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Contrast Similarity Measures of Fuzzy Sets

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Abstract. The paper proposes a new class of fuzzy set similarity measures taking into account the proximity of membership values to the border values 0 and 1. These similarity measures take values in [0,1] and generalize the crisp weak equality relation of fuzzy sets considered in the theory of fuzzy sets. The method of construction of a contrast similarity measure using a bipolar function symmetric with respect to 0.5 is presented. The similarity measure defined by the contrast intensification operation considered by Lotfi Zadeh is discussed.

Keywords: Similarity Measure, Similarity Function, Fuzzy Set

1 Introduction

Similarity and distance measures between membership values and between fuzzy sets are considered in many works. References on some of them can be found in the following recent papers [6,10]. Usually, for discrete fuzzy sets, similarity measures are based on an adaptation of Euclidean or Manhattan distances applied to membership functions, on generalization of symmetric difference of sets or on Jaccard similarity, etc [6,10]. Usually these measures compare the membership values of elements of fuzzy sets but they do not depend on the proximity of membership values to 0 and 1. But this information can be important for the following reasons. The membership function was introduced by Lotfi Zadeh [11] as a generalization of the characteristic function of a set taking the value 1 for the elements that belong to the set and the value 0 for the elements that do not belong to the set. For this reason, the proximity of membership values of elements of fuzzy sets to 1 or 0 can characterize to what certainty the element of fuzzy set "possesses" or "not possesses" the property modelled by the fuzzy set. In this case the similarity between two membership values can depend not only on difference between them but also on their proximity to 1 or 0. Similar considerations were used in the definition of the non-probabilistic entropy of fuzzy set [7] as a measure of uncertainty that decrease when the membership values of fuzzy set approach 1 or 0. The crucial point in bipolar interpretation of fuzzy set is the number 0.5. In [8], it was considered the *weak equality* of fuzzy sets that is fulfilled for two fuzzy sets A and B if and only if for all elements x of X the following is true: both membership values $A(x)$ and $B(x)$ are either greater than or equal to 0.5 or both smaller than or equal to 0.5. Another example gives the operation of *contrast intensification* of fuzzy sets [13] that makes the fuzzy set more contrast as a result of increasing or decreasing its membership values if they are greater or smaller than 0.5, respectively.

This paper proposes a new class of (dis)similarity measures between fuzzy sets taking into account the proximity of membership values to the border values 1 and 0. These similarity measures take values in [0,1] and generalize the crisp weak equality relation of fuzzy sets. Because one of these similarity and dissimilarity measures is based on the operation of contrast intensification of fuzzy sets such measures are called here as "contrast" (dis)similarity measures. More generally, the method of construction of a contrast similarity measure using a bipolar function symmetric with respect to 0.5 is considered.

2 Basic Definitions

A fuzzy subset A of the set X is defined as a function $A: X \to [0,1]$ such that for any x in X, the value $A(x)$ belongs to the interval [0,1] and denotes a degree of membership of x in the fuzzy set A. If X is a finite set with n elements, its elements will be denoted also as x_i, i= 1, …, n. In this case: $X = \{x_1, \dots, x_n\}$, and $A(x_i)$, $i = 1, \dots, n$, are membership values of the elements of X in the fuzzy set A. The membership values $A(x_i)$ in [0,1] will de denoted also by letters a, b, c.

Let us denote by Ω the set of all fuzzy sets on X. As in [4], a measure of similarity between fuzzy sets is defined as a *similarity function* $S: \Omega \times \Omega \to [0,1]$ satisfying for all fuzzy sets A and B in Ω the properties of *symmetry*:

$$S(A, B) = S(B, A),$$

and *reflexivity*:

$$S(A, A) = 1.$$

The definition of a similarity function coincides with the definition of a fuzzy (valued) proximity relation [12, 1, 9] and the properties of valued proximity relations can be extended also on similarity functions [4]. Generally, similarity functions have deep relationship with correlation functions (association measures) taking values in interval [-1,1], and similarity functions satisfying some additional properties can be used for constructing correlation functions [2,4,5].

