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# What Is the Economically Optimal Way to Guarantee Interval Bounds on Control?

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## Abstract

For control under uncertainty, interval methods enable us to find a box  $B = [\underline{u}_1, \bar{u}_1] \times \dots [\underline{u}_n, \bar{u}_n]$  for which any control  $u \in B$  has the desired properties – such as stability. Thus, in real-life control, we need to make sure that  $u_i \in [\underline{u}_i, \bar{u}_i]$  for all parameters  $u_i$  describing control. In this paper, we describe the economically optimal way of guaranteeing these bounds.

Keywords: Interval uncertainty, Actuators, Control, Economically optimal solution

## 1 Formulation of the Problem

### 1.1 Control problems: a very brief reminder

In control problems, we need to find the values of the control  $u = (u_1, \dots, u_n)$ .

Usually, there are some requirements on the control: e.g., that under this control, the system should be stable, etc. These conditions are usually described by inequalities; see, e.g., [1].

### 1.2 From optimal control to constraint satisfaction

In general, there are many different controls that satisfy all the desired constraints.

In the ideal case, when:

- we know the exact initial state of the system and
- we know the equations that describe the system's dynamics under different controls,

we can compute the exact consequences of each control. Thus, depending on what is our objective, we can select an appropriate objective functions and looks for the control that optimized this objective function.

The objective function depends on the task. For example, for selecting a plane trajectory, we can have different objective functions:

- In the situation of medical emergency, we need to find the trajectory of the plane that brings the medical team to the remote patient as soon as possible.
- For a regular passenger communications, we need to minimize expenses – and hence, instead of flying at the largest possible speed, we should fly at the speed that saves as much fuel as possible.
- For a private jet, a reasonable objective function is the ride's smoothness.

In practice, we rarely know the exact initial state and the exact system's dynamics. Often, for each of the corresponding parameters, we only know the lower and upper bounds on possible values, i.e., in other words, we only know the interval that contains all possible values of the corresponding parameter; see, e.g., [1–4]. In such cases, for each control, instead of the exact value of the objective function, we get the range of possible values  $[\underline{v}, \bar{v}]$ . Computing this range under interval uncertainty is a particular case of the main problem of *interval computations* [1–3]

In such situations, it make sense, e.g., to describe all the control  $u$  which are possibly optimal, i.e., for which  $\bar{v}(u) \geq \max_{u'} \underline{v}(u')$ .

### 1.3 What we get from interval computations

In situations of interval uncertainty, interval methods enable us to find a box  $B = [\underline{u}_1, \bar{u}_1] \times \dots [\underline{u}_n, \bar{u}_n]$  for which any control  $u \in B$  has the desired properties – such as stability or possible optimality.

## 1.4 Resulting problem

Thus, in real-life control, we need to make sure that  $u_i \in [\underline{u}_i, \bar{u}_i]$  for all parameters  $u_i$  describing control. What is the most economical way to guarantee these bounds?

## 2 Towards Formulating the Problem in Precise Terms

### 2.1 Analysis of the problem

Actuators are never precise, so we can only set up the control value  $u_i$  with some accuracy  $a_i$ . Thus, if we aim for the midpoint  $u_{mi} \stackrel{\text{def}}{=} \frac{\underline{u}_i + \bar{u}_i}{2}$ , we will get the actual value  $u_i$  within the interval  $[u_{mi} - a_i, u_{mi} + a_i]$ .

The only way to guarantee that the control value is indeed within the desired interval is to measure it. Measurement are also never absolutely precise. Let us assume that we use a measuring instrument with accuracy  $\varepsilon_i$ . This means that for each actual value  $u_i$  of the corresponding parameter, the measured value  $\tilde{u}_i$  is somewhere within the interval  $[u_i - \varepsilon_i, u_i + \varepsilon_i]$ . Based on the measurements, the only thing we can conclude about the actual (unknown) value  $u_i$  is that it belongs to the interval  $[\tilde{u}_i - \varepsilon_i, \tilde{u}_i + \varepsilon_i]$ . We want to make sure that all the values from this interval are within the desired interval  $[\underline{u}_i, \bar{u}_i]$ , i.e., that

$$\underline{u}_i \leq \tilde{u}_i - \varepsilon_i \quad (1)$$

and

$$\tilde{u}_i + \varepsilon_i \leq \bar{u}_i. \quad (2)$$

These inequalities must hold for all possible values  $\tilde{u}_i \in [u_i - \varepsilon_i, u_i + \varepsilon_i]$ . For the inequality (1) to hold for all these values, it is sufficient to require that this inequality holds for the smallest possible value  $\tilde{u}_i = u_i - \varepsilon_i$ , i.e., that we have

$$\underline{u}_i \leq (u_i - \varepsilon_i) - \varepsilon_i = u_i - 2\varepsilon_i. \quad (3)$$

