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Abstract

When we have two estimates of the same quantity, it is desirable to combine them into a single more accurate estimate. In the usual case of continuous quantities, a natural idea is to take the arithmetic average of the two estimates. If we have four estimates, then we can divide them into two pairs, average each pair, and then average the resulting averages. Arithmetic average is *consistent* in the sense that the result does not depend on how we divide the original four estimates into two pairs. For discrete quantities – e.g., quantities described by integers – the arithmetic average of two integers is not always an integer. In this case, we need to select one of the two integers closest to the average. In this paper, we show that no matter how we select – even if we allow probabilistic selection – the resulting averaging cannot be always consistent.

1 Formulation of the Problem

Need for averaging. In many practical situations, we have two (or more) estimates x_1 and x_2 of the same quantity x. In such situations, it is desirable to combine the two estimates and come up with a single – hopefully more accurate – estimate $x_1 * x_2$ of this quantity.

What operation $*$ should be use? In geometric terms, the pair (x_1, x_2) can be naturally represented by a point in a 2-D plane. If the estimates were exact, we would have the exact same number x in both components of this pair, i.e., we would have the pair (x, x). It is therefore reasonable to look for the value x for which the corresponding pair (x, x) is the closest to the pair (x_1, x_2).

The distance between the 2-D points (x, x) and (x_1, x_2) is equal to

$$\sqrt{(x - x_1)^2 + (x - x_2)^2}.$$

Minimizing this distance is equivalent to minimizing its square

$$(x - x_1)^2 + (x - x_2)^2.$$

Differentiating this expression with respect to x and equating the derivative to 0, we conclude that $x = \frac{x_1 + x_2}{2}$. Such averaging is indeed one of the main ways to combine two estimates; see, e.g., [1, 2].

Averaging is consistent. If we have four estimates x_1, x_2, x_3, and x_4, then a natural idea is:

- to divide them into two pairs; for example, we can divide into pairs (x_1, x_2) and (x_3, x_4);
- average values from each pair, coming up with combined estimates $x_1 * x_2$ and $x_3 * x_4$, and
- then average the resulting averages, combing up with the value

$$(x_1 * x_2) * (x_3 * x_4).$$

It is reasonable to require that the averaging operation is *consistent* in the sense that the result of this operation should not change if, on the first stage, we use a different division into two pairs, i.e., if

$$(x_1 * x_2) * (x_2 * x_4) = (x_1 * x_3) * (x_2 * x_4).$$

What is the corresponding quantity is discrete? Some physical quantities – like electric change – are discrete, in the sense that they can take only values $\ldots, -2e, -e, 0, e, 2e, \ldots$ proportional to some fixed value e. To make our discussion simpler, let us select this value e as a measurement unit. In this case, possible values of the quantity x are integers.

If x_1 and x_2 have the same parity, i.e., if they are either both odd or both event, then the arithmetic average $\overline{x} = \frac{x_1 + x_2}{2}$ is also an integer. However, if one of the estimates is even, and another is odd – e.g., if $x_1 = 0$ and $x_1 = 1$ – then the arithmetic average is no longer an integer. In this case, as one can easily see, we have two different integers x for which the square $(x - x_1)^2 + (x - x_2)^2$ of the distance is the smallest: the floor $\lfloor \overline{x} \rfloor$ and the ceiling $\lceil \overline{x} \rceil$ of the corresponding fraction x. For example, for $x_1 = 0$ and $x_2 = 1$, we have $\overline{x} = 0.5$, so $\lfloor \overline{x} \rfloor = 0$ and $\lceil \overline{x} \rceil = 1$.

Formulation of the problem. We would like to select, for very pair of integers (x_1, x_2), one of the two possible averages. A natural question is: can we select it in such a way that the resulting operation is consistent?

What we prove. In this paper, we prove that in the discrete case, averaging cannot be consistent.

