9-2018

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Comments:

Technical Report: UTEP-CS-18-66

Towards Parallel Quantum Computing: Standard Quantum Teleportation Algorithm Is, in Some Reasonable Sense, Unique*

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Abstract. In many practical problems, the computation speed of modern computers is not sufficient. Due to the fact that all speeds are bounded by the speed of light, the only way to speed up computations is to further decrease the size of the memory and processing cells that form a computational device. At the resulting size level, each cell will consist of a few atoms – thus, we need to take quantum effects into account. For traditional computational devices, quantum effects are largely a distracting noise, but new quantum computing algorithms have been developed that use quantum effects to speed up computations. In some problems, however, this expected speed-up may not be sufficient. To achieve further speed-up, we need to parallelize quantum computing. For this, we need to be able to transmit a quantum state from the location of one processor to the location of another one; in quantum computing, this process is known as teleportation. A teleportation algorithm is known, but it is not clear how efficient it is: maybe there are other more efficient algorithms for teleportation? In this paper, we show that the existing teleportation algorithm is, in some reasonable sense, unique – and thus, optimal.

Keywords: Data processing under uncertainty · Quantum computing · Parallel quantum computing · Teleportation · Optimization

1 Need for Parallel Quantum Computing

Need for fast data processing. In many practical problems, there is a need for fast data processing – way beyond the current data processing speeds. For example, it is known that, in principle, computational models can predict, with high probability, where a tornado will turn in the next hour. However, at present, even on modern high performance computers, the corresponding computations take significantly longer than an hour, which defeats the whole purpose of prediction.

* This work was partially supported by the US National Science Foundation via grant HRD-1242122 (Cyber-ShARE Center of Excellence).

In many such practical problems, the need for computation speed is enhanced by the need to take *uncertainty* into account. From this viewpoint, tornado prediction is a good example: if we had full information about the state of the atmosphere, we could simply solve the corresponding system of partial differential equations. However, in practice, we have only partial information, we have uncertainty – and thus, to make reasonable predictions, we need to generate many different solutions, corresponding to several different situations – and make predictions based on the frequency of solutions corresponding to different directions.

This is a general phenomenon: taking uncertainty into account drastically increases the computation time, because, crudely speaking, under uncertainty, we need to process several alternative scenarios instead of a single one.

Faster processing means smaller memory and computation cells. One of the main limits on the computation speed comes from the fact that, according to modern physics, all velocities are limited by the speed of light. The light is fast – it travels at 300 000 km/sec, but for a current computer of size 30 cm, this means that the fastest we can move information from one side of the computer to another is 1 nanosecond – and even the simplest current computers have processing speed of several Gigahertz, which means that several computation cycles take place while the information is transmitted.

To speed up computations, we therefore need to make computers much smaller – and this means that we must have much smaller memory cells and computation cells.

Need to take quantum effects into account. Already in the existing computers, a memory cell sometimes consists of several dozen atoms. For such small objects, we need to take into account the laws of quantum physics; see, e.g., [1, 5].

The main difference between traditional physics – that describes macro-size objects consisting of many molecules and atoms – and quantum physics (that describes few-atom objects) comes from the difference in our ability to measure things. Indeed, the only way to measure a physical quantity is to interact with the corresponding object. For example, to measure a distance to a faraway object, we can send a laser beam to this object and measure the time that it takes for this beam to come back – this is how we measure, e.g., the distance to the Moon.

For macro-size objects, the corresponding probe can be very small, much smaller than the object itself. Thus, we can safely ignore the effect of this probe on our object and conclude that after the measurement, the object remains the same. For example, we do not expect the distance to the Moon to change just because we hit the Moon with a laser beam. We can thus measure the Moon's location, velocity, and other characteristics with very high accuracy.

On the other hand, when we try to similarly measure the location of a micro-particle – e.g., of a proton – we can still send a photon and measure its bouncing back, but this photon is already of approximately the same size as the particle whose location we measure. As a result, every measurement changes the state of the particle – so, even if we get the particle's location, its speed changes, and

if we afterwards measure its speed, it will be different from the speed of the original particle.