Dually to the similarity function, a *dissimilarity function* is defined as a function $D: \Omega \times \Omega \to [0,1]$ satisfying for all fuzzy sets A and B in Ω the properties of *symmetry*:

$$D(A, B) = D(B, A),$$

and *irreflexivity*:

$$D(A, A) = 0.$$

A similarity function S and a dissimilarity function D are called *complementary* if for all fuzzy sets A and B in Ω we have:

$$S(A,B) + D(A,B) = 1.$$

Hence, if we define one of these functions then its complementary function can be obtained using one of these equations:

$$D(A,B) = 1 - S(A,B),\ \ S(A,B) = 1 - D(A,B).$$

Using negation [11] defined for all a in [0,1] by $N(a) = 1 - a$, we obtain for complementary similarity and dissimilarity functions: $D(A,B) = N\big(S(A,B)\big)$, and $S(A,B) = N\big(D(A,B)\big)$. If we consider *S* and *D* as fuzzy (valued) relations then *D* is the complement of *S* and vice versa. Generally, similarity and dissimilarity functions will be called (dis)similarity functions or resemblance functions [4].

The negation *N* on the set of membership values [0,1] defines the fuzzy complement $\bar{A}$ of a fuzzy set *A* for all elements *x* in *X* as follows:

$$\bar{A}(x) = N\big(A(x)\big) = 1 - A(x).$$

Resemblance functions will be called *co-symmetric* [4] if for all fuzzy sets *A* and *B* in Ω we have, respectively:

$$S(\bar{A},\bar{B}) = S(A,B), \quad D(\bar{A},\bar{B}) = D(A,B).$$

Similarity *S* and dissimilarity *D* functions defined on the set Ω of all fuzzy subsets of the set *X* will be constructed using, respectively, similarity functions $s: [0,1] \times [0,1] \to [0,1]$, and dissimilarity functions $d: [0,1] \times [0,1] \to [0,1]$, defined on the set of membership values [0,1], as follows:

$$S(A,B) = \frac{1}{n}\sum_{i=1}^{n} s\big(A(x_i), B(x_i)\big), \quad D(A,B) = \frac{1}{n}\sum_{i=1}^{n} d\big(A(x_i), B(x_i)\big). \tag{1}$$

It is clear that if *s* and *d* are symmetric, reflexive or irreflexive then for *S* and *D* the corresponding properties also fulfilled. It is clear also that if *s* and *d* are co-symmetric, i.e., for all *a*, *b* in [0,1], we have:

$$s(1-a, 1-b) = s(a,b), \quad d(1-a, 1-b) = d(a,b),$$

then the functions *S* and *D* in (1) will also be co-symmetric.

3 Constructing Contrast Fuzzy Set Similarity Measure

Consider an approach constructing contrast (dis)similarity measures between fuzzy sets that takes into account a proximity of membership values to the values 0 and 1.

Here is the simplest contrast dissimilarity function and its complementary similarity function on [0,1]:

$$d_1(a,b) = \begin{cases} 1, & \text{if } a < 0.5 < b \\ 0, & otherwise \end{cases}, \quad s_1(a,b) = \begin{cases} 0, & \text{if } a < 0.5 < b \\ 1, & otherwise \end{cases}. \tag{2}$$

The similarity function s_1 defines by (1) the corresponding similarity function between two fuzzy sets A and B:

$$S_1(A,B) = \frac{1}{n}\sum_{i=1}^{n} s_1\big(A(x_i), B(x_i)\big).$$

This similarity function related with a weak equality considered in [8] as follows. The weak equality is fulfilled for two fuzzy sets A and B if and only if $S_1(A,B) = 1$, i.e. when for all elements x of X both membership values $A(x)$ and $B(x)$ are either greater than or equal to 0.5 or both smaller than or equal to 0.5.

