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Thach N. Nguyen
Banking University of Ho Chi Minh City, ajeb@buh.edu.vn

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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Maximum Entropy Approach to Interbank Lending: Towards a More Accurate Algorithm[1]

Thach N. Nguyen†, Olga Kosheleva‡, and Vladik Kreinovich‡,2

†Banking University of Ho Chi Minh City
56 Hoang Dieu 2, Quan Thu Duc, Thu Duc
Ho Ch Minh City, Vietnam
e-mail: `Thachnn@buh.edu.vn`
‡University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
e-mails: `olgak@utep.edu, vladik@utep.edu`

Abstract : Banks loan money to each and borrow money from each other. To minimizing the risk caused by a possible default of one of the banks, a reasonable idea is to evenly spread the lending between different banks. A natural way to formalize this evenness requirement is to select the interbank amounts for which the entropy is the largest possible. The existing algorithms for solving the resulting constrained optimization problem provides only an approximate solution. In this paper, we propose a new algorithm that provides the exact solution to the maximum-entropy interbank lending problem.

Keywords : Interbank Lening; Maximum Entropy

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[2]`Corresponding author email: vladik@utep.edu (Vladik Kreinovich)`

1 Formulation of the Problem

Interbank lending: formulation of the problem. Banks lend money to each other and borrow money from each other. To minimize the risk caused by a possible default of one of the banks, a reasonable idea is to spread the lending equally between as many banks as possible.

A natural way to describe such an even spread in precise terms is to make sure that the entropy of the resulting distribution is the largest possible; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9].

Interbank lending: formulation of the problem in precise terms.

- Let a_i be the total amount of assets that the i-th bank is willing to lend.
- et ℓ_j be the amount of liabilities to cover which the j-th bank needs to borrow the money from other banks.

We assume that these amounts match, in the sense that we have a sufficient amount of available money to cover all the banks' liabilities:

$$S \stackrel{\text{def}}{=} \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} \ell_j, \tag{1}$$

where n is the total number of banks.

For simplicity, we can divide all the amounts a_i and ℓ_j by this common sum S, i.e., consider fractions of the overall lending amounts instead of the actual dollar values. After this division, the formula (1) takes a simplified form

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} \ell_j = 1. \tag{1a}$$

We want to describe, for every pairs (i, j) of the banks $(i \neq j)$, the amount x_{ij} that the i-th bank lends to the j-th one. We know the overall amount that each bank lends a_i, and the overall amount ℓ_j that each bank receives, thus:

$$\sum_{j=1}^{n} x_{ij} = a_i; \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} = \ell_j. \tag{3}$$

There are many different combinations of the values x_{ij} that satisfy the constraints (2) and (3). Among these combinations, we would like to select a combination that maximizes the entropy

$$S = -\sum_{i,j} x_{ij} \cdot \ln(x_{ij}). \tag{4}$$

How this problem is solved now. At present, instead of directly solving the above optimization problem, researchers and practitioners use the following two-step approximate solution.

- First, they solve an auxiliary problem which is similar to the above but in which we are also allowing the values $x_{ii} \neq 0$. This auxiliary problem occurs in probability theory when:
  - we know the marginal distributions a_i and ℓ_j, and
  - we want to use the maximum entropy principle to select a joint distribution which is consistent with the given marginals.

  A solution to this auxiliary problem is well known: it corresponds to the assumption that the corresponding random variables are independent:

$$x_{ij}^{(0)} = a_i \cdot \ell_j. \tag{5}$$

- Then, we adjust these values to make sure that for the resulting values x_{ij}, we have $x_{ii} = 0$. This is usually done by selecting the values x_{ij} that maximize the conditional entropy

$$\sum_{i,j} x_{ij} \cdot \ln\left(\frac{x_{ij}}{x_{ij}^{(0)}}\right). \tag{6}$$

Limitations of the existing two-stage approach. While the above two-stage approach leads to a reasonable solution, it is only an *approximate* solution to the original maximum entropy problem. It is therefore desirable to come up with the *exact* (or at least more accurate) solution to this problem.

What we do in this paper. In this paper, we provide such a more accurate algorithm for the maximum-entropy interbank lending problem.

2 Analysis of the Problem and the Resulting Algorithm

Analysis of the problem. We want to maximize the entropy (4) under the constraints (2) and (3). By applying the Lagrange multiplier method, we can reduce this constrained optimization problem to the unconstrained problem of maximizing the following expression:

$$-\sum_{ij} x_{ij} \cdot \ln(x_{ij}) + \sum_{i=1}^{n} \lambda_i \cdot \left(\sum_{j=1}^{n} x_{ij} - a_i\right) + \sum_{j=1}^{n} \mu_j \cdot \left(\sum_{i=1}^{n} x_{ij} - \ell_j\right), \tag{7}$$

for appropriate Lagrange multipliers λ_i and μ_j.

