

1-2018

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Technical Report: UTEP-CS-18-01

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### Recommended Citation

Servin, Christian; Kosheleva, Olga; and Kreinovich, Vladik, "How to Detect Crisp Sets Based on Subsethood Ordering of Normalized Fuzzy Sets? How to Detect Type-1 Sets Based on Subsethood Ordering of Normalized Interval-Valued Fuzzy Sets?" (2018). *Departmental Technical Reports (CS)*. 1217.  
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# How to Detect Crisp Sets Based on Subsethood Ordering of Normalized Fuzzy Sets? How to Detect Type-1 Sets Based on Subsethood Ordering of Normalized Interval-Valued Fuzzy Sets?

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**Abstract**—If all we know about normalized fuzzy sets is which set is a subset of which, will we be able to detect crisp sets? It is known that we can do it if we allow all possible fuzzy sets, including non-normalized ones. In this paper, we show that a similar detection is possible if we only allow normalized fuzzy sets. We also show that we can detect type-1 fuzzy sets based on the subsethood ordering of normalized interval-valued fuzzy sets.

## I. INTRODUCTION

A fuzzy set is usually defined as function  $A$  from a certain set  $U$  – known as *Universe of discourse* – to the interval  $[0, 1]$ ; see, e.g., [1], [2], [3], [5], [6], [8]. Traditional – “crisp” – sets can be viewed as particular cases of fuzzy sets, for which  $A(a) \in \{0, 1\}$  for all  $x$ .

In most applications, we consider *normalized* fuzzy sets, i.e., fuzzy sets for which  $A(x) = 1$  for some  $x \in U$ . For crisp sets, this corresponds to considering non-empty sets.

For two crisp sets,  $A$  is a subset or  $B$  if and only if  $A(x) \leq B(x)$  for all  $x$ . The same condition is used as a definition of the subsethood ordering between fuzzy sets: a fuzzy set  $A$  is a *subset* of a fuzzy set  $B$  if  $A(x) \leq B(x)$  for all  $x$ . Subsets  $B \subseteq A$  which are different from the set  $A$  are called *proper* subsets of  $A$ .

A natural question is: if we have a class of all normalized fuzzy sets with the subsethood relation, can we detect which of these fuzzy sets are crisp? It is known that:

- if we allow *all* possible fuzzy sets – even non-normalized ones,
- then we can detect crisp sets; see, e.g., [7].

In this paper, we show that such a detection is possible even if we restrict ourselves only to normalized sets.

## II. MAIN RESULT

In order to describe general crisp sets in terms of subsethood relation  $\subseteq$  between fuzzy sets, we will first describe some auxiliary notions in these terms.

In this section, we only consider normalized fuzzy sets.

**Proposition 1.** *A normalized fuzzy set is a 1-element crisp set if and only if it has no proper normalized fuzzy subsets, i.e., if and only if  $B \subseteq A$  implies  $B = A$ .*

**Proof.**

1°. Let us first prove that a 1-element crisp set  $A = \{x_0\}$  (i.e., a set for which  $A(x_0) = 1$  and  $A(x) = 0$  for all  $x \neq x_0$ ) has the desired property.

Indeed, if  $B \subseteq A$ , this means that  $B(x) \leq A(x)$  for all  $x$ . For  $x \neq x_0$ , we have  $A(x) = 0$ , so we have  $B(x) = 0$  as well.

Since  $B$  is a normalized fuzzy set, it has to attain value 1 somewhere. Since we have  $B(x) = 0$  for all  $x \neq x_0$ , the only point  $x \in U$  at which we can have  $B(x) = 1$  is the point  $x_0$ . Thus, we have  $B(x_0) = 1$ .

So, indeed, we have  $B(x) = A(x)$  for all  $x$ , i.e.,  $B = A$ .

2°. Vice versa, let us prove that each normalized fuzzy set  $A$  which is different from a 1-element crisp set has a proper normalized fuzzy subset.

Indeed, since  $A$  is normalized, we have  $A(x_0) = 1$  for some  $x_0$ . Then, we can take  $B = \{x_0\}$ . Clearly,  $B \subseteq A$ , and, since  $A$  is not a 1-element crisp set,  $B \neq A$ .

The proposition is proven.

**Definition 1.** *By a 2-element set, we mean a normalized fuzzy set  $A$  for which  $A(x) > 0$  for exactly two elements  $x \in U$ .*

**Proposition 2.** *For a normalized fuzzy set  $A$  which is not a 1-element crisp set, the following two conditions are equivalent to each other:*

- $A$  is a non-crisp 2-element set, and
- the class  $\{B : B \subseteq A\}$  of all its subsets is linearly ordered, i.e.:

if  $B_1 \subseteq A$  and  $B_2 \subseteq A$  then  $B_1 \subseteq B_2$  or  $B_2 \subseteq B_1$ .

**Proof.**

1°. Let us first prove that if  $A$  is a 2-element non-crisp set, then the class of all its subsets is linearly ordered.

Indeed, since  $A$  is a normalized fuzzy set, we must have  $A(x_0) = 1$  for some  $x_0 \in U$ . Since  $A$  is a 2-element set, there must be one more value  $x \in U$  for which  $A(x) > 0$ . Let us denote this value by  $x_1$ . So, we have:

- $A(x_0) = 1$ ,
- $A(x_1) > 0$  and
- $A(x) = 0$  for all other  $x \in U$ .

If we had  $A(x_1) = 1$ , then  $A$  would be a crisp set – namely, we would have  $A = \{x_0, x_1\}$ . Since  $A$  is a non-crisp set, we thus cannot have  $A(x_1) = 1$ , so we have  $0 < A(x_1) < 1$ .

If  $B$  is a normalized fuzzy set for which  $B \subseteq A$ , then for all  $x$  different from  $x_0$  and  $x_1$ , we have  $B(x) \leq A(x) = 0$  and thus,  $B(x) = 0$ . Since  $B$  is normalized, we have  $B(x) = 1$  for some  $x$ .

- This  $x$  cannot be different from  $x_0$  and  $x_1$  – because then  $B(x) = 0$ .
- This  $x$  cannot be equal to  $x_1$ , since then we would have  $1 = B(x_1) \leq A(x_1) < 1$  and  $1 < 1$ .