Because of this, for a micro-object, we cannot uniquely determine its state – we can only describe the probability of different measurement results.

Computer scientists managed to transform this lemon into lemonade. At first glance, this makes computations more complicated: as we decrease the size of the cells, because of the quantum effects, the resulting states become only probabilistically predictable – in other words, we have a lot of noise added to our computations, noise that makes computations difficult.

However, researchers managed to use quantum effects to speed up computations – namely, they showed that by re-arranging the corresponding computation schemes, we can reach even faster computations; see, e.g., [2, 6]. For example, while in non-quantum computing, finding an element in an unsorted database with n entries may require time n – since we may need to look at each record – in quantum computing, it is possible to find this element in much smaller time $\sqrt{n}$.

An even larger speed-up is achieved in the problem of factorizing large integers – traditional algorithms require time which is exponential in terms of the number's length (and thus, not feasible for large lengths), while quantum computing can do it in polynomial time; see, e.g., [2–4]. This application is important, since most current online encryption algorithms are based on the difficulty of factoring large integers – so once quantum computers become a reality, we will be able to read all the encrypted messages that have been sent so far.

Need for parallel quantum computing. While quantum computing is fast, its speeds are also limited. To further speed up computations, a natural idea is to have several quantum computers working in parallel, so that each of them solves a part of the problem.

This idea is similar to how we humans solve complex problems: if a task is too difficult for one person to solve – be it building a big house or proving a complex theorem – several people team up and together solve the task.

Need for teleportation. To successfully collaborate, quantum computers need to exchange intermediate states of their computations. Here lies a problem: for complex problems, we would like to use computers located in different geographic areas, but a quantum state gets changed when it is sent far away.

Researchers have come up with a way to avoid this sending, called *teleportation*. There exists a scheme for teleportation.

Problem. It is not clear how good is the current teleportation scheme: maybe there are other schemes which are faster (or better in some other sense)?

What we do in this paper. In this paper, we show that the existing teleportation scheme is, in some reasonable sense, unique – and, in this sense, is the best.

The structure of the paper. We start by a brief reminder of the basics of quantum physics – specifically, the basics that are needed for the describing the quantum teleportation algorithm. After that, we describe, in detail, the existing

teleportation scheme. Finally, we describe our main result – namely, we show that this scheme is, in some reasonable sense, unique.

2 Quantum Physics and Quantum Computing: A Brief Reminder

Basic states in quantum physics. In quantum physics, in addition to the usual (non-quantum) states s_1, s_2, ..., we also have *superpositions* of these states, i.e., states of the type

$$\alpha_1 \cdot s_1 + \alpha_2 \cdot s_2 + \ldots,$$

where α_1, α_2, ... are complex numbers for which

$$|\alpha_1|^2 + |\alpha_2|^2 + \ldots = 1.$$

The complex numbers α_i are known as *amplitudes.*

For example, a computer is formed from devices representing *binary digits* (*bits*, for short), i.e., devices that can be in two possible states: 0 and 1. In quantum physics, in addition to these two states – which is quantum physics, are denoted by $|0\rangle$ and $|1\rangle$, we also have superpositions of these states, i.e., states of the type

$$\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle,$$

where α_0 and α_1 are complex numbers for which $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The corresponding quantum system is known as a *quantum bit*, or *qubit*, for short.

Composite states in quantum physics. In classical (pre-quantum) physics, there is a straightforward way to describe a composite system consisting of two independent subsystems. Due to independence, to describe the set of the system as a whole, it is sufficient to describe the state s of the first subsystem and the state s' of the second subsystem. Thus, a state of the system as a whole is an ordered pair $\langle s, s'\rangle$ of the two states.

