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Abstract

The traditional Markowitz approach to portfolio optimization assumes that we know the means, variances, and covariances of the return rates of all the financial instruments. In some practical situations, however, we do not have enough information to determine the variances and covariances, we only know the means. To provide a reasonable portfolio allocation for such cases, researchers proposed a heuristic maximum entropy approach. In this paper, we provide an economic justification for this heuristic idea.

1 Formulation of the Problem

Portfolio optimization: general problem. What is the best way to invest money? Usually, there are several possible financial instruments; let us denote the number of available financial instruments by n. The questions is then: what portion w_i of the overall money amount should we allocate to each instrument i? Of course, these portions must be non-negative and add up to one:

$$\sum_{i=1}^{n} w_i = 1. \qquad (1)$$

The corresponding tuple $w = (w_1, \ldots, w_n)$ is known as an *investment portfolio*, or simply *portfolio*, for short.

Case of complete knowledge: Markowitz solution. If we place money in a bank, we get a guaranteed interest, with a given rate of return r. However, for most other financial instruments i, the rate of return r_i is not fixed, it changes (e.g., fluctuates) year after year. For each values of instrument returns, the corresponding portfolio return r is equal to $r = \sum\limits_{i=1}^{n} w_i \cdot r_i$.

In many practical situations, we know, from experience, the probabilistic distributions of the corresponding rates of return. Based on this past experience, for each instrument i, we can estimate the expected rate of return $\mu_i = E[r_i]$ and the corresponding standard deviation $\sigma_i = \sqrt{E[(r_i - \mu_i)^2]}$. We can also estimate, for each pair of financial instruments i and j, the covariance

$$c_{ik} \stackrel{\text{def}}{=} E[(r_i - \mu_i) \cdot (r_j - \mu_j)].$$

By using this information, for each possible portfolio $w = (w_1, \ldots, w_n)$, we can compute the expected return

$$\mu = E[r] = \sum_{i=1}^{n} w_i \cdot \mu_i \tag{2}$$

and the corresponding variance

$$\sigma^2 = \sum_{i=1}^{n} w_i^2 \cdot \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot w_i \cdot w_j. \tag{3}$$

The larger the expected rate of return μ we want, the largest the risk that we have to take, and thus, the larger the variance. It is therefore reasonable, given the desired expected rate of return μ, to find the portfolio that minimizes the variance, i.e., that minimizes the expression (3) under the constraints (1) and (2).

This problem was first considered by the future Nobelist Markowitz, who proposed an explicit solution to this problem; see, e.g., [8]. Namely, the Lagrange multiplier method enables to reduce this constraint optimization problem to the following unconstrained optimization problem: minimize the expression

$$\sum_{i=1}^{n} w_i^2 \cdot \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot w_i \cdot w_j + \lambda_1 \cdot \left(\sum_{i=1}^{n} w_i - 1 \right) +$$
$$\lambda_2 \cdot \left(\sum_{i=1}^{n} w_i \cdot \mu_i - \mu \right), \tag{4}$$

where λ_1 and λ_2 are Lagrange multipliers that need to be determined from the conditions (1) and (2).

Differentiating the expression (4) by the unknowns w_i, we get the following system of linear equations:

$$2\sigma_i \cdot w_i + 2 \sum_{j \neq i} c_{ij} \cdot w_j + \lambda_1 + \lambda_2 \cdot \mu_i = 0. \tag{5}$$

Thus,

$$w_i = \lambda_1 \cdot w_i^{(1)} + \lambda_2 \cdot w_i^{(2)}, \tag{6}$$

where $w_i^{(j)}$ are solutions to the following systems of linear equations

$$2\sigma_i \cdot w_i + 2\sum_{j\neq i} c_{ij} \cdot w_j = -1 \tag{7}$$

and

$$2\sigma_i \cdot w_i + 2\sum_{j\neq i} c_{ij} \cdot w_j = -\mu_i. \tag{8}$$

Substituting the expression (6) into the equations (1) and (2), we get a system two linear equations for two unknowns λ_1 and λ_2. From this system, we can easily find the coefficients λ_i and thus, the desired portfolio (6).