Thus, this  $x$  must be equal to  $x_0$ , i.e., we must have  $B(x_0) = 1$ . So, all fuzzy normalized subsets  $B$  of the set  $A$  have the following form:

- $B(x_0) = 1$ ,
- $B(x_1) \leq A(x_1)$ , and
- $B(x) = 0$  for all other  $x$ .

For two such subsets, we can have:

- either  $B_1(x_1) \leq B_2(x_1)$ ,
- or  $B_2(x_1) \leq B_1(x_1)$ .

One can easily check that:

- if  $B_1(x_1) \leq B_2(x_1)$ , then  $B_1(x) \leq B_2(x)$  for all  $x$  and thus,  $B_1 \subseteq B_2$ ;
- similarly, if  $B_2(x_1) \leq B_1(x_1)$ , then  $B_2(x) \leq B_1(x)$  for all  $x$  and thus,  $B_2 \subseteq B_1$ .

So, for every two normalized fuzzy subsets  $B_1$  and  $B_2$  of the set  $A$ , we have either  $B_1 \subseteq B_2$  or  $B_2 \subseteq B_1$ . Thus, the class of all such subsets is indeed linearly ordered.

2°. To complete the proof of Proposition 2, let us now prove that if a normalized fuzzy set  $A$  is not a 1-element fuzzy set and *not* a non-crisp 2-element set, then the class

$$\{B : B \subseteq A\}$$

is *not* linearly ordered, i.e., there exists normalized fuzzy subsets  $B_1 \subseteq A$  and  $B_2 \subseteq A$  for which  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ .

The fact that the set  $A$  is not a 1-element set means that  $A(x) > 0$  for at least two different values  $x$ .

By definition, a non-crisp 2-element set is a normalized fuzzy set:

- which is a 2-element set *and*
- which is not crisp.

So, if a normalized fuzzy set  $A$  is *not* a non-crisp 2-element set, this means that it is:

- either not a 2-element set
- or it is a crisp 2-element set.

Let us show that in both cases, we can find subsets  $B_1 \subseteq A$  and  $B_2 \subseteq A$  for which  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ .

2.1°. Let us first consider the case when  $A$  is not a 2-element set, i.e., when, in addition to the point  $x_0$  at which  $A(x_0) = 1$ , there exist at least two other points  $x_1$  and  $x_2$  for which  $A(x_1) > 0$  and  $A(x_2) > 0$ .

In this case, we can take the following sets  $B_1$  and  $B_2$ :

- $B_1(x_0) = B_2(x_0) = 1$ ;
- $B_1(x_1) = A(x_1)$  and  $B_2(x_1) = 0$ ;
- $B_2(x_1) = 0$  and  $B_2(x_2) = A(x_2)$ , and
- $B_1(x) = B_2(x)$  for all other  $x$ .

One can see that  $B_1(x) \leq A(x)$  and  $B_2(x) \leq A(x)$  for all  $x$ , so indeed  $B_1 \subseteq A$  and  $B_2 \subseteq A$ . However, here:

- $B_1(x_1) = A(x_1) > 0 = B_2(x_1)$ , so we cannot have  $B_1 \subseteq B_2$ , because that would imply  $B_1(x_1) \leq B_2(x_1)$ ;
- similarly,  $B_2(x_2) = A(x_2) > 0 = B_1(x_2)$ , so we cannot have  $B_2 \subseteq B_1$ , because that would imply  $B_2(x_2) \leq B_1(x_2)$ .

So, we indeed have  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ .

2.2°. Let us now consider the case when  $A$  is a 2-element crisp set, i.e., when  $A = \{x_0, x_1\}$ .

In this case, we can take  $B_1 = \{x_0\}$  and  $B_2 = \{x_1\}$ . Clearly,  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ .

So, the proposition is proven.

**Proposition 3.** *A normalized fuzzy set  $A$  is a crisp 2-element set if and only if the following two conditions are satisfied:*

- the set  $A$  itself is not a 1-element crisp set and not a 2-element non-crisp set, but
- each proper normalized fuzzy subsets  $B \subseteq A$  is either a crisp 1-element sets or a non-crisp 2-element set.

**Proof.**

1°. If  $A$  is a 2-element crisp set, i.e., if  $A = \{x_0, x_1\}$  for some  $x_0 \neq x_1$ , then it is clearly:

- not a 1-element crisp set, and
- not a non-crisp 2-element set.

Let us prove that in this case, every proper normalized fuzzy subset  $B \subseteq A$  is

- either a 1-element crisp set
- or a non-crisp 2-element set.

Since  $A(x) > 0$  for only two values  $x = x_0$  and  $x = x_1$ , and  $B(x) \leq A(x)$  for all  $x$ , the value  $B(x)$  can be positive also for at most two values  $x_i$ .

If  $B(x) > 0$  for only one value  $x$ , then, since  $B$  is normalized, for this  $x$ , we must have  $B(x) = 1$ . Thus, we have  $B = \{x\}$ , i.e.,  $B$  is a 1-element crisp set.

If  $B(x) > 0$  for two different values  $x$ , this means that we have  $B(x_0) > 0$  and  $B(x_1) > 0$ . Since the set  $B$  is normalized, one of these value must be equal to 1. If the second one is equal to 1, we will have  $B = A$  – but  $B$  is a proper subset. Thus, one of the values  $B(x_i)$  is smaller than 1 – thus,  $B$  is a non-crisp 2-element set.

2°. Let us now prove that if a normalized fuzzy set  $A$  is not a 2-element crisp set, then one of the above properties is not satisfied, i.e.,

- either  $A$  is 1-element crisp set or a 2-element non-crisp set,
- or one of its proper subsets  $B \subseteq A$  is *not* a non-crisp 2-element set.

In other words, we want to prove that if  $A$  is:

- not a crisp 1-element set,
- not a crisp 2-element set, and
- not a non-crisp 2-element set,

then one of its proper subsets  $B \subseteq A$  is *not* a non-crisp 2-element set.

The condition on  $A$  means that it is:

- not a 1-element set and
- not a 2-element set.

This means that there must exist at least three different values  $x \in U$  for which  $A(x) > 0$ . For one of these values, we have  $A(x_0) = 1$ , let us denote the other two values by  $x_1$  and  $x_2$ , then  $A(x_1) > 0$  and  $A(x_2) > 0$ .

Let us now take the following normalized fuzzy set  $B$ ;

- $B(x_1) = 0.5 \cdot A(x_1)$ ,
- $B(x_2) = 0.5 \cdot A(x_2)$ , and
- $B(x) = A(x)$  for all other  $x$ .

