end{aligned} \tag{3.28}$$

and the electron self energy takes the following form

$$\Sigma(p)\Delta(+) \;\; = \;\; -\Delta(+)e^2 \int \frac{dq^4}{(2\pi)^4} e^{-\frac{q_\perp^2}{2eH}} \frac{2M}{q^2((p_{||} - q_{||})^2 - M^2)} \tag{3.29}$$

Now, performing the Wick rotation to go from Minkowski to Euclidean space.

$$\begin{aligned} q_\mu \;\; &= \;\; (\omega_n, q_1, q_2, q_3) \\ (p_{||} - q_{||})_\mu \;\; &= \;\; (\omega_m - \omega_n - i\mu, 0, 0, p_3 - q_3) \end{aligned} \tag{3.30}$$

Introducing the Matsubara sum and the chemical potential we have

$$\Sigma(p)\Delta(+) = -2\Delta(+)e^2 \int \frac{dq^3}{(2\pi)^3} e^{-\frac{q_\perp^2}{2eH}} \sum_n M \frac{1}{(\omega_n^2 + \vec{q}^2)((\omega_m - \omega_n - i\mu)^2 + (p_3 - q_3)^2 + M^2)} \tag{3.31}$$

After performing the sum we obtain

$$\Sigma(p)\Delta(+) = -2\Delta(+)e^2 \int \frac{dq^3}{(2\pi)^3} e^{-\frac{q_\perp^2}{2eH}} \frac{M}{4E_1E_2} \sum_{r,s=\pm 1} \frac{s(n_b(E_2)+\frac{1}{2}) + r(n_f(E_1 - s\mu) - \frac{1}{2})}{i\omega_m + \mu - sE_1 - rE_2} \tag{3.32}$$

where

$$\begin{aligned} E_1^2 &= (p_3 - q_3)^2 + M^2 \\ E_2^2 &= \vec{q}^{\,2} \end{aligned} \tag{3.33}$$

We now consider the self energy at T = 0, $\mu \neq 0$, and $B \neq 0$. Taking the limit $T \to 0$ we have

$$\begin{aligned} \Sigma(B, T=0, \mu)\Delta(+) &= -\Delta(+)e^2 \int \frac{d^3q}{(2\pi)^3} e^{-\frac{q_\perp^2}{2eH}} \frac{M}{4E_1E_2} \\ &\times \sum_{r=\pm 1} \left[\frac{2r\Theta(\mu - E_1)}{i\omega_m + \mu - E_1 - rE_2} + \sum_{s=\pm 1} \frac{s - r}{i\omega_m + \mu + sE_1 - rE_2} \right] \end{aligned} \tag{3.34}$$

When we do the analytic continuation $i\omega_m \to p_0$ we get

$$\begin{aligned} \Sigma(B, T=0, \mu)\Delta(+) &= -\Delta(+)e^2 \int \frac{d^3q}{(2\pi)^3} e^{-\frac{q_\perp^2}{2eH}} \frac{M}{4E_1E_2} \\ &\times \sum_{r=\pm 1} \left[\frac{2r\Theta(\mu - E_1)}{p_0 + \mu - E_1 - rE_2} + \sum_{s=\pm 1} \frac{s - r}{p_0 + \mu + sE_1 - rE_2} \right] \end{aligned} \tag{3.35}$$

To get the dynamical mass, we perform the trace of Schwinger-Dyson Equation, Eq. (3.2), with $M_0 = 0$ and using the self-energy operator Eq. (3.35) we obtain

$$\begin{aligned} 1 &= -e^2 \int \frac{d^3q}{(2\pi)^3} e^{-\frac{q_\perp^2}{2eH}} \frac{1}{4E_1E_2} \\ &\times \sum_{r=\pm 1} \left[\frac{2r\Theta(\mu - E_1)}{p_0 + \mu - E_1 - rE_2} + \sum_{s=\pm 1} \frac{s - r}{p_0 + \mu + sE_1 - rE_2} \right] \end{aligned} \tag{3.36}$$

In Fig. 3.2 we plot the dynamical mass in the infrared limit versus the chemical potential using Eq. (3.36)