Similarly, for the inequality (2) to hold for all the values  $\tilde{u}_i \in [u_i - \varepsilon_i, u_i + \varepsilon_i]$ , it is sufficient to require that this inequality holds for the largest possible value  $\tilde{u}_i = u_i + \varepsilon_i$ , i.e., that we have

$$(u_i + \varepsilon_i) + \varepsilon_i = u_i + 2\varepsilon_i \leq \bar{u}_i. \quad (4)$$

The inequalities (3) and (4) must hold for all possible values  $u_i \in [u_{mi} - a_i, u_{mi} + a_i]$ . For the inequality (3) to hold for all these values, it is sufficient to require that this inequality holds for the smallest possible value  $u_i = u_{mi} - a_i$ , i.e., that we have

$$\underline{u}_i \leq u_{mi} - 2\varepsilon_i - a_i. \quad (5)$$

Similarly, for the inequality (4) to hold for all the values  $u_i \in [u_{mi} - a_i, u_{mi} + a_i]$ , it is sufficient to require that this inequality holds for the largest possible value  $u_i = u_{mi} + a_i$ , i.e., that we have

$$u_{mi} + 2\varepsilon_i + a_i \leq \bar{u}_i. \quad (6)$$

Let us denote the half-width of the interval  $[\underline{u}_i, \bar{u}_i]$  by  $\Delta_i \stackrel{\text{def}}{=} \frac{\bar{u}_i - \underline{u}_i}{2}$ . In terms of the half-width,  $\bar{u}_i - u_{mi} = u_{mi} - \underline{u}_i = \Delta_i$ . Thus, the inequalities (5) and (6) are equivalent to the inequality

$$2\varepsilon_i + a_i \leq \Delta_i. \quad (7)$$

### 2.2 In the optimal solution, we have equality

In general, the more accuracy we want, the more expensive will be the corresponding measurements and actuators. From this viewpoint, if  $2\varepsilon_i + a_i < \Delta_i$ , then we can use slightly less accurate actuators and/or measuring instruments and still guarantee the desired inequality (7). Thus, in the most economical solution, in the formula (7), we should have the exact equality:

$$2\varepsilon_i + a_i = \Delta_i, \quad (8)$$

i.e., equivalently,

$$a_i = \Delta_i - 2\varepsilon_i. \quad (9)$$

### 2.3 Towards resulting formulation of the problem

So, the problem is to find, among all the values  $a_i$  and  $\varepsilon_i$  that satisfy the equality (9), the values for which the overall expenses are the smallest possible.

To solve this problem, we need to know how the cost of actuators and measurements depends on accuracy. To analyze this dependence, we start with a 1-D case, when we only have a single control parameter  $u_1$ .

### 3 1-D Case, When We Have a Single Control Parameter $u_1$

#### 3.1 Cost of actuators: main idea

Actuators – such as robotic arms – are usually rather crude, so we may not be able to properly orient the robot after the first attempt. A natural way to provide a better accuracy is to repeat the attempts until we get the desired location (or, in general, the desired value of the parameter  $u_i$ ).

#### 3.2 Cost of actuators: from idea to a formula

Let us assume that the given actuator can provide the value  $u_i$  with some accuracy  $A_i$ . This means that if we aim for the midpoint  $u_{mi}$ , we will get values from the interval

$$[u_{mi} - A_i, u_{mi} + A_i]. \quad (10)$$

We do not know the relative frequency of different values within this interval. Since we have no reasons to assume that some of these values are more probable and some are less probable, it is therefore reasonable to assume that all the values from the interval (10) are equally probable, i.e., that we have a uniform probability distribution on this interval.

By repeatedly trying, we want to get the value  $u_i$  within the interval

$$[u_{mi} - a_i, u_{mi} + a_i] \quad (11)$$

for some  $a_i < A_i$ . For the uniform distribution, the probability to be within a subinterval is proportional to the width of this subinterval – namely, it is equal to the ratio between the width of the subinterval and the width of the entire interval. In particular, the probability that at each try, the value  $u_i$  is within the interval is equal to the ratio  $p = \frac{2a_i}{2A_i} = \frac{a_i}{A_i}$ . So, on average, we need  $\frac{1}{p} = \frac{A_i}{a_i}$  iterations to get into the desired interval (11).

