2-2019

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Comments:

Technical Report: UTEP-CS-19-11a

To appear in *Proceedings of the World Congress of the International Fuzzy Systems Association and the Annual Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS'2019,*, Lafayette, Louisiana, June 18-22, 2019.

Logarithms Are Not Infinity: A Rational Physics-Related Explanation of the Mysterious Statement by Lev Landau

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Abstract Nobel-prize winning physicist Lev Landau liked to emphasize that logarithms are not infinity – meaning that from the physical viewpoint, logarithms of infinite values are not really infinite. Of course, from a literally mathematical viewpoint, this statement does not make sense: one can easily prove that logarithm of infinity is infinite. However, when a Nobel-prizing physicist makes a statement, you do not want to dismiss it, you want to interpret it. In this paper, we propose a possible physical explanation of this statement. Namely, in physics, nothing is really infinite: according to modern physics, even the Universe is finite in size. From this viewpoint, infinity simply means a very large value. And here lies our explanation: while, e.g., the square of a very large value is still very large, the logarithm of a very large value can be very reasonable – and for very large values from physics, logarithms are indeed very reasonable.

1 Formulation of the Problem

Physicists use intuition. Physicists have been very successful in predicting physical phenomena. Many fundamental physical phenomena can be predicted with very high accuracy. The question is: how do physicists come up with the corresponding models?

In this, physicists often use their intuition. This intuition is, however, difficult to learn, because it is not formulated in precise terms – it is imprecise, it is intuition, after all.

Can we formalize physicists' intuition – at least some of it? It would be great to be able to emulate at least some of this intuition in a computer-based systems, so that the same successful line of reasoning can be used to solve many other problems.

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Computers, however, only understand precise terms. So, to be able to emulate physicists' intuition on a computer, we need describe it – or at least some aspects of it – in precise terms.

An example of physicists' intuition: Landau's statement about logarithms. Nobel-prize physicist Lev Landau often said that "logarithms are not infinity" – meaning that, in some sense, the logarithm of an infinite value is not really infinite; see, e.g., [4], p. 472; [10], p. 84; [12], p. 30.

Of course, this statement cannot be taken literally. From the purely mathematical viewpoint, this statement by Landau makes no sense: of course, the limit of $\ln(x)$ when x tends to infinity is infinite.

It is advisable to take this statement into account. This was a statement actively used by a Nobel-prize winning physicist, so we cannot just ignore it as a mathematically ignorant nonsense.

Formulation of the problem. But how can we make sense of this Landau's statement?

What we do in this paper. In this paper, we show how Landau's statement can be consistently formalized.

2 Why Infinities Are Important in Physics

Why are infinities important in the first place? At first glance, one may wonder why physicists are worried about infinities in the first place. In physics, everything is finite, infinities are mathematical abstractions, what is the big deal?

Alas, everything should be finite in physics, but infinities naturally appear. Yes, in physics, everything should be finite, but unfortunately, infinities creep in. Let us give a simple example of such a situation – the attempts to compute the overall mass m of an electron.

According to special relativity theory (see, e.g., [3, 11]), this mass can be obtained by dividing the total energy E of the electron by the square of the speed of light c: $m = E/c^2$. This energy, in its turn, is equal to the sum of the rest energy $E_0 = m_0 \cdot c^2$ and the overall energy E_{el} of the electron's electric field.

According to the same relativity theory, the speed of all communications is limited by the speed of light. As a result, any elementary particle must be point-wise: otherwise, we would have different parts which – due to speed-of-light bound – would not be perfectly correlated and would, thus, constitute different sub-particles. The electric field $\mathbf{E}$ of a point-wise particle is well-known: it is determined by the usual Coulomb formula

$$\mathbf{E}(x) = c_1 \cdot \frac{q}{r^2},$$

where c_1 is a constant, q is the electron's electric charge, and r is the distance from a given point x to the electron's location.