Here,  $B(x_0) = A(x_0) = 1$ , so  $B$  is indeed a normalized fuzzy set. One can easily check that  $B(x) \leq A(x)$  for all  $x$ , so it is indeed a subset of  $A$ . Since  $A(x_1) > 0$ , we have  $B(x_1) = 0.5 \cdot A(x_1) \neq A(x_1)$ , so  $B$  is a proper subset of  $A$ .

However,  $B(x_0) = 1 > 0$ ,  $B(x_1) > 0$ , and  $B(x_2) > 0$ , so  $B$  is *not* a 2-element set.

The proposition is proven.

*Comment.* Now, we are ready to show that crisp sets can be described in terms of the subsethood relation.

**Proposition 4.** *A normalized fuzzy set is crisp if and only if we have one of the following two cases:*

- $A$  is a 1-element fuzzy set, or
- for every subset  $B \subseteq A$  which is a non-crisp 2-element set, there exists a crisp 2-element set  $C$  for which

$$B \subseteq C \subseteq A.$$

**Comment.** Since Propositions 1–3 show that the properties of being a crisp 1-element set, a crisp 2-element set, and a non-crisp 2-element set can all be described in terms of the subsethood relation, this Proposition shows that crispness can indeed be described in terms of subsethood.

**Proof.**

1°. Let us first prove that if  $A$  is a crisp set, then:

- either it is a 1-element crisp set,
- or for every non-crisp 2-element set  $B \subseteq A$ , there exists a crisp 2-element set  $C$  for which  $B \subseteq C \subseteq A$ .

Indeed, let  $B$  be a non-crisp 2-element set. This means that for some elements  $x_0 \in U$  and  $x_1 \in U$ , we have:

- $B(x_0) = 1$ ,
- $0 < B(x_1) < 1$ , and
- $B(x) = 0$  for all other  $x$ .

Since  $B \subseteq A$ , we have:

- $1 = B(x_0) \leq A(x_0)$  – thus  $A(x_0) = 1$ ; and
- $0 < B(x_1) \leq A(x_1)$  – thus  $A(x_1) > 0$ .

The set  $A$  is crisp, so  $A(x_1)$  can be either 0 or 1. Since  $A(x_1) > 0$ , we must have  $A(x_1) = 1$ . Thus, for a 2-element crisp set  $C = \{x_0, x_1\}$ , we have  $B \subseteq C \subseteq A$ .

2°. To complete our proof, let us prove that if a normalized crisp set  $A$  is *not* a crisp set, then there exists a non-crisp 2-element set  $B \subseteq A$  for which no crisp 2-element set  $C$  satisfies the property  $B \subseteq C \subseteq A$ .

By definition, for a crisp set, all the values  $A(x)$  are either 0s or 1s. So, the fact that  $A$  is not crisp means that we have  $0 < A(x_1) < 1$  for some  $x_1 \in U$ .

Since  $A$  is normalized, there exists  $x_0$  for which  $A(x_0) = 1$ . Let us now take the following set  $B$ ;

- $B(x_0) = 1$ ,
- $0 < B(x_1) = A(x_1) < 1$ , and
- $B(x) = 0$  for all other  $x$ .

Clearly,  $B$  is a non-crisp 2-element set and  $B \subseteq A$ .

If we had  $B \subseteq C \subseteq A$  for some crisp 2-element set  $C$ , then due to  $1 = B(x_0) \leq C(x_0)$  and  $B(x_1) \leq C(x_1)$ , we would have  $C(x_0) = 1$  and  $C(x_1) > 0$  – hence  $C(x_1) = 1$  (since  $C$  is crisp). But in this case,  $C(x_1) = 1 > A(x_1)$ , so we cannot have  $C \subseteq A$ .

The proposition is proven.

### III. INTERVAL-VALUED CASE

**Formulation of the problem.** The traditional fuzzy logic assumes that experts can meaningfully describe their degrees of certainty by numbers from the interval  $[0, 1]$ . In practice, however, experts cannot meaningfully select a single number fuzzy describing their degree of certainty – since it is not possible to distinguish between, say, degrees 0.80 and 0.81. A more adequate description of the expert's uncertainty is when we allow to characterize the uncertainty by a whole range of possible numbers, i.e., by an interval  $[\underline{A}(x), \overline{A}(x)]$ .

This idea leads to *interval-valued* fuzzy numbers (see, e.g., [3], [4]), i.e., mappings that assign, to each element  $x$  from the Universe of discourse, an interval  $A(x) = [\underline{A}(x), \overline{A}(x)]$ .

For two interval-valued degrees  $A = [\underline{A}, \overline{A}]$  and  $B = [\underline{B}, \overline{B}]$ , it is reasonable to say that  $A \leq B$  if

$$\underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B}.$$

Thus, we can define a subsethood relation between two interval-valued fuzzy sets  $A$  and  $B$  as  $A(x) \leq B(x)$  for all  $x$ .

An interval-valued fuzzy set is normalized if  $\overline{A}(x_0) = 1$  for some  $x_0$ .

Traditional (*type-2*) fuzzy sets can be viewed as particular cases of interval-valued fuzzy sets, with “degenerate” intervals

$$[A(x), A(x)].$$

Here, we have a similar problem: can we detect traditional fuzzy sets based only on the subsethood relation between interval-valued fuzzy sets?

Let us show that this is indeed possible.

**Definition 2.** By an uncertain 1-element set, we mean a normalized interval-valued fuzzy set  $A$  for which, for some  $x_0 \in U$ , we have:

- $A(x_0) = [0, 1]$  and
- $A(x) = [0, 0]$  for all other  $x$ .

**Proposition 5.** A normalized interval-valued fuzzy set  $A$  is an uncertain 1-element set if and only if it has no proper normalized subsets.

*Comment.* So, we can determine uncertain 1-element sets based on the subsethood relation.

**Proof.**

1°. Let us first prove that for an uncertain 1-element set  $A$ , there are no proper subsets.