#### 3.3 Cost of measurements

How to gauge the cost of accurate measurements? It is known that if we start with a measuring instrument with 0 mean and standard deviation  $\sigma_i$ , then, by performing  $M$  independent measurements and averaging the results, we get a  $\sqrt{M}$  times smaller standard deviation.

Indeed, the variance of the sum of independent random variables is equal to the sum of the variances, so for the sum of  $M$  measurement errors, the variance is  $m \cdot \sigma_i^2$ , and thus, the standard deviation is  $\sqrt{m} \cdot \sigma_i$ . The arithmetic average is obtained by dividing the sum by  $M$ , so its standard deviation is

$$\frac{\sqrt{M} \cdot \sigma_i}{M} = \frac{\sigma_i}{\sqrt{M}}.$$

So, if we start with a measuring instrument with accuracy  $\sigma_i$ , and we want accuracy  $\varepsilon_i$ , we need to repeat each measurement  $M_i$  times, where  $\frac{\sigma_i}{\sqrt{M_i}} = \varepsilon_i$ , i.e.,  $M_i = \frac{\sigma_i^2}{\varepsilon_i^2}$ . The cost of measurement is proportional to the number  $M_i$  of such measurements, so it is equal to

$$m_i \cdot M_i = m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2},$$

where by  $m_i$ , we denoted the cost of a single measurement.

#### 3.4 Overall cost

Let  $T_i$  denote the cost of a single actuator try.

The overall cost of one try of an actuator is equal to the actuator trying cost  $T_i$  plus the measurement cost  $m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2}$ , i.e., it is equal to  $T_i + m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2}$ . To achieve the actuator accuracy  $a_i = \Delta_i - 2\varepsilon_i$ , we need to perform  $\frac{A_i}{a_i} = \frac{A_i}{\Delta_i - 2\varepsilon_i}$  tries. Thus, the overall cost  $C$  of all the tries is equal to

$$C = \frac{A_i}{\Delta_i - 2\varepsilon_i} \cdot \left( T_i + m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2} \right). \quad (12)$$

#### 3.5 Resulting optimization problem

In the 1-D case, we need to find the value  $\varepsilon_i$  for which the cost (12) is the smallest possible.

### 3.6 Discussion

- When  $\varepsilon_i$  is close to 0, the cost of measurement tends to infinity, so we have a very large overall cost.
- Similarly, when  $a_i$  is close to 0, i.e., when  $\varepsilon_i \approx \frac{\Delta_i}{2}$ , the actuator cost becomes very large, so the overall cost is also very large.

Thus, there should be values  $\varepsilon_i$  between 0 and  $\frac{\Delta_i}{2}$  for which the cost of maintaining  $u_i$  within the desired interval  $[\underline{u}_i, \bar{u}_i]$  is the smallest possible.

### 3.7 How to solve this optimization problem

In the 1-D case, where we have a single unknown  $\varepsilon_i$ , to find the optimal value of this unknown, we can simply differentiate the objective function (1) with respect to  $\varepsilon_i$  and equate the derivative to 0.

Minimizing the expression (12) is equivalent to minimizing its logarithm

$$\ln(A_i) - \ln(\Delta_i - 2\varepsilon_i) + \ln\left(T_i + m_i \cdot \frac{\sigma_i^2}{\varepsilon_i^2}\right).$$

Differentiating this expression with respect to  $\varepsilon_i$  and equating the derivative to 0, we get

$$\frac{2}{\Delta_i - 2\varepsilon_i} - 2 \cdot m_i \cdot \sigma_i^2 \cdot \frac{1}{\varepsilon_i^3} \cdot \frac{1}{T_i + m_i \cdot \sigma_i^2 \cdot \varepsilon_i^{-2}} = 0.$$

Dividing both sides by 2, moving the negative term to the right-hand side, and explicitly multiplying the expressions in the right-hand side, we get

$$\frac{1}{\Delta_i - 2\varepsilon_i} = \frac{m_i \cdot \sigma_i^2}{\varepsilon_i^3 \cdot T_i + m_i \cdot \sigma_i^2 \cdot \varepsilon_i}.$$

Bringing both fractions to the common denominator, we get a cubic equation

$$\varepsilon_i^3 \cdot T_i + m_i \cdot \sigma_i^2 \cdot \varepsilon_i = m_i \cdot \sigma_i^2 \cdot (\Delta_i - 2\varepsilon_i),$$

i.e., equivalently,

$$T_i \cdot \varepsilon_i^3 + 3m_i \cdot \sigma_i^2 \cdot \varepsilon_i - m_i \cdot \sigma_i^2 \cdot \Delta_i = 0. \quad (13)$$

### 3.8 Resulting algorithm

To find the optimal accuracy  $\varepsilon_i$  of the measuring instrument, we can use one of the standard methods (e.g., Newton's method) so solve the cubic equation (13).

