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Decision Making Under General Set Uncertainty: Additivity Approach

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Abstract

In many practical situations, we need to make a decision under interval or set uncertainty: e.g., we need to decide how much we are willing to pay for an option that will bring us between \$10 and \$40, i.e., for which the set of possible gains is the interval $S = [10, 40]$. To make such decisions, researcher have used the idea of additivity: that if have two independent options, then the price we pay for both should be equal to the sum of the prices that we pay for each of these options. It is known that this requirement enables us to make decisions for bounded closed sets S. In some practical situations, the set S of possible gains is not closed: e.g., we may know that the gain will be between \$10 and \$40, but always greater than \$10 and always smaller than \$40. In this case, the set of possible values in an open interval $S = (10, 40)$. In this paper, we show how to make decisions in situations of general – not necessarily closed – set uncertainty.

1 Decision Making Under Set Uncertainty: What Is Known and What Is the Remaining Problem

Need for decision making under interval uncertainty. In many practical situations, we do not know the exact consequences of different alternatives. For example, we may know that investing \$1000 into a certain project will bring us between \$10 and \$40 at the end of year, but we do not know how much exactly. On the other hand, there are usually some alternatives with known results: e.g., we can place this amount into a saving account at the bank, this will bring us exactly \$20 at the end of the year. In the first case, all we know about our gain is it is somewhere in the interval $[10, 40]$, in the second case the gain is 20. Which of these two alternatives is better?

To be able to make a choice, we must be able to compare intervals with real numbers and intervals with intervals.

From interval to set uncertainty. In some cases, we know that not all the values from the corresponding interval are possible. For example, we may know that we will either get \$10 or \$40. In this case, the set of the possible values is not the whole interval $[10, 40]$, but the 2-point set $\{10, 40\}$.

We may have more complicated situations, e.g., we may have either \$10, or some value between \$30 and \$40. In this case, the set of possible values is

$$\{10\} \cup [30, 40].$$

To make decisions in such situations, we need to be able to compare sets with intervals and numbers – and with each other.

Additivity: the main idea behind such decision making. If:

- in one situation, we have a set S_1 of possible gains s_1, and
- in another independent situation, we have a set S_2 of possible gains s_2,

then, by participating in both situation, we can gain the value $s = s_1 + s_2$. The set S of possible values of the overall gain can be obtained if we consider all possible values $s_1 \in S_1$ and $s_2 \in S_2$:

$$S = S_1 + S_2 \stackrel{\text{def}}{=} \{s_1 + s_2 : s_1 \in S_1 \text{ and } s_2 \in S_2\}. \quad (1)$$

A reasonable idea is to assign, to each set S, a numerical value $u(S)$: the price we are willing to pay to participate in this situation. In these terms, if the two sets S_1 and S_2 have the same price ($u(S_1) = u(S_2)$), we say that these two sets are *equivalent* and denote it by $S_1 \equiv S_2$.

The price to pay to participate in both events should be equal to the sum of the prices that we pay to participate in each of these events, i.e., we should have

$$u(S_1 + S_2) = u(S_1) + u(S_2). \quad (2)$$

This property is known as *additivity*.

Definition 1. *Let $\mathcal{S}$ be a class of sets which is closed under set addition. We say that a function $u : \mathcal{S} \to IR$ is* additive *if for every two sets $S_1, S_2 \in \mathcal{S}$, we have $u(S_1 + S_2) = u(S_1) + u(S_2)$.*

If we assume additivity, then we can make the following corollary.

Definition 2. *Let $\mathcal{S}$ be a class of sets which is closed under set addition. An equivalence relation $\equiv$ is called* additive *if the following condition is satisfied for all $S_1, S_1', S_2 \in \mathcal{S}$:*

$$\textit{if } S_1 + S_2 = S_1' + S_2 \textit{ then } S_1 \equiv S_1'. \quad (2)$$

Proposition 1. *For every additive function u, the relation*

$$S_1 \equiv S_2 \stackrel{\text{def}}{=} (u(S_1) = u(S_2))$$

is additive.

Proof. Indeed, if $S_1 + S_2 = S_1' + S_2$, then, due to additivity, we have

$$u(S_1) + u(S_2) = u(S_1') + u(S_2).$$

Thus, $u(S_1') = u(S_1)$ and $S_1' \equiv S_1$. The statement is proven.

Decision making under interval uncertainty: what is known. In case the set of possible gains is an interval $[\underline{a}, \overline{a}]$, no matter what happens, we will get at least $\underline{a}$ and at most $\overline{a}$. Thus, the price of this interval cannot be lower than $\underline{a}$ and cannot be higher than $\overline{a}$.

Definition 2. *We say that a real-valued function u defined on the set of all intervals is* consistent *if for each interval, we have $\underline{a} \le u([\underline{a}, \overline{a}]) \le \overline{a}$.*

Proposition 2. [2, 4] *Every consistent additive function u on the set of all intervals has the form*

$$u([\underline{a}, \overline{a}]) = \alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u}, \tag{3}$$

for some $\alpha \in [0, 1]$.