Indeed, if  $A(x_0) = [0, 1]$ ,  $A(x) = [0, 0]$  for all  $x \neq x_0$ , and  $B(x) \leq A(x)$ , then:

- for  $x \neq x_0$ , from  $\underline{B}(x) \leq \underline{A}(x) = 0$  and  $\overline{B}(x) \leq \overline{A}(x) = 0$ , it follows that  $\underline{B}(x) = \overline{B}(x) = 0$ , so

$$B(x) = [0, 0] = A(x);$$

- for  $x = x_0$ , from  $\underline{A}(x_0) \leq \overline{A}(x_0) = 0$ , it follows that

$$\underline{B}(x_0) = 0 = \underline{A}(x_0).$$

On the other hand,  $B$  is a normalized interval-valued fuzzy set, so we must have  $\overline{B}(x) = 1$  for some  $x$ . This cannot be for  $x \neq x_0$ , since then  $\overline{B}(x) = 0$ . So, the only remaining option is  $x = x_0$ . Hence,  $\overline{B}(x_0) = 1$ , thus,  $\overline{B}(x_0) = \overline{A}(x_0)$ .

Therefore, if  $B \subseteq A$ , then  $B = A$ . So, the normalized interval-valued fuzzy sets  $A$  does not have any proper subsets.

2°. To complete the proof, let us prove that if a normalized interval-valued fuzzy set has no proper subsets, then it is an uncertain 1-element set.

Indeed, since  $A$  is normalized, there exists an element  $x_0$  for which  $\overline{A}(x_0) = 1$ . Then, as one can easily check, we have  $B \subseteq A$ , where:

- $B(x_0) = [0, 1]$ , and
- $B(x) = [0, 0]$  for all other  $x$

Since  $A$  has no proper subsets, we thus conclude that  $A = B$ , i.e., that  $A$  is an uncertain 1-element set.

The proposition is proven.

**Definition 3.** By a basic 1-element set, we mean a normalized interval-valued fuzzy set  $A$  for which, for some  $x_0 \in U$ , we have:

- $A(x_0) = [a, 1]$  for some  $a > 0$ , and
- $A(x) = [0, 0]$  for all  $x \neq x_0$ .

**Definition 4.** By a basic 2-element set, we mean a normalized interval-valued fuzzy set  $A$  for which, for some  $x_0 \neq x_1$ , we have:

- $A(x_0) = [0, 1]$ ,
- $A(x_1) = [0, a]$  for some  $a \in (0, 1)$ , and
- $A(x) = [0, 0]$  for all other  $x$ .

**Proposition 6.** Let  $A$  be a normalized interval-valued fuzzy set which is not an uncertain 1-element set. Then, the following two conditions are equivalent to each other:

- the class  $\{B : B \subseteq A\}$  of all subsets of  $A$  is linearly ordered;
- $A$  is either a basic 1-element set or a basic 2-element set.

*Comment.* So, we can determine, based on the subsethood relation, whether  $A$  is a basic set.

**Proof.**

1°. Let us first prove that if  $A$  is a basic 1-element set or a basic 2-element set, then the class of all its subsets is linearly ordered.

1.1°. Let us first consider the case when  $A$  is a basic 1-element set.

In this case,  $B \subseteq A$  implies  $\underline{B}(x) = \overline{B}(x) = 0$  for all  $x \neq x_0$ . Since  $B$  is normalized, then, similarly to the proof of Proposition 5, we get  $\overline{B}(x_0) = 1$ . The final inequality  $\underline{B}(x_0) \leq \underline{A}(x_0) = a$  implies that for  $b \stackrel{\text{def}}{=} \underline{B}(x_0)$ , we have

$$b \leq a.$$

So, the set  $B$  has the following form:

- $B(x) = [0, 0]$  for all  $x \neq x_0$ , and
- $B(x_0) = [b, 1]$ , where we denoted  $b = \underline{B}(x_0)$ .

One can easily check that the class of such sets is linearly ordered: namely, if for two such sets  $B_1$  and  $B_2$ , we denote the corresponding values  $b$  by  $b_1$  and  $b_2$ , then:

- if  $b_1 \leq b_2$ , then  $B_1 \subseteq B_2$ , and
- vice versa, if  $b_2 \leq b_1$ , then  $B_2 \subseteq B_1$ .

1.2°. Let us consider the case when  $A$  is a basic 2-element set.

Let  $B \subseteq A$ . Then, from  $B(x) \leq A(x)$ , we conclude that  $B(x) = [0, 0]$  when  $x$  is different from  $x_0$  and  $x_1$ , and that  $\underline{B}(x_0) = \underline{B}(x_1) = 0$ .

The set  $B$  is normalized, so  $\overline{B}(x) = 1$  for some  $x$ .

- This  $x$  cannot be different from  $x_0$  and  $x_1$ , since for such  $x$ , we have

$$\overline{B}(x) = 0 < 1.$$

- It cannot be equal to  $x_1$ , since we have

$$\overline{B}(x_1) \leq \overline{A}(x_1) = a < 1.$$

Thus, the only possible element  $x$  is  $x = x_0$ , hence we have  $\overline{B}(x_0) = 1$ . The final inequality  $\overline{B}(x_1) \leq \overline{A}(x_1) = a$  implies that for  $b \stackrel{\text{def}}{=} \overline{B}(x_1)$ , we have  $b \leq a$ .

So, the set  $B$  has the following form:

- $B(x) = [0, 0]$  for all  $x$  which are different from  $x_0$  and  $x_1$ ;
- $B(x_0) = [0, 1]$ , and
- $B(x_1) = [0, b]$ , where  $b = \overline{B}(x_1)$ .

One can easily check that the class of such sets is linearly ordered: namely, if for two such sets  $B_1$  and  $B_2$ , we denote the corresponding values  $b$  by  $b_1$  and  $b_2$ , then:

- if  $b_1 \leq b_2$ , then  $B_1 \subseteq B_2$ , and
- vice versa, if  $b_2 \leq b_1$ , then  $B_2 \subseteq B_1$ .

2°. Let us now prove that if the class of all normalized subsets of a normalized fuzzy interval-valued set  $A$  is linearly ordered, then  $A$  is either a basic 1-element set or a basic 2-element set.

Since the set  $A$  is normalized, there exists an element  $x_0 \in U$  for which  $\overline{A}(x_0) = 1$ . Let us consider two possible cases:

- $\underline{A}(x_0) > 0$  and
- $\underline{A}(x_0) = 0$ .

2.1°. Let us first consider the case when  $\underline{A}(x_0) > 0$ . Let us prove that in this case, we have a basic 1-element set, i.e., that  $A(x) = [0, 0]$  for all  $x \neq x_0$ .