2 Definitions and the Main Result

Definition 1. *We say that an operation* $* : Z \times Z \to Z$ *that maps pairs of integers into an integer is a* discrete-case averaging *if for every pair* (x_1, x_2)*, the result* $x = x_1 * x_2$ *minimizes the sum* $(x - x_1)^2 + (x - x_2)^2$*:*

$$(x_1 * x_2 - x_1)^2 + (x_1 * x_2 - x_2)^2 = \min_{x \in Z}((x - x_1)^2 + (x - x_2)^2).$$

Definition 2. *We say that a discrete-case averaging* $*$ *is* consistent *if for every four integers* x_1*,* x_2*,* x_3*, and* x_4*, we have*

$$(x_1 * x_2) * (x_2 * x_4) = (x_1 * x_3) * (x_2 * x_4).$$

Proposition 1. *No discrete-case averaging is consistent.*

Proof. Let us assume that $*$ is a consistent discrete-case averaging, and let us get a contradiction out of this assumption.

1°. By definition of a discrete-case averaging, the value $1 * 2$ should be equal either to 1 or to 2. Let us show that in both cases, consistency is violated for some values x_1, x_2, x_3, and x_4.

2°. Let us first consider the case when $1 * 2 = 1$. Let us prove that in this case, $2 * 3 = 2$.

Indeed, by definition of a discrete-case averaging, we have $2 * 3 = 2$ or $2 * 3 = 3$. However, if $2 * 3 = 3$, then, due to consistency, we have

$$(1 * 2) * (1 * 3) = (1 * 1) * (2 * 3).$$

We consider the case when $1 * 2 = 1$; by definition, $1 * 3 = 2$, thus the left-hand side of the above formula has the form $(1 * 2) * (1 * 3) = 1 * 2$, and we already know that $1 * 2 = 1$.

On the other hand, if $2 * 3 = 3$, then the right-hand side has the form $(1 * 1) * (2 * 3) = 1 * 3 = 2$. So, if $2 * 3 = 3$, then the left-hand side and the right-hand side are different – and hence, the averaging $*$ is not consistent. Since we assumed that $*$ is consistent, this means that $2 * 3$ cannot be equal to 3 – and thus, it must be equal to 2.

Then, due to consistency, we should also have

$$(1 * 2) * (2 * 3) = (1 * 3) * (2 * 2).$$

Here, $1 * 2 = 1$ and $2 * 3 = 2$, so the left-hand side of this equality takes the form $(1 * 2) * (2 * 3) = 1 * 2 = 1$.

On the other hand, here $1 * 3 = 2$ and $2 * 2 = 2$, hence the right-hand side takes the form $(1 * 3) * (2 * 2) = 2 * 2 = 2$. So, the left-hand side and right-hand side are different – and thus, the averaging is not consistent.

3°. To complete the proof, let us consider the remaining case when $1 * 2 = 2$. Let us prove that in this case, $0 * 1 = 1$.

Indeed, by definition of a discrete-case averaging, we have $0 * 1 = 0$ or $0 * 1 = 1$. However, if $0 * 1 = 0$, then, due to consistency, we have

$$(0 * 2) * (1 * 2) = (0 * 1) * (2 * 2).$$

We consider the case when $1 * 2 = 2$; by definition, $0 * 2 = 1$, thus the left-hand side of the above formula has the form $(0 * 2) * (1 * 2) = 1 * 2$, and we already know that $1 * 2 = 2$.

On the other hand, if $0 * 1 = 0$, then the right-hand side has the form $(0 * 1) * (2 * 2) = 0 * 2 = 1$. So, if $0 * 1 = 0$, then the left-hand side and the right-hand side are different – and hence, the averaging $*$ is not consistent. Since we assumed that $*$ is consistent, this means that $0 * 1$ cannot be equal to 0 – and thus, it must be equal to 1.

Then, due to consistency, we should also have

$$(1 * 2) * (0 * 1) = (1 * 1) * (0 * 2).$$

Here, $1 * 2 = 2$ and $0 * 1 = 1$, so the left-hand side of this equality takes the form $(1 * 2) * (0 * 1) = 2 * 1 = 2$.

On the other hand, here $1 * 1 = 1$ and $0 * 2 = 1$, hence the right-hand side takes the form $(1 * 1) * (0 * 2) = 1 * 1 = 1$. So, the left-hand side and right-hand side are different – and thus, the averaging is not consistent.