Let us denote possible states of the first subsystem by s_1, s_2, ..., and possible states of the second subsystem by s'_1, s'_2, ... Since the subsystems are independent, the possible states of the first subsystem do not depend on the state of the second subsystem. Thus, the set of all states of the system as a whole is the set of all possible pairs $\langle s_i, s'_j\rangle$. The set of all such pairs is known as the *Cartesian product*; it is denoted by $\{s_1, s_2, \ldots\} \times \{s'_1, s'_2, \ldots\}$.

When the subsystems are binary, the corresponding notations are usually simplified, so that e.g., the pair $\langle 0, 1\rangle$ is denoted simply as 01.

In quantum physics, we can also have superpositions of such states, i.e., the states of the type

$$\alpha_{11} \cdot \langle s_1, s'_1\rangle + \alpha_{12} \cdot \langle s_1, s'_2\rangle + \ldots + \alpha_{21} \cdot \langle s_2, s'_1\rangle + \alpha_{22} \cdot \langle s_2, s'_2\rangle + \ldots,$$

where

$$|\alpha_{11}|^2 + |\alpha_{12}|^2 + \ldots + |\alpha_{21}|^2 + |\alpha_{22}|^2 + \ldots = 1.$$

To describe such a state, we need to known all the values α_{ij}. These values form a matrix – i.e., in mathematical terms, a *tensor*. Because of this fact, the set of all such states is known as the *tensor product* $S \otimes S'$, where S is the set of all possible quantum states of the first subsystem and S' is the set of all possible quantum states of the second subsystem. Accordingly, in quantum physics the pair $\langle s, s'\rangle$ is denoted as $s \otimes s'$ and called a *tensor product* of the states s and s'. So, if the first subsystem is in the state s_i and the second subsystem is in the state s'_j, then the state of the system as a whole is $\langle s_i, s'_j\rangle = s_i \otimes s'_j$.

Just like in the non-quantum case, for binary states, we can use a simplified notation: e.g., instead of $|0\rangle \otimes |1\rangle$, we can simply write $|01\rangle$.

If the state of the first subsystem is a superposition

$$s = \alpha_1 \cdot s_1 + \alpha_2 \cdot s_2 + \ldots,$$

and the state of the second subsystem is $s' = s_j$, then the state of the system as a whole can also be described as the superposition of the corresponding pairs, i.e., as

$$s \otimes s'_j = \alpha_1 \cdot (s_1 \otimes s'_j) + \alpha_2 \cdot (s_2 \otimes s'_j) + \ldots$$

If the state s' of the second subsystem is also a superposition, i.e., has the form

$$s' = \alpha'_1 \cdot s'_1 + \alpha'_2 \cdot s'_2 + \ldots,$$

then the joint state $s \otimes s'$ can be described as a superposition of the states $s \otimes s'_j$, i.e., as

$$\alpha'_1 \cdot (s \otimes s'_1) + \alpha'_2 \cdot (s \otimes s'_2) + \ldots$$

Substituting the known expressions for $s \otimes s'_j$ into this formula, we conclude that

$$(\alpha_1 \cdot s_1 + \alpha_2 \cdot s_2 + \ldots) \otimes (\alpha'_1 \cdot s'_1 + \alpha'_2 \cdot s'_2 + \ldots) =$$

$$\alpha'_1 \cdot (\alpha_1 \cdot (s_1 \otimes s'_1) + \alpha_2 \cdot (s_2 \otimes s'_1) + \ldots) + \alpha'_2 \cdot (\alpha_1 \cdot (s_1 \otimes s'_2) + \alpha_2 \cdot (s_2 \otimes s'_2) + \ldots) + \ldots =$$

$$\alpha_1 \cdot \alpha'_1 \cdot (s_1 \otimes s'_1) + \alpha_1 \cdot \alpha'_2 \cdot (s_1 \otimes s'_2) + \ldots + \alpha_2 \cdot \alpha'_1 \cdot (s_2 \otimes s'_1) + \alpha_2 \cdot \alpha'_2 \cdot (s_2 \otimes s'_2) + \ldots$$

One can see that this formula is similar to the formula for the product of two linear combinations, with the tensor product playing the role of the product.