We will prove this by contradiction. Let us assume that  $\overline{A}(x) > 0$  for some  $x \neq x_0$ . Then, we can consider the following two subsets of  $A$ :

- $B_1(x_0) = A(x_0)$ ,  $B_2(x_0) = [0, 1]$ ;
- $B_2(x_1) = [0, 0]$ ,  $B_1(x_1) = A(x_1)$ , and
- $A(x) = B(x) = [0, 0]$  for all other  $x \in U$ .

One can easily check that  $B_1 \subseteq A$  and  $B_2 \subseteq A$ . However:

- we have  $\underline{B}_1(x_0) = \underline{A}(x_0) > 0 = \underline{B}_2(x_0)$ , hence we cannot have  $B_1 \subseteq B_2$ ;
- on the other hand,  $\overline{B}_2(x_1) = \overline{A}(x_1) > 0 = \overline{B}_1(x_1)$ , hence we cannot have  $B_2 \subseteq B_1$ .

The fact that here  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$  shows that  $\overline{A}(x) > 0$  is impossible. Thus,  $\overline{A}(x) = 0$  for all  $x \neq x_0$ , so  $A$  is indeed a basic 1-element set.

2.2°. Let us first consider the case when  $\underline{A}(x_0) = 0$ . Let us prove that in this case, we have a basic 2-element set, i.e., that:

- $A(x_1) = [0, a]$  for some  $x_1 \in U$  and some  $a \in (0, 1)$ , and
- $A(x) = [0, 0]$  for all other  $x$ .

Indeed, since  $A(x_0) = [0, 1]$ , but the set  $A$  is not an uncertain 1-element set, there exists some  $x_1 \neq x_0$  for which  $\overline{A}(x_1) > 0$ .

2.2.1°. Let us prove that in this case,  $A(x) = [0, 0]$  for all other  $x$ .

We prove this by contradiction. Let us assume that for some  $x_2$ , we have  $x_2 \neq x_0$ ,  $x_2 \neq x_1$  and  $\overline{A}(x_2) > 0$ . In this case, we can form the following two subsets  $B_1$  and  $B_2$ :

- $B_1(x_0) = B_2(x_0) = [0, 1]$ ;
- $B_1(x_1) = A(x_1)$ ,  $B_2(x_1) = [0, 0]$ ;
- $B_1(x_2) = [0, 0]$ ,  $B_2(x_2) = A(x_2)$ ; and
- $B_1(x) = B_2(x) = [0, 0]$  or all other  $x$ .

Clearly,  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but:

- $\overline{B}_1(x_1) > 0 = \overline{B}_2(x_1)$ , so we cannot have  $B_1 \subseteq B_2$ , and
- $\overline{B}_2(x_2) = \overline{A}(x_2) > 0 = \overline{B}_1(x_2)$ , so we cannot have

$$B_2 \subseteq B_1.$$

This contradicts to our assumption that the class of all subsets of  $A$  is linearly ordered. Thus,  $A(x) = [0, 0]$  for all element  $x$  which are different from  $x_0$  and  $x_1$ .

2.2.2°. Let us prove, by contradiction, that  $\underline{A}(x_1) = 0$ .

Indeed, if  $\underline{A}(x_1) > 0$ , then we can form the following sets  $B_1$  and  $B_2$ :

- $B_1(x_0) = B_2(x_0) = [0, 1]$ ;
- $B_1(x_1) = [0, \overline{A}(x_1)]$ ,  $B_2(x_1) = 0.5 \cdot \underline{A}(x_1)$ .
- $B_1(x) = B_2(x) = [0, 0]$  for all other  $x$ .

One can easily check that  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but:

- $\overline{B}_1(x_1) = \overline{A}(x_1) \geq \underline{A}(x_1) > 0.5 \cdot \underline{A}(x_1) = \overline{B}_2(x_1)$ , so we do not have  $B_1 \subseteq B_2$ ;
- on the other hand,  $\underline{B}_2(x_1) = 0.5 \cdot \underline{A}(x_1) > 0 = \underline{B}_1(x_1)$ , so we do not have  $B_2 \subseteq B_1$  either.

This contradicts to our assumption that the class of all subsets of  $A$  is linearly ordered. This contradiction shows that

$$\underline{A}_1(x_1) = 0.$$

2.2.3°. Finally, let us prove that  $\overline{A}(x_1) < 1$ .

Indeed, if  $\overline{A}(x_1) = 1$ , i.e., if  $A(x_1) = [0, 1]$ , then we can find the following two sets  $B_1 \subseteq A$  and  $B_2 \subseteq A$  for which  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ :

- $B_1(x_0) = [0, 1]$ ,  $B_2(x_0) = [0, 0]$ ;
- $B_1(x_1) = [0, 0]$ ,  $B_2(x_1) = A(x_1) = [0, 1]$ , and
- $B_1(x) = B_2(x) = [0, 0]$  for all other  $x$ .

Then:

- $\overline{B}_1(x_0) = 1 > \overline{B}_2(x_0)$ , so we cannot have  $B_1 \subseteq B_2$ ;
- $\overline{B}_2(x_1) = 1 > 0 = \overline{B}_1(x_1)$ , so we cannot have  $B_2 \subseteq B_1$ .

Contradiction show that we cannot have  $\overline{A}(x_1) = 1$ , thus

$$\overline{A}(x_1) < 1.$$

Thus, in this case,  $A$  is a basic 2-element set. The proposition is proven.

**Proposition 7.** If  $A$  is a basic 1-element set or a basic 2-element set, then the following two properties are equivalent to each other:

- $A$  is a crisp 1-element set;
- no proper superset of  $A$  is a basic 1-element set or a basic 2-element set.

*Comment.* So, we can determine crisp 1-element sets based only on the subethood relation.

**Proof.** If  $A = \{x_0\}$ , then clearly  $A$  cannot have any proper supersets which are basic 1-element or basic 2-element sets.

Vice versa, if  $A$  is a basic 1-element set with  $\underline{A}(x_0) < 1$ , then  $B = \{x_0\}$  is its proper superset which is a 1-element basic set.

Similarly, if  $A$  is a basic 2-element set, with  $A(x_0) = [0, 1]$ ,  $\underline{A}(x_1) = 0$ , and  $\overline{A}(x_1) < 1$ , then we can have the following proper superset  $B \supseteq A$  which is also a basic 2-element set:

- $B(x_0) = [0, 1]$ ;
- $B(x_1) = \left[0, \frac{1 + \overline{A}(x_1)}{2}\right]$ ; and
- $B(x) = 0$  for all other  $x$ .