Case of complete information: modifications of Markowitz solution. Some researchers argue that variance may be not the best way to describe the intuitive notion of risk. Instead, they propose to use other statistical characteristics, e.g., the quantile q_α corresponding to a certain small probability α – i.e., a value for which the probability that the returns are very low ($r \le q_\alpha$) is equal to α.

Instead of the original Markowitz problem, we thus have a problem of maximizing q_α – or another characteristic – under the given expected return μ. Computationally, the resulting constraint optimization problems are no longer quadratic and thus, more complex to solve, but they are still well formulated and thus, solvable.

Case of partial information: formulation of the general problem. In many practical situations, we only have partial information about the probabilities of different rates of return r_i.

For example, in some cases, we know the expected returns μ_i, but we do not have any information about the standard deviations and covariances. What portfolio should we select in such situations?

Maximum Entropy approach: reminder. Situations in which we only have partial information about the probabilities – and thus, several different probability distributions are consistent with the available information – such situations are ubiquitous.

Usually, some of the consistent distributions are more precise, some are more uncertain. We do not want to pretend that we know more than we actually do, so in such situations of uncertainty, a natural idea is to select a distribution which has the largest possible degree of uncertainty. A reasonable way to describe the uncertainty of a probability distribution with the probability density $\rho(x)$ is by its *entropy*

$$S = -\int \rho(x) \cdot \ln(\rho(x))\, dx. \tag{9}$$

So, we select the distribution whose entropy is the largest; see, e.g., [5].

In many cases, this *Maximum Entropy* approach makes perfect sense. For example, if the only information that we have about a probability distribution is that it is located on an interval $[\underline{x}, \overline{x}]$, then out of all possible distributions, the Maximum Entropy approach selects the uniform distribution $\rho(x) = \text{const}$ on this interval. This makes perfect sense – if we do not have any reason to believe that one of the values from the interval is more probable than other values, then it makes sense to assume that all the values from this interval are equally probable, which is exactly $\rho(x) = \text{const}$.

In situations when we know marginal distributions of each of the variables, but we do not have any information about the dependence between these variables, the Maximum Entropy approach concludes that these variables are independent. This also makes perfect sense: if we have no reason to believe that the variables are positively or negatively correlated, it makes sense to assume that they are not correlated at all.

If all we know is the mean and the standard deviation, then the Maximum Entropy approach leads to the normal (Gaussian) distribution – which is in good accordance with the fact that such distributions are indeed ubiquitous.

So, in situations when we only have a partial information about the probabilities of different return values, it makes sense to select, out of all possible probability distributions, the one with the largest entropy, and then use this selected distribution to find the corresponding portfolio.

Problem: Maximum Entropy approach is not applicable to the case when only know μ_i. In many practical situations, the Maximum Entropy approach leads to reasonable results. However, it is not applicable to the situation when we only know the expected rates of return μ_i.

This impossibility can be illustrated already on the case when we have a single financial instrument. Its rate of return r_1 can take any value, positive or negative, the only information that we have about the corresponding probability distribution $\rho(x)$ is that

$$\mu_1 = \int x \cdot \rho(x)\, dx \tag{10}$$

and, of course, that $\rho(x)$ is a probability distribution, i.e., that

$$\int \rho(x)\, dx = 1. \tag{11}$$

The constraint optimization problem of maximizing the entropy (9) under the constraints (10) and (11) can be reduced to the following unconstrained optimization problem: maximize

$$-\int \rho(x) \cdot \ln(\rho(x))\, dx + \lambda_1 \cdot \left(\int x \cdot \rho(x)\, dx - \mu_1\right) + \lambda_2 \cdot \left(\int \rho(x)\, dx - 1\right). \tag{12}$$

Differentiating the expression (12) with respect to the unknown $\rho(x)$ and equating the derivative to 0, we get

$$-\ln(\rho(x)) - 1 + \lambda_1 \cdot x + \lambda_2 = 0,$$

hence

$$\ln(\rho(x)) = (\lambda_2 - 1) + \lambda_1 \cdot x$$

and $\rho(x) = C \cdot \exp(\lambda_1 \cdot x)$, where $C = \exp(\lambda_2 - 1)$. The problem is that the integral of this exponential function over the real line is always infinite, we cannot get it to be equal to 1 – which means that it is not possible to attain the maximum, entropy can be as large as we want.

So how do we select a portfolio in such a situation?