Transformations. In quantum physics, physically possible transformation are the mappings from state to state that satisfy the following two properties:

- superpositions get transformed into similar superpositions:

  $$T(\alpha_1 \cdot s_1 + \alpha_2 \cdots s_2 + \ldots) = \alpha_1 \cdot T(s_1) + \alpha_2 \cdot T(s_1) + \ldots,$$

  and
- the property $\sum |\alpha_i|^2 = 1$ is preserved, i.e., if $\sum |\alpha_i|^2 = 1$, then, for $T\left(\sum \alpha_i \cdot s_i\right) = \sum \beta_i \cdot s_i$, we have $\sum |\beta_i|^2 = 1$.

Because of the first property, transformations are linear: $\sum \alpha_i \cdot s_i \to \sum \beta_i \cdot s_i$, with $\beta_i = \sum\limits_j t_{ij} \cdot \alpha_j$. Because of the second property, the matrix $T = (t_{ij})$ is *unitary*, i.e., $TT^\dagger = \mathbf{1}$, where $\mathbf{1}$ is a unit matrix and $T^\dagger \stackrel{\text{def}}{=} (t^*_{ji})$, with z^* denoting the complex conjugate number $(a + b \cdot \text{i})^* \stackrel{\text{def}}{=} a - b \cdot \text{i}$.

Measurement process in quantum physics. For binary states $\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle$, if we want to measure whether the state is 0 or 1, then:

- with probability $|\alpha_0|^2$, we get the result 0 – and the state turns into $|0\rangle$; and
- with probability $|\alpha_1|^2$, we get the result 1 – and the state turns into $|1\rangle$.

Since the result is either 0 or 1, the probabilities should add up to 1; this explains why physically possible states should satisfy the condition $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

In general, if we have n classical states $s_1, \ldots, s_n$, and we want to detect, in a quantum state $\sum \alpha_i \cdot s_i$, which of these states we are in, we get each s_i with probability $|\alpha_i|^2$ – and once the measurement process detects the state s_i, the actual state turns into s_i.

Instead of the classical states, we can use any other sequence of states $s'_i = \sum\limits_j t_{ij} \cdot s_j$, as long as they are *orthonormal* (= orthogonal and normal) in the sense that:

- for each i, we have $\|s'_i\|^2 = 1$, where $\|s'_i\|^2 \stackrel{\text{def}}{=} \sum\limits_j |t_{ij}|^2$ (*normal*), and
- for each i and i', we have $s'_i \perp s'_{i'}$, i.e., $\langle s'_i | s'_{i'} \rangle = 0$, where $\langle s'_i | s'_{i'} \rangle \stackrel{\text{def}}{=} \sum\limits_j t_{ij} \cdot t^*_{i'j}$ (*orthogonal*).

In this case, if we have a state $\sum \alpha'_i \cdot s'_i$, then with probability $|\alpha'_i|^2$, the measurement result is s'_i and the state turns into s'_i.

In general, instead of a sequence of orthogonal vectors, we can have a sequence of orthogonal linear spaces $L_1, L_2, \ldots$ – where $L_i \perp L_j$ means that $s_i \in L_i$ and $s_j \in L_j$ implies $s_i \perp s_j$. In this case, every state s can be represented as a sum $s = \sum s_i$ of the vectors $s_i \in L_i$. As a result of the measurement, with probability $\|s_i\|^2$, we conclude that the state is in the space L_i, and the original state turns into a new state $s_i / \|s_i\|$.

3 Standard Quantum Teleportation Algorithm: Reminder

Need for communication. At one location, we have a particle in a certain state; we want to send this state to some other location.

Usually, the sender is denoted by A and the receiver by B. In communications, it is common to call the sender Alice, and to call the receiver Bob. States corresponding to Alice are usually described by using a subscript A, and states corresponding to Bob are usually described by using a subscript B.