It is known that the field's energy density $\rho(x)$ is proportional to the square of the field: $\rho(x) = c_2 \cdot (\mathbf{E}(x))^2$, i.e.,

$$\rho(x) = c_3 \cdot \frac{1}{r^4},$$

where $c_3 \stackrel{\text{def}}{=} c_2 \cdot (c_1 \cdot q)^2$. Thus, the overall energy of the electric field can be found if we integrate this density over the whole space:

$$E_{\text{el}} = \int \rho(x)\,dx = c_3 \cdot \int \frac{1}{r^4}\,dx.$$

Since the density function depends only on the distance r – i.e., is spherically symmetric – we can use the usual formulas of integrating spherically symmetric functions. Namely:

- First, for each radius r, we integrate over the sphere of this radius – whose area is $4\pi \cdot r^2$. On this sphere, the function is constant, so we simply multiply the expression by $4\pi \cdot r^2$.
- Then, we integrate the result over all possible values r.

In our case, the result is

$$E_{\text{el}} = c_3 \cdot \int_0^\infty \frac{4\pi \cdot r^2}{r^4}\,dr = c_4 \cdot \int \frac{1}{r^2}\,dr,$$

where $c_4 \stackrel{\text{def}}{=} c_3 \cdot 4\pi$. This integral is well know, so we get

$$E_{\text{el}} = -c_4 \cdot \left.\frac{1}{r}\right|_0^\infty.$$

For $r = \infty$, the expression $1/r$ is 0, but at the limit $r = 0$, we get a physically meaningless infinity!

This infinity problem is ubiquitous. The problem is not just in the specific formulas for the Coulomb law, the problem is much deeper: it can be traced to the fact that electromagnetic interactions – and many other physical interactions, e.g., gravitational ones – are *scale-invariant* in the sense that they have no physically preferable unit of length.

If we change from the original unit of length to a new one which is λ times smaller, then all numerical values of distance r will get multiplied by λ, so that the new values get the form $r' = \lambda \cdot r$. Scale-invariance means that all the physical equations - e.g., the equation that describes how the field energy density ρ depends on the distance r – remain the same after this change – provided, of course, that we appropriately change the unit for measuring energy density, to $\rho \to \rho' = c(\lambda) \cdot \rho$.

So, if in the original units, we have $\rho(r) = f(r)$ for some function f, then in the new units, we will have $\rho'(r') = f(r')$ for the exact same function $f(r)$. Here, $\rho' = c(\lambda) \cdot \rho$ and $r' = \lambda \cdot r$, so we conclude that $c(\lambda) \cdot \rho(r) = f(\lambda \cdot r)$. Since $\rho(r) = f(r)$,

we thus conclude that

$$c(\lambda)\cdot f(r) = f(\lambda\cdot r).$$

It is known (see, e.g., [1]) that every measurable solution of this equation has the form $f(r) = c\cdot r^\alpha$ for some c and α. Thus, $\rho(r) = c\cdot r^\alpha$ and therefore, the overall energy of the corresponding field is equal to

$$\int \rho(x)\,dx = \int c\cdot r^\alpha\,dx = \int_0^\infty c\cdot r^\alpha\cdot 4\pi\cdot r^2\,dr = c'\cdot\int_0^\infty r^{2+\alpha}\,dr,$$

where we denoted $c' \stackrel{\text{def}}{=} 4\pi\cdot c$.

When $\alpha \neq -3$, this integral is proportional to $r^{3+\alpha}|_0^\infty$:

- When $\alpha < -3$, this value is 0 at infinity, but infinite at $r = 0$.
- When $\alpha > -3$, this value is 0 for $r = 0$, but infinite for $r = \infty$.

In both cases, we get infinite energy.

When $\alpha = -3$, the integral is proportional to $\ln(x)|_0^\infty$. Logarithm is infinite both for $r = 0$ (when it is $-\infty$) and for $r = \infty$ (when it is $+\infty$), so the difference is infinite as well.