This formula was first proposed by the future Nobel prize winner Leo Hurwicz and is, thus, known as *Hurwicz optimism-pessimism criterion* [1, 3].

- Optimism in this name corresponds to the case $\alpha = 1$, when a decision maker values the interval as much as its largest value – i.e., in effect, considers only the best value from this interval to be possible.

- Similarly, pessimism corresponds to the case $\alpha = 0$, when a decision maker values the interval as much as its smallest value – i.e., in effect, considers only the worst value from this interval to be possible.

Decision making under set uncertainty: what is known. What is known is how to make a decision when the set S is bounded and closed – i.e., contains all its limit points.

In this case, we have the following result.

Proposition 3. *For every additive equivalence relation on the set of all bounded closed sets, each such set S is equivalent to the corresponding interval* $[\inf(S), \sup(S)]$.

Corollary. *For each additive function on the set of all bounded closed sets, the utility of each set S is equal to the utility of the corresponding interval*

$$[\inf(S), \sup(S)].$$

Proof of Proposition 3. Every bounded closed sets contains its limit points; in particular, it contains the points $\inf(S)$ and $\sup(S)$. Thus,

$$\{\inf(S), \sup(S)\} \subseteq S \subseteq [\inf(S), \sup(S)].$$

So, by a clear set-inclusion monotonicity of set addition, we conclude that

$$\{\inf(S), \sup(S)\} + [\inf(S), \sup(S)] \subseteq S + [\inf(S), \sup(S)] \subseteq$$

$$[\inf(S), \sup(S)] + [\inf(S), \sup(S)].$$

However, one can easily check that

$$\{\inf(S), \sup(S)\} + [\inf(S), \sup(S)] = [\inf(S), \sup(S)] + [\inf(S), \sup(S)] =$$

$$[2\inf(S), 2\sup(S)].$$

Thus, we the intermediate set $S + [\inf(S), \sup(S)]$ should be equal to the same interval:

$$S + [\inf(S), \sup(S)] = [\inf(S), \sup(S)] + [\inf(S), \sup(S)] = [2\inf(S), 2\sup(S)].$$

Since the equivalence relation is assumed to be additive, we conclude that

$$S \equiv [\inf(S), \sup(S)].$$

The proposition is proven.

Remaining problem. Boundedness is reasonable: in all real-life situations, we have lower and upper bounds on possible gains:

- in usual investments, we do not expect to gain millions, and
- we do not exact to lose millions – since usually, we just do not have these millions to lose.

However, the requirement that the set be closed may be too restrictive.

For example, we may know that the gain will be between 0 and \$100, but we are sure that the gain cannot be zero and cannot be exactly \$100. In this case, the set S of possible values of gain is an open interval $(0, 100)$, an interval that does not contain its limit points 0 and 100.

How can we make decision under such general (not necessarily closed) set uncertainty? This is a question that we analyze in this paper.

2 Main Result

Proposition 4. *For every additive equivalence relation on the set of all bounded sets, each such set S is equivalent to the corresponding interval* $[\inf(S), \sup(S)]$.

Comment. In other words, not only every bounded closed set is equivalent to the corresponding interval: every bounded set S (not necessarily closed one) is equivalent to the interval $[\inf(S), \sup(S)]$.

Proof.

1°. Let us first show that each open or semi-open interval is equivalent to the corresponding closed interval. Indeed, one can easily check that

$$(\underline{a}, \overline{a}) + (\underline{a}, \overline{a}) = [\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = (2\underline{a}, 2\overline{a}),$$

thus, by definition of additivity of an equivalence relation, we get

$$(\underline{a}, \overline{a}) \equiv [\underline{a}, \overline{a}].$$

Similarly, from

$$(\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = [\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = (2\underline{a}, 2\overline{a}),$$

we conclude that

$$(\underline{a}, \overline{a}] \equiv [\underline{a}, \overline{a}],$$

and from

$$[\underline{a}, \overline{a}) + (\underline{a}, \overline{a}) = [\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = (2\underline{a}, 2\overline{a}),$$

we conclude that

$$[\underline{a}, \overline{a}) \equiv [\underline{a}, \overline{a}].$$

2°. Let us now consider a general bounded set S. If this set contains both points $\inf(S)$ and $\sup(S)$, then the equivalence of the set S and the corresponding interval follows from the proof of Proposition 3. Thus, to complete our proof, it is sufficient to consider the case when either $\inf(S) \notin S$ or $\sup(S) \notin S$. Without losing generality, let us consider the case when $\inf(S) \notin S$.