Communication is straightforward is classical physics but a challenge in quantum physics. In classical (pre-quantum) physics, the communication

problem has a straightforward solution: if we want to communicate a state, we measure all possible characteristics of this state, send these values to Bob, and let Bob reproduce the object with these characteristics. This is how, e.g., 3D printing works. This solution is based on the fact that in classical (non-quantum) physics we can, in principle, measure all characteristic of a system without changing it.

The problem is that in quantum physics, such a straightforward approach is not possible: as we have mentioned, in quantum physics, every measurement changes the state – and moreover, irreversibly deletes some information about the state. For example, if we start with a state $\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle$, all we get after the measurement is either 0 or 1, with no way to reconstruct the values α_0 and α_1 that characterize the original state. Since we cannot use the usual straightforward approach for communicating a state, we need to use an indirect approach. This approach is known as *teleportation*.

What we consider in this section. In this section, we consider the simplest possible quantum state – namely, the quantum analogue of the simplest possible non-quantum state. In the non-quantum case, a system can be in several different states. The state passing problem makes sense only when the system can be in at least two different states – otherwise, if we know beforehand what state we want to send, there is no need to send any information, Bob can simply reproduce the known state. The simplest case when communication is needed is when the number of possible states is as small as possible but still larger than 1 – i.e., the case when the system can be in two different states. In the computer, such situation can be naturally described if we associate these two possible states with 0 and 1.

In these terms, the problem is as follows:

- Alice has a state
$$\alpha_0 \cdot |0\rangle + \alpha_1 \cdot |1\rangle \tag{1}$$
that she wants to communicate to Bob – a person at a different location.
- As a result of this process, Bob should have the same state.

Notations. Let us indicate states corresponding to Alice with a subscript A, and states corresponding to Bob with a subscript B. The state (1) is not exclusively Alice's and it is not exclusively Bob's, so to describe this state, we will use the next letter – letter C. In these terms, Alice has a state

$$\alpha_0 \cdot |0\rangle_C + \alpha_1 \cdot |1\rangle_C \tag{2}$$

that she wants to communicate to Bob.

Preparing for teleportation: an entangled state. To make teleportation possible, Alice and Bob prepare a special *entangled* state:

$$\frac{1}{\sqrt{2}} \cdot |0_A 1_B\rangle + \frac{1}{\sqrt{2}} \cdot |1_A 0_B\rangle. \tag{3}$$

This state is a superposition of two classical states:

– the state $0_A 1_B$ in which A is in state 0 and B is in state 1, and
– the state $1_A 0_B$ in which A is in state 1 and B is in state 0.

What is the joint state of A, B, and C at the beginning of the procedure. In the beginning, the state C is independent of A and B. So, the joint state is a tensor product of the AB-state (3) and the C-state (2):

$$\frac{\alpha_0}{\sqrt{2}} \cdot |0_A 1_B 0_C\rangle + \frac{\alpha_1}{\sqrt{2}} \cdot |0_A 1_B 1_C\rangle + \frac{\alpha_0}{\sqrt{2}} \cdot |1_A 0_B 0_C\rangle + \frac{\alpha_1}{\sqrt{2}} \cdot |1_A 0_B 1_C\rangle. \tag{4}$$

First stage: measurement. In the first stage of the standard teleportation algorithm, Alice performs a measurement procedure on the parts A and C which are available to her. In general, to describe the possible results of measuring a state s with respect to linear spaces L_i, we need to represent s as the sum

$$s = \sum s_i, \tag{5}$$

with $s_i \in L_i$.