A heuristic idea. In the situation in which we only know the means μ_i, we cannot use the Maximum Entropy approach to find the most appropriate probability distribution. However, here, the portions w_i – since they add up to 1 – can also be viewed as kind of probabilities. It therefore makes sense to look for a portfolio for which the corresponding entropy

$$-\sum_{i=1}^{n} w_i \cdot \ln(w_i) \tag{13}$$

attains the largest possible value under the constraints (1) and (2); see, e.g., [1, 3, 9, 10, 11, 12].

This heuristic idea sometimes leads to reasonable results. Here, entropy can be viewed as a measure of diversity. Thus, the idea to bring more diversity to one's portfolio makes perfect sense. However, there is a problem.

Remaining problem. The problem is that while the weights w_i do add up to one, they are *not* probabilities. So, in contrast to the probabilistic case, where the Maximum Entropy approach has many justifications, for the weights, there does not seem to be any reasonable justification. It is therefore desirable to either justify this heuristic method – or provide a justified alternative.

What we do in this paper. In this paper, we provide a justification for the Maximum Entropy approach. We also show that a similar idea can be applied to a slightly more complex – and more realistic – case, when we only know bounds $\underline{\mu}_i$ and $\overline{\mu}_i$ on the values μ_i.

2 Case When We Only Know the Expected Rates of Return μ_i: Economic Justification of the Maximum Entropy Approach

General definition. We want, given n expected return rates $\mu_1, \ldots, \mu_n$, to generate the weights $w_1 = f_{n1}(\mu_1, \ldots, \mu_n), \ldots, w_n = f_{nn}(\mu_1, \ldots, \mu_n)$ depending on μ_i for which the sum of the weights is equal to 1.

Definition 1. *By a* portfolio allocation scheme*, we mean a family of functions $f_{ni}(\mu_1, \ldots, \mu_n) \neq 0$ of non-negative variables μ_i, where n is arbitrary integer larger than 1, and $i = 1, 2, \ldots, n$, such that for all n and for all $\mu_i \geq 0$, we have*

$$\sum_{i=1}^{n} f_{ni}(\mu_1, \ldots, \mu_n) = 1.$$

Symmetry. Of course, the portfolio allocation should not depend on the order in which we list the instrument.

Definition 2. *We say that a portfolio allocation scheme is* symmetric *if for each n, for each $\mu_1, \ldots, \mu_n$, for each $i \le n$, and for each permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, we have*

$$f_{ni}(\mu_1, \ldots, \mu_n) = f_{n,\pi(i)}(\mu_{\pi(1)}, \ldots, \mu_{\pi(n)}).$$

Pairwise comparison. If we only have two financial instruments ($n = 2$) with expected rates μ_1 and μ_2, then we assign weights w_1 and $w_2 = 1 - w_1$ depending on the known values μ_1 and μ_2: $w_1 = f_{21}(\mu_1, \mu_2)$ and $w_2 = f_{22}(\mu_1, \mu_2)$.

In the general case, if we have n instruments including these two, then the amount $f_{n1}(\mu_1, \ldots, \mu_n) + f_{n2}(\mu_1, \ldots, \mu_n)$ is allocated for these two instruments. Once this amount is decided on, we should divide it optimally between these two instruments. The optimal division means that the first instrument gets the portion $f_{21}(w_1, w_2)$ of this overall amount, so we must have

$$f_{n1}(\mu_1, \mu_2, \ldots) = f_{21}(\mu_1, \mu_2) \cdot (f_{n1}(\mu_1, \ldots, \mu_n) + f_{n2}(\mu_1, \ldots, \mu_n)). \tag{14}$$

Thus, we arrive at the following definition.

Definition 3. *We say that a portfolio allocation scheme is* consistent *if for every $n > 2$ and for all $i \ne j$, we have*

$$f_{ni}(\mu_1, \ldots, \mu_n) = f_{21}(\mu_i, \mu_j) \cdot (f_{ni}(\mu_1, \ldots, \mu_n) + f_{nj}(\mu_1, \ldots, \mu_n)). \tag{15}$$

Proposition 1. *A portfolio allocation scheme is symmetric and consistent if and only if there exists a function $f(\mu) \ge 0$ for which*

$$f_{ni}(\mu_1, \ldots, \mu_n) = \frac{f(\mu_i)}{\sum\limits_{j=1}^{n} f(\mu_j)}. \tag{16}$$

Proof. It is easy to check that the formula (16) describes a symmetric and consistent portfolio allocation scheme. So, to complete the proof, it is sufficient to show that every symmetric and consistent portfolio allocation scheme has the form (16).