The proposition is proven.

**Proposition 8.** For a normalized interval-valued fuzzy set, the following two conditions are satisfied:

- $A$  is either an uncertain 1-element set or a basic 1-element set;
- $A$  is a subset of a crisp 1-element set.

**Proof:** straightforward.

*Comment.* Since we know how to describe, based on the subethood relation,

- when  $A$  is an uncertain 1-element set, and
- when  $A$  is a basic set,

we can therefore determine:

- basic 1-element sets and
- basic 2-element sets

based on subethood relation only.

**Definition 5.** Let  $A$  be a basic 2-element set, with:

- $A(x_0) = [0, 1]$ ,
- $A(x_1) = [0, a]$  for some  $a \in (0, 1)$ , and
- $A(x) = [0, 0]$  for all other  $x$ .

Then, by its type-1 cover, we mean a normalized interval-valued fuzzy set  $A'$  for which:

- $A'(x_0) = [1, 1]$ ,
- $A'(x_1) = [a, a]$ , and
- $A'(x) = [0, 0]$  for all other  $x$ .

Let us show that the type-1 cover can be determined in terms of the subethood relation.

**Proposition 9.** Let  $A$  be a basic 2-element set. Then, its type-1 cover  $A'$  is the  $\subseteq$ -smallest normalized interval-valued fuzzy set that contains all the normalized interval-valued sets  $B \supseteq A$  for which the following four conditions are satisfied:

- the set  $B$  is not a basic 2-element set;
- the class of all basic 2-element subsets of  $B$  is linearly ordered;
- the class  $\{C : A \subseteq C \subseteq B\}$  of all normalized interval-valued set between  $A$  and  $B$  is linearly ordered; and
- the set  $B$  has only one uncertain 1-element subset.

**Proof.**

1.1°. Let us first prove that a set  $B$  satisfies the above four conditions if and only if it has one the following two forms:

- either it has the form  $B(x_0) = [b, 1]$  for some  $b > 0$ ,  $B(x_1) = A(x_1)$ , and  $B(x) = [0, 0]$  for all other  $x$ ; we will call these  $B$  of the first form;
- or it has the form  $B(x_0) = A(x_0)$ ,  $B(x_1) = [b, a]$  for some  $b > 0$ , and  $B(x) = [0, 0]$  for all other  $x$ ; we will call these  $B$  of the second form.

1.1°. Let us first prove that the all the sets  $B$  of the first form satisfy all the above four conditions.

1.1.1°. Indeed, clearly, such  $B$  is not a basic 2-element set.

1.1.2°. If  $C$  is a basic 2-element set for which  $C \subseteq B$ , then we have:

- $C(x_0) = [0, 1]$ ,
- $C(x) = [0, 0]$  for all  $x$  different from  $x_0$  and  $x_1$ , and
- $C(x_1) = [0, c]$  for some  $c \leq a$ .

Clearly, the set of all such  $C$  is linearly ordered: if we have two such sets, corresponding to elements  $c_1$  and  $c_2$ , then:

- if  $c_1 \leq c_2$ , then we have  $C_1 \subseteq C_2$ , and
- if  $c_2 \leq c_1$ , then we have  $C_2 \subseteq C_1$ .

1.1.3°. If  $A \subseteq C \subseteq B$ , then we have:

- $C(x_0) = [c, 1]$  for some  $c \in [b, 1]$ ,
- $C(x_1) = A(x_1)$ , and
- $C(x) = [0, 0]$  for all other  $x$ .

Thus, if we have two such sets, corresponding to elements  $c_1$  and  $c_2$ , then:

- if  $c_1 \leq c_2$ , then we have  $C_1 \subseteq C_2$ , and
- if  $c_2 \leq c_1$ , then we have  $C_2 \subseteq C_1$ .

1.1.4°. Of course, the only uncertain 1-element set contained in  $B$  is the set corresponding to  $x_0$ .

All four conditions are proven.

1.2°. Let us now prove that the all the sets  $B$  of the second form satisfy all the above four conditions.

1.2.1°. Indeed, clearly, such  $B$  is not a basic 2-element set.

1.2.2°. If  $C \subseteq B$  is a basic 2-element set, then we have:

- $C(x_0) = [0, 1]$ ,
- $C(x) = [0, 0]$  for all  $x$  different from  $x_0$  and  $x_1$ , and
- $C(x_1) = [0, c]$  for some  $c \leq a$ .

Clearly, the set of all such  $C$  is linearly ordered: if we have two such sets, corresponding to elements  $c_1$  and  $c_2$ , then:

- if  $c_1 \leq c_2$ , then we have  $C_1 \subseteq C_2$ , and

- if  $c_2 \leq c_1$ , then we have  $C_2 \subseteq C_1$ .

1.2.3°. If  $A \subseteq C \subseteq B$ , then we have:

- $C(x_0) = A(x_0)$ ,
- $C(x_1) = [c, a]$  for some  $c \in [b, a]$ , and
- $C(x) = [0, 0]$  for all other  $x$ .

Thus, if we have two such sets, corresponding to elements  $c_1$  and  $c_2$ , then:

- if  $c_1 \leq c_2$ , then we have  $C_1 \subseteq C_2$ , and
- if  $c_2 \leq c_1$ , then we have  $C_2 \subseteq C_1$ .

1.2.4°. Of course, the only uncertain 1-element set contained in  $B$  is the set corresponding to  $x_0$ .

All four conditions are proven.

1.3°. Let us now prove that if a set  $B$  satisfies the above four conditions, then  $B$  is either of the first form or of the second form.

1.3.1°. Let us first prove that we must have  $B(x) = [0, 0]$  for all elements  $x$  which are different from  $x_0$  and  $x_1$ .

We will prove this by contradiction. Assume that  $\overline{B}(x_2) > 0$  for some element  $x_2$  which is different from  $x_0$  and  $x_1$ . Then, in addition to a basic 2-element set  $A \subseteq B$ , we also have another basic 2-element set  $C \subseteq B$  for which:

- $C(x_0) = [0, 1]$ ,
- $C(x_2) = [0, \overline{B}(x_2)]$ , and
- $C(x) = [0, 0]$  for all other elements  $x$ .