Below there is a modification of the dissimilarity function (2):

$$d_2(a,b) = \begin{cases} 0, & \text{if } (a,b < 0.5) \text{ or } (0.5 < a,b) \text{ or } (a = b = 0.5) \\ 0.5, & \text{if } a \neq b \text{ and } (a = 0.5 \text{ or } b = 0.5) \\ 1, & \text{if } a < 0.5 < b \end{cases} . \tag{3}$$

This definition reflects the situation of uncertainty appearing for dissimilarity function (2) when one of two numbers a,b equals to 0.5 and another not, for example when $a = 0.5$ and $b > 0.5$. Suppose for two fuzzy sets A and B we have for some element x in X: $A(x) = a = 0.5$ and $B(x) = b > 0.5$. In we interpret this situation as bipolar with "x rather belongs to A" and "x rather belongs to B" then $d_1\big(A(x), B(x)\big) = 0$, $s_1\big(A(x), B(x)\big) = 1$. The opposite interpretation of $A(x) = a = 0.5$ as "x rather do not belong to A" will give $d_1\big(A(x), B(x)\big) = 1$, $s_1\big(A(x), B(x)\big) = 0$. The dissimilarity function (3) tries to correct such ambiguity of dissimilarity function (2).

It is easy to check that both dissimilarity functions in (2) and (3) are irreflexive, symmetric and co-symmetric. Below we consider the method of construction of *contrast* co-symmetric dissimilarity functions using bipolar transformations of membership values. The idea is based on the operation of *contrast intensification* introduced by Zadeh [13] that makes the membership function more contrast transforming them in the directions to the borders 0 and 1 of the interval [0,1] of membership values. Some results related with bipolar functions can be found also in [3].

Definition 1. An increasing real-valued function $f: [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$ is called a *bipolar transformation* of [0,1] if for all a in [0,1] it satisfies the following condition:

$$f(a) + f\big(N(a)\big) = 1. \tag{4}$$

For negation $N(a) = 1 - a$ we obtain from (4): $f(a) + f(1 - a) = 1$,

Proposition 1. Let f be a bipolar transformation of [0,1] then the function defined for all a,b in [0,1] by:

$$d(a,b) = |f(a) - f(b)|, \tag{5}$$

is the co-symmetric dissimilarity function.

It is easy to see that the dissimilarity function d_2 defined by (3) can be constructed by (5) using the following bipolar transformation:

$$f_2(a) = \begin{cases} 1, & if\ a > 0.5 \\ 0.5, & if\ a = 0.5. \\ 0, & if\ a < 0.5 \end{cases}$$

Definition 2. Let q be a positive real constant such that q < 0.5 and *a*,*b*,*c* be the elements of [0,1] satisfying the following properties:

$$|a - b| = |b - c| = q,$$

then a dissimilarity function $d: [0,1] \times [0,1] \rightarrow [0,1]$ will be referred to as a *contrast dissimilarity function* if it is co-symmetric, and we have the following:

$$d(a, b) \leq d(b, c) \quad if\ a < b < c < 0.5\ ,$$

$$d(a, b) < d(b, c) \quad if\ \mathrm{a} < b < 0.5 < c\ , \tag{6}$$

$$d(a, b) > d(b, c) \quad if\ a < 0.5 < b < c, \tag{7}$$

$$d(a, b) \geq d(b, c) \quad if\ 0.5 < a < b < c.$$

Definition 3. A bipolar function $f: [0,1] \rightarrow [0,1]$ is called a *contrast transformation* of [0,1] if the dissimilarity function:

$$d(a, b) = |f(a) - f(b)|$$

is contrast.

Consider an example of contrast transformation that is not so drastic as f_2:

$$f_3(a) = \begin{cases} (1 - 2r)a + 2r, & if\ a > 0.5 \\ 0.5, & if\ a = 0.5\ , \\ (1 - 2r)a, & if\ a < 0.5 \end{cases}$$

where *r* is a nonnegative parameter such that $r \leq 0.5$. When $r = 0.5$ we have $f_3 = f_2$.

Another example of the contrast transformation gives the *contrast intensification* operation proposed by Zadeh [13] by:

$$f_4(a) = \begin{cases} 2a^2, & if\ a \leq 0.5 \\ 1 - 2(1 - a)^2, & if\ a \geq 0.5\text{'} \end{cases}$$

where $0 \leq a \leq 1$.

4 Conclusion and future work

The paper introduced the new (dis)similarity measures between membership functions called "contrast" that take into account the proximity of membership values to the border values 0 and 1. The method of construction of such measures using bipolar contrast transformation of membership values is proposed. An example of such transformation is the contrast intensification operator introduced by Zadeh. In the future work we plan to describe the general form of contrast transformations and to introduce parametric families of such transformations.

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