Then, we can find the optimal value  $a_i$  of the actuator accuracy as  $a_i = \Delta_i - 2\varepsilon_i$ .

## 4 General Case

### 4.1 Notations

In general, we may have several actuators. Let us denote the number of actuators by  $A$ . For each actuator  $a = 1, \dots, A$ , let us denote:

- the cost of one try by  $T_a$ , and
- the number of the corresponding control parameters by  $n_a$ .

For each actuator  $a$ , and for each  $i$  from 1 to  $n_a$ , let us denote:

- the  $i$ -th control parameter by  $u_{ai}$ ;
- the bounds of the corresponding control parameter by  $\underline{u}_{ai}$  and  $\bar{u}_{ai}$ ;
- the midpoint of the resulting interval by  $u_{mai} \stackrel{\text{def}}{=} \frac{\underline{u}_{ai} + \bar{u}_{ai}}{2}$ ;
- the half-width of the corresponding interval by  $\Delta_{ai} \stackrel{\text{def}}{=} \frac{\bar{u}_{ai} - \underline{u}_{ai}}{2}$ ;
- the bounds achievable on one try by  $A_{ai}$ ,

- the desired actuator accuracy by  $a_{ai}$ ,
- the desired measurement accuracy by  $\varepsilon_{ai}$ ,
- the accuracy of the corresponding measuring instrument by  $\sigma_{ai}$ , and
- the cost of a single measurement with that accuracy by  $m_{ai}$ .

#### 4.2 Relation between accuracies of actuator and measurement

Similarly to the 1-D case, we can conclude that in the general case, for each  $a$  and  $i$ , we have  $2\varepsilon_{ai} + a_{ai} = \Delta_{ai}$ , i.e.,  $a_{ai} = \Delta_{ai} - 2\varepsilon_{ai}$ .

#### 4.3 Cost of actuators

We assume that the actuator  $a$  can provide the value  $u_{ai}$  with some accuracy  $A_{ai}$ . This means that if we aim for the midpoint  $u_{mai}$ , we will get values from the interval  $[u_{mai} - A_{ai}, u_{mai} + A_{ai}]$ . Similar to the 1-D case, it is reasonable to assume that all the combinations  $(u_{a1}, \dots, u_{an_a})$  from the corresponding box

$$[u_{ma1} - A_{a1}, u_{ma1} + A_{a1}] \times \dots \times [u_{man_a} - A_{an_a}, u_{man_a} + A_{an_a}]$$

are equally probable, i.e., that we have a uniform probability distribution on this box.

By repeatedly trying, we want to get the value  $(u_{a1}, \dots, u_{an_a})$  within the smaller box

$$[u_{ma1} - a_{a1}, u_{ma1} + a_{a1}] \times \dots \times [u_{man_a} - a_{an_a}, u_{man_a} + a_{an_a}].$$

For the uniform distribution, the probability to be within a sub-box is proportional to the volume of this sub-box – namely, it is equal to the ratio between the volume of the sub-box and the volume of the entire box. In particular, the probability that at each try, the values  $a_{ai}$  are within the desired box is equal to the ratio

$$p = \frac{(2a_1) \cdot \dots \cdot (2a_{n_a})}{(2A_1) \cdot \dots \cdot (2A_{n_a})} = \frac{a_{a1} \cdot \dots \cdot a_{an_a}}{A_{a1} \cdot \dots \cdot A_{an_a}}.$$

So, on average, we need

$$\frac{1}{p} = \frac{A_{a1} \cdot \dots \cdot A_{an_a}}{a_{a1} \cdot \dots \cdot a_{an_a}} = \prod_{i=1}^{n_a} \frac{A_{ai}}{a_{ai}}$$

iterations to get into the desired box

$$[u_{ma1} - a_{a1}, u_{ma1} + a_{a1}] \times \dots \times [u_{man_a} - a_{an_a}, u_{man_a} + a_{an_a}].$$