Indeed, let us assume that the portfolio allocation scheme satisfies the formula (15). If we write the formulas (15) for i and j and then divide the i-formula by the j-formula, we get the following equality:

$$\frac{f_{ni}(\mu_1, \ldots, \mu_n)}{f_{nj}(\mu_1, \ldots, \mu_n)} = \Phi(\mu_i, \mu_j) \stackrel{\text{def}}{=} \frac{f_{21}(\mu_i, \mu_j)}{f_{21}(\mu_j, \mu_i)}. \tag{17}$$

Due to symmetry, $f_{22}(\mu_i, \mu_j) = f_{21}(\mu_j, \mu_i)$, so we have

$$\Phi(\mu_i, \mu_j) = \frac{f_{21}(\mu_i, \mu_j)}{f_{21}(\mu_j, \mu_i)} \tag{18}$$

and

$$\Phi(\mu_j, \mu_i) = \frac{f_{21}(\mu_j, \mu_i)}{f_{21}(\mu_i, \mu_j)}, \tag{19}$$

thus

$$\Phi(\mu_j, \mu_i) = \frac{1}{\Phi(\mu_i, \mu_j)}. \tag{20}$$

Now, for each i, j, and k, we have

$$\frac{f_{ni}(\mu_1, \ldots, \mu_n)}{f_{nj}(\mu_1, \ldots, \mu_n)} = \frac{f_{ni}(\mu_1, \ldots, \mu_n)}{f_{nk}(\mu_1, \ldots, \mu_n)} \cdot \frac{f_{nk}(\mu_1, \ldots, \mu_n)}{f_{nj}(\mu_1, \ldots, \mu_n)},$$

thus

$$\Phi(\mu_i, \mu_j) = \Phi(\mu_i, \mu_k) \cdot \Phi(\mu_k, \mu_j).$$

In particular, for $\mu_k = 1$, we have

$$\Phi(\mu_i, \mu_j) = \Phi(\mu_i, 1) \cdot \Phi(1, \mu_j). \tag{21}$$

Due to (20), this means that

$$\Phi(\mu_i, \mu_j) = \frac{\Phi(\mu_i, 1)}{\Phi(\mu_j, 1)}, \tag{22}$$

i.e.,

$$\Phi(\mu_i, \mu_j) = \frac{f(\mu_i)}{f(\mu_j)}, \tag{23}$$

where we denoted $f(\mu) \stackrel{\text{def}}{=} F(\mu, 1)$. Substituting this expression (23) into the formula (17) and taking $j = 1$, we conclude that

$$\frac{f_{ni}(\mu_1, \ldots, \mu_n)}{f_{n1}(\mu_1, \ldots, \mu_n)} = \frac{f(\mu_i)}{f(\mu_1)}, \tag{24}$$

i.e.,

$$f_{ni}(\mu_1, \ldots, \mu_n) = C \cdot f(\mu_i), \tag{25}$$

where we denoted

$$C \stackrel{\text{def}}{=} \frac{f_{n1}(\mu_1, \ldots, \mu_n)}{f(\mu_1)}.$$

From the condition that the values f_{nj} corresponding to $j = 1, \ldots, n$ should add up to 1, we conclude that $C \cdot \sum\limits_{j=1}^{n} f(\mu_j) = 1$, hence

$$C = \frac{1}{\sum\limits_{j=1} f(\mu_j)}$$

and thus, the expression (25) takes exactly the desired form.

The proposition is proven.

Monotonicity. If all we know about each financial instruments is their expected rate of return, then it is reasonable to assume that the larger the expected rate of return, the better the instrument. It is therefore reasonable to require that the larger the rate of return, the larger portion of the original amount should be invested in this instrument.

Definition 4. *We say that a portfolio allocation scheme is* monotonic *if for each n and each μ_i, if $\mu_i \geq \mu_j$, then $f_{ni}(\mu_1, \ldots, \mu_n) \geq f_{nj}(\mu_1, \ldots, \mu_n)$.*

One can easily check that a symmetric and consistent portfolio allocation scheme is monotonic if and only if the corresponding function $f(\mu)$ is non-decreasing.