Then:

- $\overline{A}(x_1) = a > 0 = \overline{C}(x_1)$ , so we cannot have  $A \subseteq C$ ;  
and
- $\overline{C}(x_2) > 0 = \overline{A}(x_2)$ , so we cannot have  $C \subseteq A$  either.

This contradicts to the condition that set of all basic 2-element sets which are subsets of  $B$  is linearly ordered.

Thus,  $\overline{B}(x) > 0$  is impossible. So, indeed,  $B(x) = [0, 0]$  for all elements  $x$  which are different from  $x_0$  and  $x_1$ .

1.3.2°. Due to Part 1.3.1 of this proof, the set  $B$  is uniquely described by its values  $B(x_0)$  and  $B(x_1)$ . The condition that  $A \subseteq B$  implies that  $\overline{A}(x_0) = 1$  and that:

- $\overline{B}(x_0) \geq 0$ ,
- $\underline{B}(x_1) \geq 0$ , and
- that  $\overline{B}(x_1) \geq a = \overline{A}(x_1)$ .

Since  $B$  is not a basic 2-element set and  $A$  is such a set, we have  $B \neq A$ . Thus, at least one of the above inequalities must be strict. Let us consider these three inequalities one by one.

1.3.3°. Let us first consider the case when  $\overline{B}(x_0) > 0$ . Let us prove that in this case, we have  $B(x_1) = A(x_1)$ , i.e., that we have a set of the first form.

We will first prove, by contradiction, that  $\underline{B}(x_1) = 0$ . Indeed, if  $\underline{B}(x_1) > 0$ , then we can form the following two sets  $C_1$  and  $C_2$  for which  $A \subseteq C_1 \subseteq B$ ,  $A \subseteq C_2 \subseteq B$ , but  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ :

- $C_1(x_0) = A(x_0) = [0, 1]$ ,  $C_1(x_1) = B(x_1)$ , and  $C_1(x) = [0, 0]$  for all other  $x$ ;

- $C_2(x_0) = B(x_0)$ ,  $C_2(x_1) = A(x_1)$ , and  $C_2(x) = [0, 0]$  for all other  $x$ .

Here:

- $\underline{C}_1(x_1) = \underline{B}(x_1) > 0 = \underline{C}_2(x_1)$ , so we cannot have

$$C_1 \subseteq C_2;$$

- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$ , so we cannot have

$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets  $C$  is linearly ordered. Thus, we must have  $\underline{B}(x_1) = 0$ .

Let us now prove, by contradiction, that  $\overline{B}(x_1) = \overline{A}(x_1)$ . Indeed, suppose that  $\overline{B}(x_1) > \overline{A}(x_1)$ . Then we can form the following two sets  $C_1$  and  $C_2$  for which  $A \subseteq C_1 \subseteq B$ ,  $A \subseteq C_2 \subseteq B$ , but  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ :

- $C_1(x_0) = A(x_0) = [0, 1]$ ,  $C_1(x_1) = B(x_1)$ , and  $C_1(x) = [0, 0]$  for all other  $x$ ;
- $C_2(x_0) = B(x_0)$ ,  $C_2(x_1) = A(x_1)$ , and  $C_2(x) = [0, 0]$  for all other  $x$ .

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$ , so we cannot have

$$C_1 \subseteq C_2;$$

- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$ , so we cannot have

$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets  $C$  is linearly ordered. Thus, we must have

$$\overline{B}(x_1) = \overline{A}(x_1).$$

So, in this case, we indeed have a set of the first form.

1.3.4°. Let us now consider the case when  $\underline{B}(x_1) > 0$ . Let us prove that in this case, we have  $\overline{B}(x_0) = 0$  and  $\overline{B}(x_1) = \overline{A}(x_1)$ , i.e., that we have a set of the second form.

We will first prove, by contradiction, that  $\underline{B}(x_0) = 0$ . Indeed, if  $\underline{B}(x_0) > 0$ , then we can form the following two sets  $C_1$  and  $C_2$  for which  $A \subseteq C_1 \subseteq B$ ,  $A \subseteq C_2 \subseteq B$ , but  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ :

- $C_1(x_0) = A(x_0) = [0, 1]$ ,  $C_1(x_1) = B(x_1)$ , and  $C_1(x) = [0, 0]$  for all other  $x$ ;
- $C_2(x_0) = B(x_0)$ ,  $C_2(x_1) = A(x_1)$ , and  $C_2(x) = [0, 0]$  for all other  $x$ .

Here:

- $\underline{C}_1(x_1) = \underline{B}(x_1) > 0 = \underline{C}_2(x_1)$ , so we cannot have

$$C_1 \subseteq C_2;$$

- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$ , so we cannot have

$$C_2 \subseteq C_1.$$



This contradicts to our assumption that the class of all intermediate fuzzy sets  $C$  is linearly ordered. Thus, we must have

$$\underline{B}(x_0) = 0.$$

Let us now prove, by contradiction, that  $\overline{B}(x_1) = \overline{A}(x_1)$ . Indeed, suppose that  $\overline{B}(x_1) > \overline{A}(x_1)$ . Then we can form the following two sets  $C_1$  and  $C_2$  for which  $A \subseteq C_1 \subseteq B$ ,  $A \subseteq C_2 \subseteq B$ , but  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ :

- $C_1(x_0) = [0, 1]$ ,  $C_1(x_1) = B(x_1)$ , and  $C_1(x) = [0, 0]$  for all other  $x$ ;
- $C_2(x_0) = B(x_0)$ ,  $C_2(x_1) = A(x_1)$ , and  $C_2(x) = [0, 0]$  for all other  $x$ .

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$ , so we cannot have

$$C_1 \subseteq C_2;$$

- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$ , so we cannot have

$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets  $C$  is linearly ordered. Thus, we must have

$$\overline{B}(x_1) = \overline{A}(x_1).$$

So, in this case, we indeed have a set of the second form.

1.3.5°. Finally, let us prove that the case when  $\overline{B}(x_1) > \overline{A}(x_1)$  is not possible.

