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Why Bellman-Zadeh Approach to Fuzzy Optimization

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Abstract

In many cases, we need to select the best of the possible alternatives, but we do not know for sure which alternatives are possible and which are not possible. Instead, for each alternative x, we have a subjective probability $p(x)$ that this alternative is possible. In 1970, Richard Bellman and Lotfi Zadeh proposed a heuristic method for selecting an alternative under such uncertainty. Interestingly, this method works very well in many practical applications, while similarly motivated alternative formulas do not work so well. In this paper, we explain the empirical success of the Bellman-Zadeh approach by showing that its formulas can be derived from the general decision theory recommendations.

1 Formulation of the Problem

Need for optimization. In many practical problems, we need to select between several different alternatives. Often, we know exactly what we want: e.g., we want to get from point A to point B the earlier the better, or we want to buy the cheapest possible air ticket, etc. In such situations, there is a clearly defined objective function, and we want to minimize it or maximize it. This can be travel time, travel cost, etc.

In all these cases, to find the best alternative, we need to solve the corresponding optimization problem: finding the alternative x for which the objective function $f(x)$ attains its largest (or smallest) value.

Comment. Minimizing an objective function $f(x)$ is equivalent to maximizing the function $-f(x)$. Thus, without losing generality, we can safely assume that we want to maximize the objective function.

What if the set of possible alternatives is fuzzy? The usual optimization approach implicitly assumes that we know exactly which alternatives are possible and which alternatives are not possible. In practice, this is not always

the case. Instead, for each alternative x, we have a degree of confidence (= subjective probability) $p(x)$ that this alternative is possible.

Which alternative should we then select? Often, if we simply look for an alternative for which the objective function $f(x)$ attains its largest possible value, we get an alternative with a very low probability $p(x)$ of being possible. So what do we do?

Bellman-Zadeh approach: descriptions, successes, and challenge. In their pioneering paper [1], Richard Bellman, a well-known specialist in control, and Lotfi Zadeh, a well-known specialist in uncertainty, proposed to select an alternative x for which the following function attains the largest posisble value:

$$J(x) \stackrel{\text{def}}{=} p(x) \cdot \frac{f(x) - \underline{f}}{\overline{f} - \underline{f}},$$

where:

- $\underline{f}$ is the smallest possible value of $f(x)$ over all alternatives which are, in principle, possible, i.e. for which $p(x) > 0$, and
- $\overline{f}$ is the smallest possible value of $f(x)$ over all alternatives which are, in principle, possible.

This idea works well in many practical applications; see, e.g., [2, 5, 7, 9, 10, 12]. However, it is not clear why this particular formula works well, and other similar ideas – e.g., maximizing the minimum

$$\min\left(p(x), \frac{f(x) - \underline{f}}{\overline{f} - \underline{f}}\right)$$

do not work so well in practice.

What we do in this paper. In this paper, we provide an explanation for the empirical success of the Bellman-Zadeh formula.

2 Analysis of the Problem and the Resulting Solution

Let us recall general decision theory. Since we are talking about decision making, let us recall the results from the general decision theory; see, e.g., [3, 4, 6, 8, 11].

To describe people's preferences, decision theory uses the notion of *utility.* To describe what is utility, we need to select two alternatives (which are *not* among the ones we choose between):

- a very good alternative A_1, an alternative which is much better than any of the currently available alternatives, and

- a very bad alternative A_0, an alternative which is much worse than any of the currently available alternatives.

For each alternative A and for each value p from the interval $[0, 1]$, we can then ask the decision maker to select between:

- the alternative A and
- a lottery $L(p)$ in which he/she gets A_1 with probability p and A_0 with the remaining probability $1 - p$.

When p is small, clearly, $L(p)$ is close to A_0 and is, thus, much worse than A: $L(p) < A$. When p is close to 1, the lottery $L(p)$ is close to A_1 and is, thus, much better than A: $A < L(p)$.

The larger p, the better the lottery $L(p)$. Thus, there exists a threshold value $u(A)$ such that:

- for all $p < u(A)$, we have $L(p) < A$, and
- for all $p > u(A)$, we have $A < L(p)$.

We will denote this relation by $A \equiv L(u(A))$. This threshold value is what is called the *utility* of the alternative A.

Each alternative A is in this sense equivalent to the lottery $L(u(A))$. Between the two or more lotteries with the same two possible outcomes, we should select the one in which the probability of getting the better outcome is the largest. Thus, if we have several alternatives, we need to select the one for which the utility is the largest.

Suppose now that we choose between several actions. Each possible action a could lead to several possible situations $S_1, \ldots, S_n$ with known probabilities $p_1, \ldots, p_n$. We can determine the utility $u(S_i)$ of each of the possible situations. To select an action, we need to know the utility of each action. What is the utility of an action?

Each situation S_i is equivalent to a lottery $L(u(S_i))$ in which we get A_1 with probability $u(S_i)$ and A_0 with the remaining probability $1 - u(S_i)$. Thus, the action is equivalent to a complex lottery, in which:

- we first select i with probability p_i, and then
- depending on the selectied i, choose A_1 with probability $u(S_i)$ and A_0 with probability $1 - u(S_i)$.

As a result of this complex lottery, we get either A_1 or A_0. The probability $u(a)$ of getting A_1 can be computed by using the formula for full probability:

$$u(a) = \sum_{i=1}^{n} p_i \cdot u(S_i). \quad (1)$$

So, the action a is equivalent to a lottery in which we get A_1 with probability $u(a)$ and A_0 with the remaining probability $1 - u(a)$. By definition of the utility,

this means that the utility of the action a is equal to the expression (1). So, we can use the expression (1) to estimate the utility of each action.

Let us apply decision theory to optimization under uncertainty. According to decision theory, our goal is to maximize the utility $u(x)$ of an alternative x. Thus, we can assume that the objective function $f(x)$ is the utility $u(x)$.

In general, if we select an alternative x for which we are not certain that it is possible (i.e., for which $p(x) < 1$), then we will encounter one of the following two situations:

- it may turn out the the selected alternative x is possible; in this case, we get the utility $u(x)$;
- it may also turn out that the alternative x is not possible; in this case, all we can get is the guaranteed minimal value

$$\underline{u} \stackrel{\text{def}}{=} \min_y u(y).$$

The probability of the first alternative is $p(x)$, the probability of the second alternative is, correspondingly, $1-p(x)$. Thus, according to the general decision-theory formula (1), the utility of selecting the alternative x is equal to

$$p(x) \cdot u(x) + (1 - p(x)) \cdot \underline{u}.$$

This expression can be equivalently rewritten as

$$p(x) \cdot u(x) + \underline{u} - op(x) \cdot \underline{u} = p(x) \cdot (u(x) - \underline{u}) + \underline{u}.$$

Adding a constant $\underline{u}$ to all the values of the objective function does not change which alternatives have smaller values of the objective function and which have larger values. Thus, optimizing the above expression is equivalent to maximizing a simpler expression

$$p(x) \cdot (u(x) - \underline{u}).$$

Similarly, dividing all the values of the objective function by a constant $\overline{u}-\underline{u}$ does not change which alternatives have smaller values of the objective function and which have larger values. Thus, optimizing the above expression is equivalent to maximizing the expression

$$p(x) \cdot \frac{u(x) - \underline{u}}{\overline{u} - \underline{u}}.$$

This is exactly Bellman-Zadeh formula!

Thus, we have indeed provided an explanation for why this formula works so well – since it follows directly from the general decision theory recommendations.

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