Let us prove that in this case, we have

$$S + (\inf(S), \sup(S)) = (2\inf(S), 2\sup(S)). \tag{4}$$

Since it is easy to check that

$$(\inf(S), \sup(S)) + (\inf(S), \sup(S)) = (2\inf(S), 2\sup(S)),$$

the equality (4) would imply that

$$S + (\inf(S), \sup(S)) = (\inf(S), \sup(S)) + (\inf(S), \sup(S))$$

and thus, by additivity of the equivalence relation, $S \equiv (\inf(S), \sup(S))$. Since in Part 1 of this proof, we have shown that $(\inf(S), \sup(S)) \equiv [\inf(S), \sup(S)]$, we will thus be able to conclude that $S \equiv [\inf(S), \sup(S)]$, which is exactly what we want to prove. So, all we need to do is prove the equality (4).

The two sets are equal if the first is contained in the second one, and vice versa. Here, $S \subseteq (\inf(S), \sup(S)]$, thus

$$S + (\inf(S), \sup(S)) \subseteq (\inf(S), \sup(S)] + (\inf(S), \sup(S)) = (2\inf(S), 2\sup(S)).$$

Thus, to complete the proof, it is sufficient to prove that, vice versa, every number s from the interval $(2\inf(S), 2\sup(S))$ belongs to the sum

$$S + (\inf(S), \sup(S)),$$

i.e., that this number s can be represented as $s_1 + s_2$, where

$$s_1 \in S \text{ and } s_2 \in (\inf(S), \sup(S)).$$

To prove this, let us consider two possible cases: $s \leq \inf(S) + \sup(S)$ and $\inf(S) + \sup(S) < s$.

2.1°. Let us first consider the case when $s \leq \inf(S) + \sup(S)$. Since s is in the open interval $(2\inf(S), 2\sup(S))$, we have

$$2\inf(S) < s \leq \inf(S) + \sup(S).$$

In this case, for $s' \stackrel{\text{def}}{=} s - \inf(S)$, we get the inequality $\inf(S) < s' \leq \sup(S)$. By definition of $\inf(S)$, for every $s' > \inf(S)$, there exists a point $s_1 \in S$ for which $s_1 < s'$, i.e., a point s_1 for which $\inf(S) < s_1 < s - \inf(S)$ (the first inequality is strict since $s_1 \in S$ and we consider the case when $\inf(S) \notin S$). From the inequality $s_1 < s - \inf(S)$, we conclude that $\inf(S) < s - s_1$, i.e., that the value $s_2 \stackrel{\text{def}}{=} s - s_1$ is larger than $\inf(S)$.

On the other hand, from the inequalities $s \leq \inf(S) + \sup(S)$ and $\inf(S) < s_1$, we conclude that

$$s_2 = s - s_1 < (\inf(S) + \sup(S)) - \inf(S) = \sup(S).$$

So, $s_2 \in (\inf(S), \sup(S))$. Thus, indeed, $s = s_1 + s_2$, where

$$s_1 \in S \text{ and } s_2 \in (\inf(S), \sup(S)).$$

2.2°. Let us now consider the case when $\inf(S) + \sup(S) < s$, i.e., when

$$\inf(S) + \sup(S) < s < 2\sup(S).$$

From this inequality, it follows that

$$\inf(S) < s - \sup(S) < \sup(S).$$

By definition of $\sup(S)$, for each value smaller than $\sup(S)$, in particular, for the value $s - \sup(S)$, there exists a larger value from the set S. Let us denote this larger value by s_1: $s - \sup(S) < s_1$. Thus, for $s_2 \stackrel{\text{def}}{=} s - s_1$, we get $s_2 < \sup(S)$.

On the other hand, from $\inf(S) + \sup(S) < s$ and $s_1 \le \sup(S)$, it follows that

$$(\inf(S) + \sup(S)) - \sup(S) = \inf(S) < s_2 = s - s_1.$$

So, $s_2 \in (\inf(S), \sup(S))$. Thus, indeed, $s = s_1 + s_2$, where

$$s_1 \in S \text{ and } s_2 \in (\inf(S), \sup(S)).$$

The proposition is proven.

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References

[1] L. Hurwicz, *Optimality Criteria for Decision Making Under Ignorance*, Cowles Commission Discussion Paper, Statistics, No. 370, 1951.

[2] V. Kreinovich, "Decision making under interval uncertainty (and beyond)", In: P. Guo and W. Pedrycz (eds.), *Human-Centric Decision-Making Models for Social Sciences,* Springer Verlag, 2014, pp. 163–193.

[3] R. D. Luce and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.

[4] Tran Anh Tuan, V. Kreinovich, and Thach Ngoc Nguyen, "Decision making under interval uncertainty: beyond hurwicz pessimism-optimism criterion", In: V. Kreinovich, Nguyen Ngoc Thach, Nguyen Duc Trung, and Dang Van Thanh (eds.), *Beyond Traditional Probabilistic Methods in Economics*, Springer, Cham, Switzerland, 2019, pp. 176–184.