Shift-invariance. Suppose that, in addition to the return from the investment, a person also get some additional fixed income, which when divided by the amount of money to be invested, translates into the rate r_0. This situation can be described in two different ways:

- we can consider r_0 separately from the investment; in this case, we should allocate, to each financial instrument i, the portion $f_i(\mu_1, \ldots, \mu_n)$;
- alternatively, we can combine both incomes into one and say that for each instrument i, we will get the expected rate of return $\mu_i + r_0$; in this case, to each financial instrument i, we allocate a portion $f_i(\mu_1 + r_0, \ldots, \mu_n + r_0)$.

Clearly, this is the same situations described in two different ways, so the portfolio allocation should not depend on how exactly we represent the same situation. Thus, we arrive at the following definition.

Definition 5. *We say that a portfolio allocation scheme is* shift-invariant *if for all n, for all $\mu_1, \ldots, \mu_n$, for all i, and for all r_0, we have*

$$f_{ni}(\mu_1, \ldots, \mu_n) = f_{ni}(\mu_1 + r_0, \ldots, \mu_n + r_0).$$

Proposition 2. *For each portfolio allocation scheme, the following two conditions are equivalent to each other:*

- *the scheme is symmetric, consistent, monotonic, and shift-invariant, and*
- *the scheme has the form*

$$f_{ni}(\mu_1, \ldots, \mu_n) = \frac{\exp(\beta \cdot \mu_i)}{\sum\limits_{j=1}^{n} \exp(\beta \cdot \mu_j)}. \tag{26}$$

for some $\beta \geq 0$.

Proof. It is clear that the scheme (26) has all the desired properties. Vice versa, let us assume that a scheme has all the desired properties. Then, from shift-invariance, for each i and j, we get

$$\frac{f_{ni}(\mu_1, \ldots, \mu_n)}{f_{nj}(\mu_1, \ldots, \mu_n)} = \frac{f_{ni}(\mu_1 + r_0, \ldots, \mu_n + r_0)}{f_{nj}(\mu_1 + r_0, \ldots, \mu_n + r_0)}. \tag{27}$$

Substituting the formula (16), we conclude that

$$\frac{f(\mu_i)}{f(\mu_j)} = \frac{f(\mu_i + r_0)}{f(\mu_j + r_0)}, \tag{28}$$

which implies that

$$\frac{f(\mu_i + r_0)}{f(\mu_i)} = \frac{f(\mu_j + r_0)}{f(\mu_j)}. \tag{29}$$

The left-hand side of this equality does not depend on μ_j, the right-hand side does not depend on μ_i. Thus, the ratio depends only on r_0. Let us denote this ratio by $R(r_0)$. Then, we get $f(\mu + r_0) = R(r_0) \cdot f(\mu)$.

It is known (see, e.g., [2]) that every non-decreasing solution to this functional equation has the form $\text{const} \cdot \exp(\beta \cdot \mu)$ for some $\beta \geq 0$. The proposition is proven.

Main result. Now, we are ready to formulate our main result – an economic justification of the above heuristic method.

Proposition 3. *Let μ be the desired expected return rate, and assume that we only consider allocation schemes providing this expected return rate, i.e., schemes for which*

$$\sum_{i=1}^{n} \mu_i \cdot w_i = \sum_{i=1}^{n} \mu_i \cdot f_{ni}(\mu_1, \ldots, \mu_n) = \mu. \tag{30}$$

Then, the following two conditions on a portfolio allocation schemes are equivalent to each other:

- *the scheme is symmetric, consistent, monotonic, and shift-invariant, and*
- *the scheme has the largest possible entropy $-\sum\limits_{i=1}^{n} w_i \cdot \ln(w_i)$ among all the schemes with the given expected return rate.*

Proof. Maximizing entropy under the constraints $\sum w_i \cdot \mu_i = \mu_0$ and $\sum w_i = 1$ is, due to Lagrange multiplier method, equivalent to maximizing the expression

$$-\sum_{i=1}^{n} w_i \cdot \ln(w_i) + \lambda_1 \cdot \left(\sum_{i=1}^{n} w_i \cdot \mu_i - \mu \right) + \lambda_2 \cdot \left(\sum_{i=1}^{n} w_i - 1 \right). \tag{31}$$