In the standard teleportation algorithm, we perform the measurement with respect to the following four linear spaces $L_i = L_B \otimes t_i$, where L_B is the set of all possible linear combinations of $|0\rangle_B$ and $|1\rangle_B$, and the states t_i have the following form:

$$\begin{aligned} t_1 &= \frac{1}{\sqrt{2}} \cdot |0_A 0_C\rangle + \frac{1}{\sqrt{2}} \cdot |1_A 1_C\rangle; \\ t_2 &= \frac{1}{\sqrt{2}} \cdot |0_A 0_C\rangle - \frac{1}{\sqrt{2}} \cdot |1_A 1_C\rangle; \\ t_3 &= \frac{1}{\sqrt{2}} \cdot |0_A 1_C\rangle + \frac{1}{\sqrt{2}} \cdot |1_A 0_C\rangle; \\ t_4 &= \frac{1}{\sqrt{2}} \cdot |0_A 1_C\rangle - \frac{1}{\sqrt{2}} \cdot |1_A 0_C\rangle. \end{aligned} \tag{6}$$

One can easily check that the states t_i are orthonormal, hence the spaces L_i are orthogonal.

To describe the result of measuring the state (4) with respect to these linear spaces, we must first represent the state (4) in the form $s = \sum s_i$, with $s_i \in L_i$. For this purpose, we can use the fact that, due to the formulas (6), we have

$$\begin{aligned} |0_A 0_C\rangle &= \frac{1}{\sqrt{2}} \cdot t_1 + \frac{1}{\sqrt{2}} \cdot t_2; \\ |1_A 1_C\rangle &= \frac{1}{\sqrt{2}} \cdot t_1 - \frac{1}{\sqrt{2}} \cdot t_2; \\ |0_A 1_C\rangle &= \frac{1}{\sqrt{2}} \cdot t_3 + \frac{1}{\sqrt{2}} \cdot t_4; \end{aligned} \tag{7}$$

$$|1_A 0_C\rangle = \frac{1}{\sqrt{2}} \cdot t_3 - \frac{1}{\sqrt{2}} \cdot t_4.$$

Substituting the expressions (7) into the formula (4), we get

$$\frac{\alpha_0}{\sqrt{2}} \cdot |1\rangle_B \otimes \left(\frac{1}{\sqrt{2}} \cdot t_1 + \frac{1}{\sqrt{2}} \cdot t_2\right) + \frac{\alpha_1}{\sqrt{2}} \cdot |1\rangle_B \otimes \left(\frac{1}{\sqrt{2}} \cdot t_3 + \frac{1}{\sqrt{2}} \cdot t_4\right) +$$

$$\frac{\alpha_0}{\sqrt{2}} \cdot |0\rangle_B \otimes \left(\frac{1}{\sqrt{2}} \cdot t_3 - \frac{1}{\sqrt{2}} \cdot t_4\right) + \frac{\alpha_1}{\sqrt{2}} \cdot |0\rangle_B \otimes \left(\frac{1}{\sqrt{2}} \cdot t_1 - \frac{1}{\sqrt{2}} \cdot t_2\right),$$

thus

$$\left(\frac{\alpha_0}{2}|1_B\rangle + \frac{\alpha_1}{2}|0_B\rangle\right) \otimes t_1 + \left(\frac{\alpha_0}{2}|1_B\rangle - \frac{\alpha_1}{2}|0_B\rangle\right) \otimes t_2 +$$

$$\left(\frac{\alpha_1}{2}|1_B\rangle + \frac{\alpha_0}{2}|0_B\rangle\right) \otimes t_3 + \left(\frac{\alpha_1}{2}|1_B\rangle - \frac{\alpha_0}{2}|0_B\rangle\right) \otimes t_4.$$

So, we get a representation of the type (5), with

$$s_1 = \left(\frac{\alpha_0}{2} \cdot |1_B\rangle + \frac{\alpha_1}{2}|0_B\rangle\right) \otimes t_1, \quad s_2 = \left(\frac{\alpha_0}{2} \cdot |1_B\rangle - \frac{\alpha_1}{2} \cdot |0_B\rangle\right) \otimes t_2,$$

$$s_3 = \left(\frac{\alpha_1}{2} \cdot |1_B\rangle + \frac{\alpha_0}{2} \cdot |0_B\rangle\right) \otimes t_3, \quad s_4 = \left(\frac{\alpha_1}{2} \cdot |1_B\rangle - \frac{\alpha_0}{2} \cdot |0_B\rangle\right) \otimes t_4.$$

Here, for each i, we have

$$\|s_i\|^2 = \left|\frac{\alpha_0}{2}\right|^2 + \left|\frac{\alpha_1}{2}\right|^2 = \frac{1}{4} \cdot (|\alpha_0|^2 + |\alpha_1|^2) = \frac{1}{4},$$

thus $\|s_i\| = \dfrac{1}{2}$.