The proposition is proven.

3 What If We Allow Probabilistic Averaging

Idea. The above result is about a *deterministic* averaging, when to every pair (x_1, x_2), we assign a single value $x_1 * x_2$. For example, for $x_1 = 0$ and $x_2 = 1$, we have two possible values x, for which the sum $(x - x_1)^2 + (x - x_2)^2$ is the smallest – namely, the values 0 and 1, and we pick one of these values.

But if 0 and 1 are equally good, why not select each of them with some probability, e.g., with probability 1/2 each? In this case, we get a *probabilistic* averaging, for which, for each x_1 and x_2, the value $x_1 * x_2$ is a random variable.

Natural question. Will the resulting probabilistic averaging be consistent – in the sense that for every x_1, x_2, x_3, and x_4, the random variables $(x_1 * x_2) * (x_2 * x_4)$ and $(x_1 * x_3) * (x_2 * x_4)$ have the same distribution?

What we prove. We prove that the answer is still "no" – but at least the above two random variables have the same mean.

Definition 3. *By a* probabilistic averaging*, we mean an operation $*$ that assigns, to every pair of integers (x_1, x_2), the following random variable:*

- *when the sum* $x_1 + x_2$ *is even, the random variable* $x_1 * x_2$ *is equal to* $\overline{x} \stackrel{\text{def}}{=} \dfrac{x_1 + x_2}{2}$ *with probability 1;*

- *when the sum* $x_1 + x_2$ *is odd, the random variable is equal either to* $\lfloor \overline{x} \rfloor$ *or to* $\lceil \overline{x} \rceil$*, with some probability.*

Definition 4. *We say that a probabilistic averaging is* consistent *if for for every* x_1*,* x_2*,* x_3*, and* x_4*, the random variables*

$$(x_1 * x_2) * (x_3 * x_4) \text{ and } (x_1 * x_3) * (x_2 * x_4)$$

have the same distribution, where different $*$ *operations are assumed to be independent.*

Definition 5. *We say that a probabilistic averaging is* weakly consistent *if for for every* x_1*,* x_2*,* x_3*, and* x_4*, the random variables* $(x_1 * x_2) * (x_3 * x_4)$ *and* $(x_1 * x_3) * (x_2 * x_4)$ *have the same mean.*

Proposition 2. *No probabilistic averaging is consistent.*

Proposition 3. *There exists a probabilistic averaging which is weakly consistent.*

Proof of Proposition 2. Let us assume that $*$ is a consistent probabilistic averaging, and let us get a contradiction out of this assumption.

1°. Let us first prove that for all pairs $(n, n+1)$, the probability p of selecting n as $n * (n+1)$ is equal to either 0, or 0.5, or 1.

Indeed, by definition of consistency, we should have

$$(n * n) * ((n+1) * (n+1)) = (n * (n+1)) * (n * (n+1)).$$

The left-hand side is equal to $n*(n+1)$ and is, thus, equal to n with probability p.

In the right-hand side, each of the two terms $n * (n+1)$ is equal to n with probability p and to $n+1$ with the remaining probability $1-p$. Since different $*$-operations are assumed independent, we therefore have four possible cases:

- the first case is when both terms $n * (n+1)$ are equal to n; the probability of this case is $p \cdot p = p^2$;

- the second case is when the first term is equal to n and the second term is equal to $n+1$; the probability of this case is equal to $p \cdot (1-p)$;

- the third case is when the first term is equal to $n+1$ and the second term is equal to n; the probability of this case is equal to $(1-p) \cdot p$;

- finally, the fourth case is when both terms $n * (n+1)$ are equal to $n+1$; the probability of this case is $(1-p) \cdot (1-p) = (1-p)^2$.