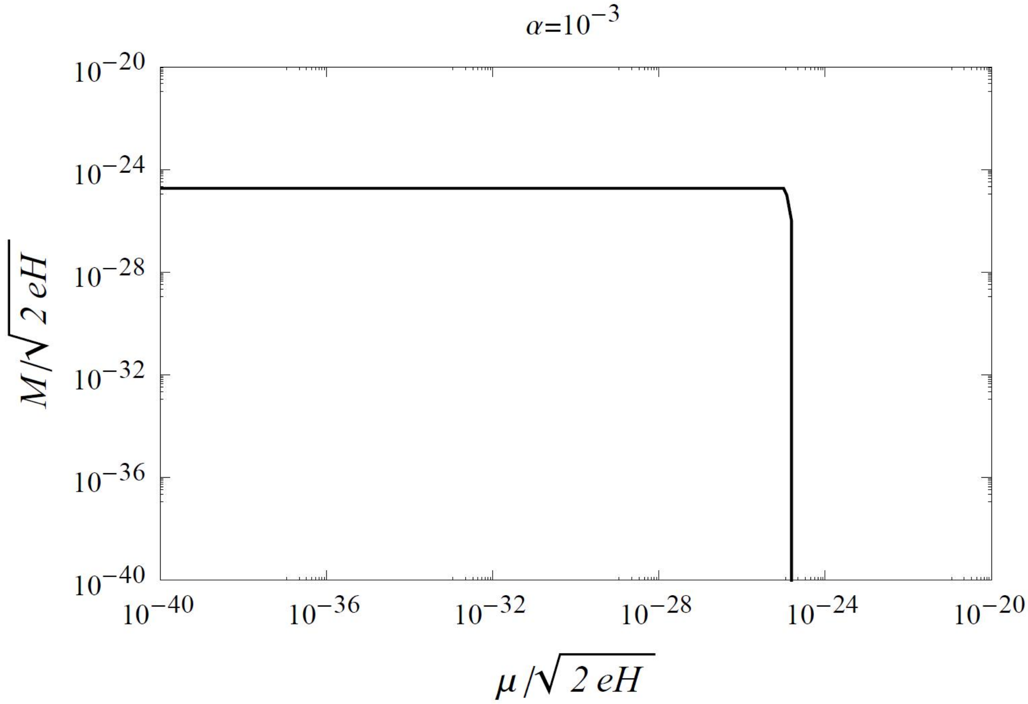

Figure 3.2: Graph of the Dynamical Mass vs chemical potential

As can be seen in Fig 3.2, the dynamical mass remains almost constant until it has reached the point around where the chemical potential equals the dynamical mass at zero chemical potential is reached [13] [18].

$$M_0 = \sqrt{2eH}e^{-\sqrt{\frac{\pi}{\alpha}}} \tag{3.37}$$

where α is the fine-structure in natural units. There is a first-order phase transition at that critical point, where the system regains the chiral symmetry.

Compairing this result with that at finite temperature [14], we find that both parameters, μ and T, contribute to erase the chiral condensate. The only difference is that the chemical potential produces a first-order phase transition while the phase transition of thermal effects is of second-order.

Chapter 4

Summary and Concluding Remarks

We started our exposition by going over the basic occurances in nature of a strong magnetic field. Having the possibility to have in the core of neutron stars fields of the order $\sim 10^{20}$G is a little different than the 1G we are more acccustomed to on Earth. Not only that, but the immense pressure and density which can take place in compact astronomical objects require a special treatment of QFT under external conditions. After covering the motivation for why we would need such QED methods, we went into a review of the main surrounding ideas and theories serving as basis for our work. In this sense Noether's theorem, Goldstone's Theorem and the Higgs Mechanism were reviewed with a special emphasis of chiral symmetry breaking produced by the induction of a dynamical mass. Then, we also reviewed some basic calculation techniques in QFT, as the Feynman rules, that we specifically used in the calculations for the a dense and strongly magnetized medium.

In the second chapter we went over the one-loop PPO. First, the PPO was calculated at finite temperature with zero magnetic field and then with a strong magnetic field. As it is common in these calculations the vacuum and medium contributions separate. After that, the infrared limit of the PPO was calculated for a neutral positron-electron plasma. In Fig. 2.3 the graph of the electric susceptibility as a function of temperature was given. It shows that the electric susceptibility decreases with the temperature. For T$\simeq$M the susceptibility is practically zero. Then, the system's electrical susceptibility was obtained in Fig. 2.4 as a function of the chemical potential and at zero temperature. The graph for the electric susceptibility showed that for $\mu < m$ the susceptibility is almost constant and beyond the critical point at μ=m, where appears a singularity, it has a sharp decrease to zero. In Fig. 2.5 we showed how the Debye mass depends on the chemical potential at zero temperature.

There we also notice the singularity at $\mu = m$ that separates a zero value of the Debye mass for $\mu < m$ from a small valve different from zero. Both singularities indicate a possible phase transition.

In the third chapter we were concerned with the fermion self energy operator and its contribution to the Shwinger-Dyson equation. There, we calculated in the one-loop approximation the self-energy of an electron in vacuum. We then added temperature and density effects. In Fig. 3.2 we graphed the dynamical mass vs. chemical potential.