Comment. The situation is not limited to our 3-dimensional proper space (corresponding to 4-dimensional space-time), it can be observed in space-time of any dimension. Indeed, no matter what dimension d we assume for the proper space, the area of the sphere is proportional to r^{d-1}, thus the overall energy is proportional to the integral of $r^\alpha\cdot r^{d-1} = r^{\alpha+d-1}$. So:

- if $\alpha \neq -d$, this integral is proportional to $r^{\alpha+d}$ and is, thus, infinite either for $r = 0$ (when $\alpha < -d$) or for $r = \infty$ (when $\alpha > -d$);
- if $\alpha = -d$, the the integral is proportional to $\ln(x)|_0^\infty$ and is, thus, infinite as well.

3 Towards Possible Physical Explanation of Landau's Statement

In reality, infinities are an idealization. In the above computations, we assumed that the distance r can take any value from 0 to infinity. In reality, the distance r cannot be too large: according to modern physics, a distance cannot be too large – it cannot exceed the current radius R of the Universe.

Similarly, the distance r cannot be too small: when the distance becomes too small, of order $r_0 \approx 10^{-33}$ cm, quantum effects become so relatively large that the notion of exact distance becomes impossible [3, 11].

In physics, infinite usually means "very large", 0 often means "very small". In reality, when physicists talk about infinite value, what they mean is that in reality, the value is very large – so large that we can safely replace it with infinity. Indeed, the size of an electron is so small in comparison with the size R of the Universe that in most physical problems, we can safely assume that the Universe is infinite – just like when we measure short distances on Earth, we can safely ignore the fact that

we are on a surface of a finite sphere, and use formulas of planar geometry – i.e., in effect, assume that the Earth is an infinite plane.

Similarly, when physicists talk about 0 values, what they mean is that the corresponding values are so small, then we can safely ignore this value. Indeed, in most physical problems, the quantum-effects distance 10^{-33} cm is so much smaller than anything we measure that we can safely take this distance to be 0.

What should we do. The notions "very large" and "very small" are clearly imprecise. So, to properly describe these notions – and to properly describe how physicists use them – it makes sense to use techniques specifically designed for dealing with such notions – namely, the techniques of fuzzy logic; see, e.g., [2, 5, 6, 7, 8, 9, 13].

This is something we will try to do, and this is something that we encourage interested readers to try. While such a formalization is still not done, what can we do?

Since there are no infinities, what is the problem? Why are mathematical infinities – which are not really infinite – still bothering physicists?

For example, if instead of using $r = 0$ as the lower bound on the integral, we use the quantum distance $r_0 = 10^{-33}$ cm, we will get a finite value proportional to $1/r_0$. The problem is that this value, while not infinite, is still too large to be physically meaningful. Indeed, the value r_0 is approximately 10^{-20} of the observed electron radius. Since the overall energy of the electric field is proportional to $1/r_0$, this means that the overall energy of the electron's electric field is 10^{20} times larger than we expected – too large.

Similarly, in all other cases: if we take a very large value, and raise it to a power, we still get a very large value.

But with logarithms it is different: a physical explanation of Landau's statement. Interestingly enough, the situation with logarithms is drastically different. Indeed, if we have a term proportional to $\ln(x)$, then, even if $x \approx 10^{20}$, this term is only proportional to $\ln(10^{20}) = 20 \cdot \ln(10) \approx 46$. If the coefficient of proportionality is 0.01 – as often happens in physics – the resulting term is smaller than 1!

This is probably what Landau had in mind when he made this statement:

- that when you have a power law like $y = r^\alpha$, then mathematical infinity usually means that the value of the quantity y is indeed too large to be meaningful;
- on the other hand, if we have a logarithmic dependence like $y = \ln(r)$, then, while mathematically we still have an infinity, in practice, even if we substitute a very large value r, we still get a very reasonable – and very finite – value of the corresponding quantity y.

Comment. Of course, this is just a qualitative explanation. To get a quantitative explanation, we need – as we have mentioned earlier – to further develop fuzzy (or similar) formalization of this idea.

Acknowledgments

This work was supported in part by the US National Science Foundation via grant HRD-1242122 (Cyber-ShARE Center of Excellence). The authors are thankful to the anonymous referees for valuable suggestions.

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