So, with equal probability of $\dfrac{1}{4}$, we get one of the following four states – and Alice knows which one it is:

$$\begin{aligned}
&(\alpha_0 \cdot |1_B\rangle + \alpha_1 \cdot |0_B\rangle) \otimes t_1;\\
&(\alpha_0 \cdot |1_B\rangle - \alpha_1 \cdot |0_B\rangle) \otimes t_2;\\
&(\alpha_1 \cdot |1_B\rangle + \alpha_0 \cdot |0_B\rangle) \otimes t_3;\\
&(\alpha_1 \cdot |1_B\rangle - \alpha_0 \cdot |0_B\rangle) \otimes t_4.
\end{aligned} \tag{8}$$

Second stage: communication. On the second stage, Alice sends to Bob the measurement result. As a result, Bob knows in which the four states (8) the system is.

Final stage: Bob "rotates" his state and thus, get the original state teleported to him. On the final stage, Bob performs an appropriate transformation of his state B.

– In the first case, he uses a unitary transformation that swaps $|0\rangle_B$ and $|1\rangle_B$, for which $t_{01} = t_{10} = 1$ and $t_{00} = t_{11} = 0$.

– In the second case, he uses a unitary transformation for which $t_{01} = 1$, $t_{10} = -1$ and $t_{00} = t_{11} = 0$.
– In the third case, he already has the desired state.
– In the fourth case, he uses a unitary transformation for which $t_{00} = -1$, $t_{11} = 1$, and $t_{01} = t_{10} = 0$.

As a result, in all fours cases, he gets the original state $\alpha_0 \cdot |0\rangle_B + \alpha_1 \cdot |1\rangle_B$.

4 Our Main Result: Standard Quantum Teleportation Algorithm Is, in Some Reasonable Sense, Unique

Formulation of the problem. Teleportation is possible because we have prepared an *entangled* state (3), i.e., a state s_{AB} in which the states of Alice and Bob are not independent, i.e., a state that does not have a form $s_A \otimes s_B$. However, (3) is not the only possible entangled state. Let us consider, instead, a general joint state of two qubits:

$$a_{00} \cdot |0_A 0_B\rangle + a_{01} \cdot |0_A 1_B\rangle + a_{10} \cdot |1_A 0_B\rangle + a_{11} \cdot |1_A 1_B\rangle. \tag{3a}$$

What will happen if we use this more general entangled state instead of the one that is used in the known teleportation algorithm?

Analysis of the problem. For the state (3a), the joint state of all three subsystems has the form

$$\begin{gathered}\alpha_0 \cdot a_{00} \cdot |0_A 0_B 0_C\rangle + \alpha_1 \cdot a_{00} \cdot |0_A 0_B 1_C\rangle + \\ \alpha_0 \cdot a_{01} \cdot |0_A 1_B 0_C\rangle + \alpha_1 \cdot a_{01} \cdot |0_A 1_B 1_C\rangle + \\ \alpha_0 \cdot a_{10} \cdot |1_A 0_B 0_C\rangle + \alpha_1 \cdot a_{10} \cdot |1_A 0_B 1_C\rangle + \\ \alpha_0 \cdot a_{11} \cdot |1_A 1_B 0_C\rangle + \alpha_1 \cdot a_{11} \cdot |1_A 1_B 1_C\rangle.\end{gathered} \tag{4a}$$