We found that there is a critical value of the chemical potential of the order of the dynamical mass at zero density and where a first-order phase transition takes place and the system regains the chiral symmetry. Nevertheless, the existence of a singular behaviour at that critical point could be signalizing the existence of a new condensate. This condensate would be the one corresponding to an inhomogeneous particle-hole pair, that can appear beyond that critical point.

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Appendix A

Appendix A

A.1 Generic Sums over Matsubara Frequencies

Let us perform the Matsubara sums of the different terms contributing to the one-loop PPO in Eq. (2.19).

We start by summing

$$Sum_1 \equiv \sum_{n=-\infty}^{\infty} \frac{1}{(p_{||}^2 + m^2)[(p-q)_{||}^2 + m^2]} \tag{A.1}$$

Using Eq. (2.20) in Eq. (A.1) we get

$$Sum_1 = \sum_{n=-\infty}^{\infty} \frac{1}{[(\omega_n - i\mu)^2 + E_1^2][(\omega_n - \omega_m - i\mu)^2 + E_2^2]} \tag{A.2}$$

where

$$E_1^2 \equiv p_3^2 + m^2 \ , \quad E_2^2 \equiv (p_3 - q_3)^2 + m^2. \tag{A.3}$$

and

$$\begin{aligned} \omega_n &= \frac{(2n+1)\pi}{\beta}, \quad n = 0, \pm 1, \pm 2, \pm 3 \\ \omega_m &= \frac{2m\pi}{\beta}, \quad m = 0, \pm 1, \pm 2, \pm 3 \end{aligned} \tag{A.4}$$

Using partial fractions, it is easy to see that the sum can be rewritten as follows

$$Sum_1 = \sum_{n=-\infty}^{\infty} \sum_{\sigma,\sigma'=\pm 1} \frac{1}{4E_1E_2} \frac{\sigma\sigma'}{(i(\omega_n - i\mu) + \sigma E_1)(i(\omega_n - \omega_m - i\mu) + \sigma' E_2)} \tag{A.5}$$

Using again partial fractions, we rewrite the above equation as

$$
\begin{aligned}
Sum_1 &= \frac{1}{4E_1E_2} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma'}{-i\omega_m + \sigma' E_2 - \sigma E_1} \times \\
&\times \sum_{n=-\infty}^{\infty} \left[\frac{1}{i(\omega_n - i\mu) + \sigma E_1} - \frac{1}{i(\omega_n - \omega_m - i\mu) + \sigma' E_2} \right] \\
&= \frac{1}{4E_1E_2} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma'}{-i\omega_m + \sigma' E_2 - \sigma E_1} \times \\
&\times \sum_{n=-\infty}^{\infty} \left[\frac{-\sigma}{i\omega_n - (E_1 + \sigma\mu)} - \frac{-\sigma'}{i(\omega_n + \sigma'\omega_m) - (E_2 + \sigma'\mu)} \right]
\end{aligned} \tag{A.6}
$$

where in the last line of the above equation we have used the fact that the sum over n does not change when we replace n by $-n$. Next, we use the identity

$$
\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{i\omega_n - \xi_p} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)\frac{i\pi}{\beta} - \xi_p} = n_f(\xi_p) - \frac{1}{2} \tag{A.7}
$$

with $n_f(\xi_p) = \frac{1}{e^{\beta(\xi_p - \mu)} + 1}$ the Fermi-Dirac distribution, this changes Sum_1 into

$$
\begin{aligned}
Sum_1 &= \frac{-\beta}{4E_1E_2} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma'}{-i\omega_m + \sigma' E_2 - \sigma E_1} \times \\
&\times \left(\sigma \left[n_f(E_1 + \sigma\mu) - \frac{1}{2} \right] - \sigma' \left[n_f(E_2 + \sigma'\mu) - \frac{1}{2} \right] \right)
\end{aligned} \tag{A.8}
$$

Finally, this can be rewriteen as

$$
Sum_1 = \frac{-\beta}{4E_1E_2} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma' n_f(E_1 + \sigma\mu) - \sigma n_f(E_2 + \sigma'\mu) - \frac{1}{2}(\sigma' - \sigma)}{-i\omega_m + \sigma' E_2 - \sigma E_1} \tag{A.9}
$$