Differentiating this expression by w_i and equating the derivative to 0, we conclude that

$$-\ln(w_i) - 1 + \lambda_1 \cdot \mu_1 + \lambda_2 = 0, \tag{32}$$

i.e., that

$$w_i = \text{const} \cdot \exp(\lambda_1 \cdot \mu_i).$$

This is exactly the expression (26) which, as we have proved in Proposition 2, is indeed equivalent to symmetry, consistency, monotonicity, and shift-invariance. The proposition is proven.

Discussion. What we proved, in effect, is that maximizing diversity is a great idea, be it diversity when distributing money between financial instrument, or – when the state invests in its citizens – when we allocate the budget between cities, between districts, between ethic groups, or when a company is investing in its future by hiring people of different backgrounds.

3 Case When We Only Know the Intervals $[\underline{\mu}_i, \overline{\mu}_i]$ Containing the Actual (Unknown) Expected Return Rates

Description of the case. Let us now consider an even more realistic case, when we take into account that the expected rates of return μ_i are only approximately known. To be precise, we assume that for each i, we only know the interval $[\underline{\mu}_i, \overline{\mu}_i]$ containing the actual (unknown) expected return rates μ_i. How should we then distribute the investments?

Definition 6. *By an* interval-based portfolio allocation scheme*, we mean a family of functions $f_{ni}(\underline{\mu}_1, \overline{\mu}_1 \ldots, \underline{\mu}_n, \overline{\mu}_n) \neq 0$ of non-negative variables μ_i, where n is an arbitrary integer larger than 1, and $i = 1, 2, \ldots, n$, such that for all n and for all $0 \le \underline{\mu}_i \le \overline{\mu}_i$, we have $\sum\limits_{i=1}^{n} f_{ni}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n) = 1$.*

Definition 7. *We say that an interval-based portfolio allocation scheme is* symmetric *if for each n, for each $\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n$, for each $i \le n$, and for each permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, we have*

$$f_{ni}(\underline{\mu}_1, \overline{\mu}_1 \ldots, \underline{\mu}_n, \overline{\mu}_n) = f_{n,\pi(i)}(\underline{\mu}_{\pi(1)}, \overline{\mu}_{\pi(1)}, \ldots, \underline{\mu}_{\pi(n)}, \overline{\mu}_{\pi(n)}).$$

Definition 8. *We say that an interval-based portfolio allocation scheme is* consistent *if for every $n > 2$ and for all $i \neq j$, we have*

$$f_{ni}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n) =$$
$$f_{21}(\underline{\mu}_i, \overline{\mu}_i, \underline{\mu}_j, \overline{\mu}_j) \cdot (f_{ni}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n) + f_{nj}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n)).$$

Proposition 4. *An interval-based portfolio allocation scheme is symmetric and consistent if and only if there exists a function $f(\underline{\mu}, \overline{\mu}) \ge 0$ for which*

$$f_{ni}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n) = \frac{f(\underline{\mu}_i, \overline{\mu}_i)}{\sum\limits_{j=1}^{n} f(\underline{\mu}_j, \overline{\mu}_j)}.$$

Proof is similar to the proof of Proposition 1.

Definition 9. *We say that an interval-based portfolio allocation scheme is* monotonic *if for each n and each $\underline{\mu}_i$ and $\overline{\mu}_i$, if $\underline{\mu}_i \ge \underline{\mu}_j$ and $\overline{\mu}_i \ge \overline{\mu}_j$, then*

$$f_{ni}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n) \ge f_{nj}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n).$$

One can easily check that a symmetric and consistent portfolio allocation scheme is monotonic if and only if the corresponding function $f(\underline{\mu}, \overline{\mu})$ is non-decreasing in both variables.