#### 4.4 Overall cost for each actuator

Similarly to the 1-D case, for each  $i$ , the cost of measuring the value  $u_{ai}$  with accuracy  $\varepsilon_{ai}$  is equal to  $m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2}$ . Thus, the overall cost of measuring all these values is equal to the sum

$$\sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2}.$$

The overall cost of one try of an actuator is equal to the actuator trying cost  $T_a$  plus the measurement cost:

$$T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2}.$$

To achieve the actuator accuracy  $a_{ai} = \Delta_{ai} - 2\varepsilon_{ai}$ , we need to perform

$$\prod_{i=1}^{n_a} \frac{A_{ai}}{a_{ai}} = \prod_{i=1}^{n_a} \frac{A_{ai}}{\Delta_{ai} - 2\varepsilon_{ai}}$$

tries. Thus, the overall cost  $C_a$  of all the tries is equal to

$$C_a = \prod_{i=1}^{n_a} \frac{A_{ai}}{\Delta_{ai} - 2\varepsilon_{ai}} \cdot \left( T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2} \right). \quad (14)$$

#### 4.5 The overall cost of all the actuators

The overall cost  $C$  of all the actuators can be obtained by adding up all the costs of all the actuators:

$$C = \sum_{a=1}^A C_a = \sum_{a=1}^A \left( \prod_{i=1}^{n_a} \frac{A_{ai}}{\Delta_{ai} - 2\varepsilon_{ai}} \cdot \left( T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2} \right) \right). \quad (15)$$

#### 4.6 Resulting optimization problem

In the general case, we need to find the values  $\varepsilon_{ai}$  for which the cost (15) is the smallest possible.

#### 4.7 Towards solving the optimization problem

First, one can notice that each cost  $C_a$  depends only on the parameters corresponding to this actuator. Thus, to optimize the overall cost  $C$ , it is sufficient to optimize the cost  $C_a$  for each actuator  $a$ .

For each  $a$ , minimizing  $C_a$  is equivalent to minimizing its logarithm

$$\sum_{i=1}^{n_a} \ln(A_{ai}) - \sum_{i=1}^{n_a} \ln(\Delta_{ai} - 2\varepsilon_{ai}) + \ln \left( T_a + \sum_{i=1}^{n_a} m_{ai} \cdot \frac{\sigma_{ai}^2}{\varepsilon_{ai}^2} \right).$$

Differentiating this expression with respect to  $\varepsilon_{ai}$  and equating the derivative to 0, we get

$$\frac{2}{\Delta_{ai} - 2\varepsilon_{ai}} - 2 \cdot m_{ai} \cdot \sigma_{ai}^2 \cdot \frac{1}{\varepsilon_{ai}^3} \cdot \frac{1}{m_a} = 0, \quad (16)$$

where we denoted

$$m_a \stackrel{\text{def}}{=} T_a + \sum_{j=1}^{n_a} m_{aj} \cdot \frac{\sigma_{aj}^2}{\varepsilon_{aj}^2}. \quad (17)$$

Dividing both sides of the formula (16) by 2, moving the negative term to the right-hand side, and explicitly multiplying the expressions in the right-hand side, we get

$$\frac{1}{\Delta_{ai} - 2\varepsilon_{ai}} = \frac{m_{ai} \cdot \sigma_{ai}^2}{\varepsilon_{ai}^3 \cdot m_a}.$$

Bringing both fractions to the common denominator, we get a cubic equation

$$\varepsilon_{ai}^3 \cdot m_a = m_{ai} \cdot \sigma_{ai}^2 \cdot (\Delta_{ai} - 2\varepsilon_{ai}),$$

i.e., equivalently,

$$m_a \cdot \varepsilon_{ai}^3 + 2m_{ai} \cdot \sigma_{ai}^2 \cdot \varepsilon_{ai} - m_{ai} \cdot \sigma_{ai}^2 \cdot \Delta_{ai} = 0. \quad (18)$$

#### 4.8 Resulting algorithm

For each actuator  $a$ , once we fixed the value  $m_a$ , we can find each value  $\varepsilon_{ai}$  ( $i = 1, \dots, n_a$ ) by solving the cubic equation (18).

We can then check whether our guess was correct by checking whether the formula (17) is satisfied for the resulting values  $\varepsilon_{ai}$ . By using bisection, we can find the value  $m_a$  for which the equality (17) is satisfied.

Then, we can find the optimal value  $a_{ai}$  of the actuator accuracy as  $a_{ai} = \Delta_{ai} - 2\varepsilon_{ai}$ .

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