The two other sums which are going to appear in our calculations are

$$
\begin{aligned}
Sum_2 &\equiv \sum_{n=-\infty}^{\infty} \frac{(\omega_n - i\mu)}{[(\omega_n - i\mu)^2 + E_1^2][(\omega_n - \omega_m - i\mu)^2 + E_2^2]} \\
Sum_3 &\equiv \sum_{n=-\infty}^{\infty} \frac{(\omega_n - \omega_m - i\mu)}{[(\omega_n - i\mu)^2 + E_1^2][(\omega_n - \omega_m - i\mu)^2 + E_2^2]}
\end{aligned} \tag{A.10}
$$

Using partial fractions we rewrite Sum_2 as follows

$$\begin{aligned} Sum_2 &= \sum_{\sigma,\sigma'=\pm 1} \frac{1}{-4iE_2} \sum_{n=-\infty}^{\infty} \left(\frac{\sigma}{\sigma i(\omega_n - i\mu) + E_1} \right) \left(\frac{1}{\sigma' i(\omega_n - \omega_m - i\mu) + E_2} \right) \\ &= \sum_{\sigma,\sigma'=\pm 1} \frac{-\sigma'}{4iE_2} \sum_{n=-\infty}^{\infty} \frac{1}{-i\omega_m + \sigma' E_2 - \sigma E_1} \times \\ &\times \left(\frac{-\sigma}{i\omega_n - (E_1 + \sigma\mu)} - \frac{-\sigma'}{i(\omega_n + \sigma'\omega_m) - (E_2 + \sigma'\mu)} \right) \end{aligned} \quad \text{(A.11)}$$

Finally, substituting Eq. (A.7) in Eq. (A.11), we have after summing in n,

$$\begin{aligned} Sum_2 &= \frac{\beta}{i4E_2} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma'[n_f(E_1 + \sigma\mu) - \frac{1}{2}] - [n_f(E_2 + \sigma'\mu) - \frac{1}{2}]}{-i\omega_m + \sigma' E_2 - \sigma E_1} \\ &= \frac{\beta}{i4E_2} \sum_{\sigma,\sigma'=\pm 1} \frac{\sigma\sigma' n_f(E_1 + \sigma\mu) - n_f(E_2 + \sigma'\mu) - \frac{1}{2}(\sigma\sigma' - 1)}{-i\omega_m + \sigma' E_2 - \sigma E_1} \end{aligned} \quad \text{(A.12)}$$

Since Sum_3 is similar to Sum_2, we just do the same steps as we did for Sum_2 giving us

$$Sum_3 = \frac{\beta}{i4E_1} \sum_{\sigma,\sigma'=\pm 1} \frac{n_f(E_1 + \sigma\mu) - \sigma\sigma' n_f(E_2 + \sigma'\mu) - \frac{1}{2}(1 - \sigma\sigma')}{-i\omega_m + \sigma' E_2 - \sigma E_1} \quad \text{(A.13)}$$

Let us consider now the following sum over Matsubara frequencies Sum_4, defined by

$$Sum_4 = \sum_{n=-\infty}^{\infty} \frac{(\omega_n - i\mu)[(\omega_n - \omega_m) - i\mu]}{[(\omega_n - i\mu)^2 + E_1^2][(\omega - \omega_m - i\mu)^2 + E_2^2]} \quad \text{(A.14)}$$

Note that each factor in the above equation can be rewritten as follows

$$\begin{aligned} \frac{(\omega_n - i\mu)}{[(\omega_n - i\mu)^2 + E_1^2]} &= -\frac{1}{2i} \left(\frac{1}{i(\omega_n - i\mu) + E_1} - \frac{1}{-i(\omega_n - i\mu) + E_1} \right) \\ \frac{\omega_n - \omega_m - i\mu}{[(\omega_n - \omega_m - i\mu)^2 + E_2^2]} &= -\frac{1}{2i} \left(\frac{1}{i(\omega_n - \omega_m - i\mu) + E_2} \right. \\ &\left. - \frac{1}{-i(\omega_n - \omega_m - i\mu) + E_2} \right) \end{aligned} \quad \text{(A.15)}$$

Then substituting Eq. (A.15) in Eq. (A.14), we get

$$\begin{aligned} Sum_4 &= \sum_{\sigma,\sigma'=\pm 1} \frac{1}{(-2i)(-2i)} \sum_{n=-\infty}^{\infty} \left(\frac{\sigma}{i\sigma(\omega - i\mu) + E_1} \right) \left(\frac{\sigma'}{i\sigma'(\omega_n - \omega_m - i\mu) + E_2} \right) \\ &= -\frac{1}{4} \sum_{\sigma,\sigma'=\pm 1} \sum_{n=-\infty}^{\infty} \left(\frac{1}{i(\omega_n - i\mu) + \sigma E_1} \right) \left(\frac{1}{i(\omega_n - \omega_m - i\mu) + \sigma' E_2} \right) \end{aligned} \quad \text{(A.16)}$$

Where we multiplied the expression inside the first parenthesis by $\frac{\sigma}{\sigma}$ and that inside the second parenthesis by $\frac{\sigma'}{\sigma'}$ and took into account that $\sigma^2 = \sigma'^2 = 1$.