Additivity. Let us assume that in year 1, we have instruments with bounds $\underline{\mu}_i$ and $\overline{\mu}_i$, and in year 2, we have a different set of instruments, with bounds $\underline{\mu}'_j$ and $\overline{\mu}'_j$. Then, we can view this situation in two different ways:

- we can view it as two different portfolio allocations, with allocations w_i in the first year and independently, allocations w'_j in the second year; since these two years are treated independently, the portion of money that goes into the i-th instrument in the first year and in the j-th instrument in the second year can be simply computed as a product $w_i \cdot w'_j$ of the corresponding portions;
- alternatively, we can consider portfolio allocation as a 2-year problem, with $n \cdot m$ possible options, so that for each option (i, j), the expected return is the sum $\mu_i + \mu'_j$ of the corresponding expected returns; since μ_i is in the interval $[\underline{\mu}_i, \overline{\mu}_i]$ and μ'_j is in the interval $[\underline{\mu}'_j, \overline{\mu}'_j]$, the sum $\mu_i + \mu'_j$ can take all the values from $\underline{\mu}_i + \underline{\mu}'_i$ to $\overline{\mu}_i + \overline{\mu}'_j$.

It is reasonable to require that the resulting portfolio allocation not depend on how exactly we represent this situation.

Definition 10. *An interval-based portfolio allocation scheme is called* additive *if for every n and m, for all values $\underline{\mu}_i$, $\overline{\mu}_i$, $\underline{\mu}'_i$, and $\overline{\mu}'_i$, and for every i and j, we have*

$$f_{n \cdot m, i, j}(\underline{\mu}_1 + \underline{\mu}'_1, \overline{\mu}_1 + \overline{\mu}'_1, \underline{\mu}_1 + \underline{\mu}'_2, \overline{\mu}_1 + \overline{\mu}'_2, \ldots, \underline{\mu}_n + \underline{\mu}'_m, \overline{\mu}_n + \overline{\mu}'_m) =$$

$$f_{ni}(\underline{\mu}_1, \overline{\mu}_1, \ldots, \underline{\mu}_n, \overline{\mu}_n) \cdot f_{mj}(\underline{\mu}'_1, \overline{\mu}'_1, \ldots, \underline{\mu}'_n, \overline{\mu}'_n).$$

Proposition 5. *A symmetric and consistent interval-based portfolio allocation scheme is additive if and only if the corresponding function $f(\underline{u}, \overline{u})$ has the form*

$$f(\underline{u}, \overline{u}) = \exp(\underline{\beta} \cdot \underline{u} + \overline{\beta} \cdot \overline{u})$$

for some $\underline{\beta} \ge 0$ and $\overline{\beta} \ge 0$.

Proof. In terms of the function $f(\underline{u},\overline{u})$, additivity takes the form

$$f(\underline{u}+\underline{u}',\overline{u}+\overline{u}')=C\cdot f(\underline{u},\overline{u})\cdot f(\underline{u}',\overline{u}').$$

For $F\stackrel{\rm def}{=}\ln(f)$, this equation has the form

$$F(\underline{u}+\underline{u}',\overline{u}+\overline{u}')=c+F(\underline{u},\overline{u})+F(\underline{u}',\overline{u}'),$$

where $c\stackrel{\rm def}{=}\ln(C)$. For $G\stackrel{\rm def}{=}F+c$, we have

$$G(\underline{u}+\underline{u}',\overline{u}+\overline{u}')=G(\underline{u},\overline{u})+G(\underline{u}',\overline{u}').$$

According to [2], the only monotonic solution to this equation is a linear function. Thus, the function $f=\exp(F)=\exp(G-c)=\exp(-c)\cdot\exp(G)$ has the desired form. The proposition is proven.

Relation to Hurwicz approach to decision making under interval uncertainty. The above formula has the form $\exp(\beta\cdot(\alpha_H\cdot\overline{u}+(1-\alpha_H)\cdot\underline{u}))$, where $\beta\stackrel{\rm def}{=}\underline{\beta}+\overline{\beta}$ and $\alpha_H\stackrel{\rm def}{=}\overline{\beta}/\beta$.

Thus, it is equivalent to using the non-interval formula with

$$u=\alpha_H\cdot\overline{u}+(1-\alpha_H)\cdot\underline{u}.$$

This is exactly the utility equivalent to an interval proposed by a Nobelist Leo Hurwicz; see, e.g., [4, 6, 7].

Relation to maximum entropy. This formula corresponds to maximizing entropy under the constraint that the expected value of the Hurwicz combination $u=\alpha_H\cdot\overline{u}+(1-\alpha_H)\cdot\underline{u}$ takes a given value.

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