In the first case, the value $(n*(n+1))*(n*(n+1))$ is always equal to n, and, as we recall, this case occurs with probability p^2. In the second and third cases, the value n appears with probability p; thus, the overall probability of getting n in these cases is $2p\cdot(1-p)\cdot p = 2p^2\cdot(1-p)$. In the fourth case, we always get $n+1$. So, the overall probability of getting n is

$$p^2 + 2p^2\cdot(1-p) = p^2 + 2p^2 - 2p^3 = 3p^2 - 2p^3.$$

Since the operation $*$ is consistent, the probability of getting n on both sides should be equal, so we must get $p = 3p^2 - 2p^3$. The first possibility to get this equality is to have $p = 0$. If $p \neq 0$, then we can divide both sides by p and get $1 = 3p - 2\cdot 2p^2$, i.e., a quadratic equation $2p^2 - 3p + 1 = 0$, whose solutions are $p = 0.5$ and $p = 1$.

2°. From Part 1 of this proof, it follows that the probability p_{12} of getting 1 as a result of $1*2$ is either 0, or 0.5, or 1. Let us first consider the case when $p_{12} > 0$.

In this case, let us consider another consistency requirement:

$$(1*2)*(1*3) = (1*1)*(2*3).$$

In the left-hand side, $1*2$ is equal to 1 with probability $p_{12} > 0$, and to 2 with the remaining probability $1-p_{12}$. Here, $1*3 = 2$, so $(1*2)*(1*3)$ is equal to $1*2$ with probability p_{12} and to $2*2$ with probability $1-p_{12}$. In the first case, we get 1 in p_{12} of the cases, so the overall probability that the left-hand side is 1 is equal to p_{12}^2.

In the right-hand side, $1*1 = 1$, and $2*3$ is equal to 2 with some probability p_{23} and to 3 with the remaining probability $1-p_{23}$. Thus, the right-hand side is equal to $1*2$ with probability p_{23} and to $1*3 = 2$ with probability $1-p_{23}$. In the first case, we get 1 in p_{12} of the cases, so the overall probability that the right-hand side is 1 is equal to $p_{12}\cdot p_{23}$.

Due to consistency, the probability that the left-hand side is 1 and that the right-hand side is 1 should be the same, so we get $p_{12}^2 = p_{12}\cdot p_{23}$. Since $p_{12} > 0$, we can conclude that $p_{12} = p_{23}$.

Now, let us consider yet another particular case of consistency:

$$(1*2)*(2*3) = (1*3)*(2*2).$$

The right-hand side is always equal to $2*2 = 2$, while in the left-hand side, we have $1*2 = 1$ with probability p_{12}, $2*3 = 2$ with probability $p_{23} = p_{12}$ and thus, $(1*2)*(2*3) = 1*2$ with probability p_{12}^2. Out of these cases, we get $(1*2)*(2*3) = 1$ with probability $p_{12}\cdot p_{12}^2 > 0$.

So, in the right-hand side, we never get 1, but in the left-hand side, we get 1 with positive probability – which contradicts to the consistency assumption.

3°. Thus, the case $p_{12} > 0$ is impossible, and so, we always have $1*2 = 2$.

In this case, consistency implies that $(1*2)*(0*2) = (0*1)*(2*2)$. Here, $1*2 = 2$ and $0*2 = 1$, and thus, the left-hand side is equal to $2*1 = 2$.

The value $0 * 1$ is equal to 0 with some probability p_{01} and to 1 with the remaining probability $1 - p_{01}$. Since $2 * 2 = 2$, the right-hand side is equal to $0 * 2 = 1$ with probability p_{01} and to $1 * 2 = 2$ with probability $1 - p_{01}$. The left-hand side is always equal to 2, hence the right-hand side cannot be equal to 1, and so $p_{01} = 0$.

Thus, we always have $0 * 1 = 1$. In this case, we can use another particular case of consistency: $(1 * 2) * (0 * 1) = (1 * 1) * (0 * 2)$. Here, since $1 * 2 = 2$ and $0 * 1 = 1$, the left-hand side is equal to $2 * 1 = 2$, while the right-hand side is equal to $1 * 1 = 1$ – a contradiction.

Thus, the proposition is proven.

Proof of Proposition 3. One can easily check that, as the desired probabilistic averaging, we can take the averaging in which, for each pair with non-integer $\overline{x}$, we return both the floor and the ceiling of $\overline{x}$ with equal probability $1/2$. In this case, the mean is simply the usual arithmetic average, and we know that the arithmetic average is consistent.

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