We will first prove, by contradiction, that in this case,  $\underline{B}(x_0) = 0$ . Indeed, if  $\underline{B}(x_0) > 0$ , then we can form the following two sets  $C_1$  and  $C_2$  for which  $A \subseteq C_1 \subseteq B$ ,  $A \subseteq C_2 \subseteq B$ , but  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ :

- $C_1(x_0) = A(x_0) = [0, 1]$ ,  $C_1(x_1) = B(x_1)$ , and  $C_1(x) = [0, 0]$  for all other  $x$ ;
- $C_2(x_0) = B(x_0)$ ,  $C_2(x_1) = A(x_1)$ , and  $C_2(x) = [0, 0]$  for all other  $x$ .

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$ , so we cannot have

$$C_1 \subseteq C_2;$$

- $\underline{C}_2(x_0) = \underline{B}(x_0) > 0 = \underline{C}_1(x_0)$ , so we cannot have

$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets  $C$  is linearly ordered. Thus, we must have

$$\underline{B}(x_0) = 0.$$

Let us now prove, by contradiction, that  $\underline{B}(x_1) = 0$ . Indeed, suppose that  $\underline{B}(x_1) > 0$ . Then we can form the following two

sets  $C_1$  and  $C_2$  for which  $A \subseteq C_1 \subseteq B$ ,  $A \subseteq C_2 \subseteq B$ , but  $C_1 \not\subseteq C_2$  and  $C_2 \not\subseteq C_1$ :

- $C_1(x_0) = A(x_0) = [0, 1]$ ,  $C_1(x_1) = [0, \overline{B}(x_1)]$ , and  $C_1(x) = [0, 0]$  for all other  $x$ ;
- $C_2(x_0) = A(x_0) = [0, 1]$ ,  $C_2(x_1) = [\underline{B}(x_1), \overline{A}(x_1)]$ , and  $C_2(x) = [0, 0]$  for all other  $x$ .

Here:

- $\overline{C}_1(x_1) = \overline{B}(x_1) > \overline{A}(x_1) = \overline{C}_2(x_1)$ , so we cannot have

$$C_1 \subseteq C_2;$$

- $\underline{C}_2(x_1) = \underline{B}(x_1) > 0 = \underline{C}_1(x_1)$ , so we cannot have

$$C_2 \subseteq C_1.$$

This contradicts to our assumption that the class of all intermediate fuzzy sets  $C$  is linearly ordered. Thus, we must have

$$\underline{B}(x_1) = 0.$$

Finally,  $\overline{B}(x_1) < 1$ , since otherwise  $B$  would have two uncertain 1-element subsets:

- a subset corresponding to  $x_0$ , and
- a subset corresponding to  $x_1$ ,

Then, since we know that  $\overline{B}(x_0) = 1$  and we have proved that  $\underline{B}(x_0) = \underline{B}(x_1) = 0$  and  $\overline{B}(x_1) < 1$ , we conclude that the set  $B$  is a basic 2-element set – and we explicitly assumed that it is not.

Thus, the third inequality cannot be strict, so  $B$  is indeed either of the first form, or of the second form. Once can check that the smallest set containing all such sets is indeed the set  $A'$ .

The proposition is proven.

**Definition 6.** Let  $A$  be an uncertain 1-element set, with:

- $A(x_0) = [0, 1]$ , and
- $A(x) = [0, 0]$  for all other  $x$ .

Then, by its type-1 cover, we mean a crisp set  $A' = \{x_0\}$ .

**Proposition 10.** A normalized interval-valued fuzzy set is a type-1 set if and only if the following two conditions are satisfied:

- if  $B \subseteq A$  for some uncertain 1-element set, then  $B' \subset A$ , and
- if  $B \subseteq A$  for some basic 2-element set, then  $B' \subseteq A$ .

*Comment.* Since we have shown that:

- the operation  $B'$ ,
- uncertain 1-element sets, and
- basic 2-element sets

can all be described in terms of the subsethood relation, we can thus conclude that we can detect type-1 sets based on the subsethood relation between normalized interval-valued fuzzy sets.

## Proof.

1°. One can see that the type-1 cover of a set  $A(x) = [\underline{A}(x), \overline{A}(x)]$  has the form  $A'(x) = [\overline{A}(x), \overline{A}(x)]$ .

For a type-1 set,  $\underline{A}(x) = \overline{A}(x)$ , thus  $A' = A$ , and clearly,  $A \subseteq B$  implies  $A' \subseteq B$ .

2°. Vice versa, let us prove that if the above two conditions are satisfied, then  $A$  is a type-1 set, i.e., that  $\underline{A}(x) = \overline{A}(x)$  for all  $x$ .

To prove this, let us consider two possible cases:

- elements  $x$  for which  $\overline{A}(x) = 1$ , and
- elements  $x$  for which  $\overline{A}(x) < 1$ .

2.1°. Let us first consider an element  $x$  for which  $\overline{A}(x) = 1$ . In this case,  $B \subseteq A$  for the uncertain 1-element set  $B$  for which  $B(x) = [0, 1]$  and  $B(y) = [0, 0]$  for all  $y \neq x$ . Then,  $B' = \{x\}$ , i.e.,  $B'(x) = [1, 1]$ . Thus, from  $B' \subseteq A$  it follows that  $1 = \underline{B}'(x) \leq \underline{A}(x)$ , so  $\underline{A}(x) = 1 = \overline{A}(x)$ . So, for such elements  $x$ , we indeed have  $\underline{A}(x) = \overline{A}(x)$ .

2.2°. Finally, let us consider an element  $x$  for which  $\overline{A}(x) < 1$ . Since  $A$  is normalized, there exists an element  $x_0$  for which  $\overline{A}(x_0) = 1$ . Now, we can form the following basic 2-element set  $B$ :

- $B(x_0) = [0, 1]$ ,
- $B(x) = [0, \overline{A}(x)]$ , and
- $B(y) = [0, 0]$  for all other elements  $y$ .

Clearly,  $B \subseteq A$ , hence  $B' \subseteq A$ . Here,  $B'(x) = [\underline{B}'(x), \overline{B}'(x)] = [\overline{A}(x), \overline{A}(x)]$ . So,  $B' \subseteq A$  implies  $\underline{B}'(x) = \overline{A}(x) \leq \underline{A}(x)$ , thus  $\underline{A}(x) = \overline{A}(x)$ .

The proposition is proven.

## ACKNOWLEDGMENTS

This work was supported in part by the US National Science Foundation grant HRD-1242122.

The authors are thankful to all the participants of the 2017 Annual Conference of the North American Fuzzy Information Processing Society NAFIPS'2017 (Cancun, Mexico, October 16–18, 2017) for valuable suggestions.

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