After using partial fractions, we obtain

$$\begin{aligned}
Sum_4 &= -\frac{1}{4}\sum_{\sigma,\sigma'=\pm1}\sum_{n=-\infty}^{\infty}\frac{1}{-i\omega_m+\sigma' E_2-\sigma E_1}\left(\frac{1}{i(\omega_n-i\mu)+\sigma E_1}-\frac{1}{i(\omega_n-\omega_m-i\mu)+\sigma' E_2}\right)\\
&= -\frac{1}{4}\sum_{\sigma,\sigma'=\pm1}\frac{1}{-i\omega_m+\sigma' E_2-\sigma E_1}\times\\
&\quad\times\sum_{n=-\infty}^{\infty}\left(\frac{-\sigma}{i\omega_n-(E_1+\sigma\mu)}-\frac{-\sigma'}{i(\sigma'\omega_n+\sigma'\omega_m)-(E_2+\sigma'\mu)}\right) \qquad \text{(A.17)}
\end{aligned}$$

Finally, using the identity of Eq. (A.7) in Eq. (A.17), it reduces to

$$Sum_4 = \frac{\beta}{4}\sum_{\sigma,\sigma'=\pm1}\frac{\sigma n_f(E_1+\sigma\mu)-\sigma' n_f(E_2+\sigma'\mu)-\frac{1}{2}(\sigma-\sigma')}{-i\omega_m+\sigma' E_2-\sigma E_1} \qquad \text{(A.18)}$$

For the sake of ease of use we will suppose that there is an ω >0 then suppose we have the equations

$$\begin{aligned}
\sum_{-\infty}^{\infty}\frac{1}{i\omega_n-\omega} &= \beta[n_b(\omega)-\frac{1}{2}]\\
\sum_{-\infty}^{\infty}\frac{1}{i\omega_n+\omega} &= -\beta[n_b(\omega)-\frac{1}{2}] \qquad \text{(A.19)}
\end{aligned}$$

With this we may suppose that

$$\sum_{-\infty}^{\infty}\sum_{r=\pm1}\frac{1}{i\omega_n-r\omega} = r(n_b(\omega)-\frac{1}{2}) \qquad \text{(A.20)}$$

This little trick helps us to calculate the Matsubara summations resulting in fermion and boson distributions.

Appendix B

Fermi-Dirac Distribution zero-Temperature limits

There are some anomolies that are present for the Fermi-Dirac distibution function as the limit as $T \to 0$.

Let $\beta = \frac{1}{T}$ so $\lim_{T\to 0} \beta = \infty$

With this in mind we know that the Fermi distribution function at finite density is given by

$$n_f(E_1 - \mu) = \frac{1}{e^{\beta(E_1-\mu)} + 1} \tag{B.1}$$

with $E_1 > 0$; $\mu > 0$

Case 1: $E - \mu < 0$

$$\lim_{T\to 0} n_f(E_1 - \mu) = \lim_{T\to 0} \frac{1}{e^{\beta(E_1-\mu)} + 1} = 1 \tag{B.2}$$

Case 2: $E_1 - \mu > 0$

$$\lim_{T\to 0} n_f(E_1 - \mu) = \lim_{T\to 0} \frac{1}{e^{\beta(E_1-\mu)} + 1} = 0 \tag{B.3}$$

Case 3: $E_1 - \mu = 0$

$$\lim_{T\to 0} n_f(E_1 - \mu) = \lim_{T\to 0} \frac{1}{e^{\beta(E_1-\mu)} + 1} = \lim_{T\to 0} \frac{1}{e^0 + 1} = \frac{1}{2} \tag{B.4}$$

We can represent this behaivor through the Heavyside step function

$$\Theta(\mu - E_1) = \begin{cases} 1, & \mu > E_1 \\ \frac{1}{2}, & \mu = E_1 \\ 0, & \mu < E_1 \end{cases} \tag{B.5}$$

Curriculum Vitae

Paul Lee Springsteen was born on September 15, 1985. He is the son of Larry L Springsteen and Chong Ok Springsteen. He graduated from Andress High School in the spring of 2004. He enlisted out of high school into the United States Air Force in 2004. After being honorably discharged he entered college at the University of Texas at El Paso in the spring of 2007 as a physics major. In spring of 2012 he earned his Bachelors of Science in Physics at UTEP.

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