Substituting expressions (7) into this formula, we get

$$\begin{gathered}\frac{\alpha_0}{\sqrt{2}} \cdot a_{00} \cdot |0\rangle_B \otimes (t_1 + t_2) + \frac{\alpha_1}{\sqrt{2}} \cdot a_{00} \cdot |0\rangle_B \otimes (t_3 + t_4) + \\ \frac{\alpha_0}{\sqrt{2}} \cdot a_{01} \cdot |1\rangle_B \otimes (t_1 + t_2) + \frac{\alpha_1}{\sqrt{2}} \cdot a_{01} \cdot |1\rangle_B \otimes (t_3 + t_4) + \\ \frac{\alpha_0}{\sqrt{2}} \cdot a_{10} \cdot |0\rangle_B \otimes (t_3 - t_4) + \frac{\alpha_1}{\sqrt{2}} \cdot a_{10} \cdot |0\rangle_B \otimes (t_1 - t_2) + \\ \frac{\alpha_0}{\sqrt{2}} \cdot a_{11} \cdot |1\rangle_B \otimes (t_3 - t_4) + \frac{\alpha_1}{\sqrt{2}} \cdot a_{11} \cdot |1\rangle_B \otimes (t_1 - t_2),\end{gathered}$$

thus $s = S_1 \otimes t_1 + S_2 \otimes t_2 + \ldots$, where

$$S_1 = \left(\frac{\alpha_0 \cdot a_{00}}{\sqrt{2}} + \frac{\alpha_1 \cdot a_{10}}{\sqrt{2}}\right) \cdot |0\rangle_B + \left(\frac{\alpha_0 \cdot a_{01}}{\sqrt{2}} + \frac{\alpha_1 \cdot a_{11}}{\sqrt{2}}\right) \cdot |1\rangle_B,$$

and $S_2, \ldots$ are described by similar expressions.

This means that after the measurement, Bob will have the normalized state $S_1/\|S_1\|$. To perform teleportation, we need to transform this state into the original state $\alpha_0 \cdot |0\rangle_B + \alpha_1 \cdot |1\rangle_B$. Thus, the transformation from the resulting state $S_1/\|S_1\|$ to the original state must be unitary. It is known that the inverse transformation to a unitary one is also unitary. In general, a unitary transformation transforms orthonormal states into orthonormal ones.

So, the inverse transformation that:

- maps the state $|0\rangle_B$ (corresponding to $\alpha_0 = 1$ and $\alpha_1 = 0$) into a new state $|1'\rangle_B \stackrel{\rm def}{=} \text{const} \cdot (a_{00} \cdot |0\rangle_B + a_{01} \cdot |1\rangle_B$, and
- maps the state $|1\rangle_B$ (corresponding to $\alpha_0 = 0$ and $\alpha_1 = 1$) into a new state $|0'\rangle_B \stackrel{\rm def}{=} \text{const} \cdot (a_{00} \cdot |0\rangle_B + a_{01} \cdot |1\rangle_B$,

transforms two original orthonormal vectors $|0\rangle_B$ and $|1\rangle_B$ into two new orthonormal ones $|0'\rangle_B$ and $|1'\rangle_B$.

In terms of these new states, the entangled state (3a) takes the form

$$\text{const} \cdot (|0\rangle_A \otimes |1'\rangle_B + |1\rangle_B \otimes |0'\rangle_B).$$

From the requirement that the sum of the squares of absolute values of al the coefficients add up to 0, we conclude that $2 \cdot \text{const}^2 = 1$. Then $\text{const} = \dfrac{1}{\sqrt{2}}$ and the entangled state takes the familiar form

$$\frac{1}{\sqrt{2}} \cdot (|0\rangle_A \otimes |1'\rangle_B + |1\rangle_B \otimes |0'\rangle_B). \tag{3}$$

This is exactly the entangled state used in the standard teleportation algorithm. So, we can make the following conclusion.

Conclusion. The only entangled state that leads to a successful teleportation is the state (3) corresponding to the standard quantum teleportation algorithm – for some orthornomal states $|0'\rangle_B$ and $|1'\rangle_B$. In this sense, the existing teleportation algorithm is unique.

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