Differentiating the expression (7) with respect to x_{ij} and equating the derivative to 0, we conclude that

$$-\ln(x_{ij}) - 1 + \lambda_i + \mu_j = 0.$$

hence

$$\ln(x_{ij}) = (\lambda_i - 1) + \mu_j.$$

By applying $\exp(x)$ to both sides, we conclude that

$$x_{ij} = b'_i \cdot c'_j, \tag{8}$$

where we denoted $b'_i \stackrel{\text{def}}{=} \exp(\lambda_i - 1)$ and $c'_j \stackrel{\text{def}}{=} \exp(\mu_j)$.

We can somewhat simplify this expression if we normalize the values b'_i by dividing them by the sum

$$b' = \sum_{j=1}^{n} b'_j$$

of these values, i.e., by considering the new values

$$b_i \stackrel{\text{def}}{=} \frac{b'_i}{b'}$$

for which

$$\sum_{i=1}^{n} b_i = 1. \tag{9}$$

Then, $b'_i = b' \cdot b_i$ and thus, the product $b'_i \cdot c'_j$ can be equivalently described as $b_i \cdot b' \cdot c'_j$, i.e., as

$$x_{ij} = b_i \cdot c_j \tag{10}$$

for $i \neq j$, where we denoted $c_j \stackrel{\text{def}}{=} b' \cdot c'_j$.

For these values (10), the condition (4) takes the form

$$\ell_j = \sum_{i=1}^{n} x_{ij} = \sum_{i \neq j} b_i \cdot c_j = c_j \cdot \sum_{i \neq j} b_i. \tag{11}$$

Due to condition (1), the sum in the right-hand side of the formula (11) takes the form $1 - b_j$, thus the condition (4) takes the form

$$c_j \cdot (1 - b_j) = \ell_j. \tag{12}$$

Similarly, the condition (3) takes the form

$$a_i = \sum_{j=1}^{n} x_{ij} = \sum_{j \neq i} b_i \cdot c_j = b_i \cdot \sum_{j \neq i} c_j. \tag{13}$$

Let us denote

$$C \stackrel{\text{def}}{=} \sum_{j=1}^{n} c_j. \tag{14}$$

In this case, the sum in the right-hand side of the formula (13) takes the form $C - c_i$, hence the condition (13) takes the form

$$b_i \cdot (C - c_i) = a_i. \tag{15}$$

From (12), we can express c_i as

$$c_i = \frac{\ell_i}{1 - b_i}. \tag{16}$$

substituting this expression into the formula (15), we conclude that

$$b_i \cdot \left(C - \frac{\ell_i}{1 - b_i}\right) = a_i. \tag{16}$$

Multiplying both sides of this equality by $1 - b_i$, we conclude that

$$b_i \cdot C \cdot (1 - b_i) - \ell_i \cdot b_i = a_i \cdot (1 - b_i). \tag{17}$$

Opening parentheses and moving all the term to the right-hand side, we get the following quadratic equation for determining b_i:

$$b_i^2 - b_i \cdot (C + a_i - \ell_i) + a_i = 0, \tag{18}$$

hence

$$b_i = \frac{C + a_i - \ell_i}{2} \pm \sqrt{\left(\frac{C + a_i - \ell_i}{2}\right)^2 - a_i}. \tag{19}$$

For the large number of equal-size banks, the effect of n terms x_{ii} in the approximate solution is much smaller than the effect of $n^2 - n$ other terns, so we should have $C \approx 1$ and $b_i \approx a_i$. In the formula (19), only the solution corresponding to minus has this asymptotics, so we should have minus:

$$b_i = \frac{C + a_i - \ell_i}{2} - \sqrt{\left(\frac{C + a_i - \ell_i}{2}\right)^2 - a_i}. \tag{20}$$

Thus, once we know the value C:

- we can find all the values b_i by busing the formula (20),
- then we can find the values c_j by using the formula (16), and
- hence, we can compute the values $x_{ij} = b_i \cdot c_j$.

We need to find the value C for which, for the values b_i from the formula (20), we have $\sum\limits_{i=1}^n b_i = 1$. Since $\sum a_i = \sum b_j$, the sum of the first (pre-square root) terms in the formula (20) is equal to $\dfrac{C \cdot n}{2}$, so the desired equality takes the form

$$\frac{C \cdot n}{2} - \sum_{i=1}^n \sqrt{\left(\frac{C + a_i - \ell_i}{2}\right)^2 - a_i} = 1. \tag{21}$$

Thus, we arrive at the following algorithm.

Resulting algorithm. First, we find the unknown value C by solving the equation

$$\frac{C \cdot n}{2} - \sum_{i=1}^n \sqrt{\left(\frac{C + a_i - \ell_i}{2}\right)^2 - a_i} = 1. \tag{21}$$

with a single unknown C.

Once we know C, we compute the auxiliary quantities

$$b_i = \frac{C + a_i - \ell_i}{2} - \sqrt{\left(\frac{C + a_i - \ell_i}{2}\right)^2 - a_i} \tag{20}$$

and

$$c_i = \frac{\ell_i}{1 - b_i}, \tag{16}$$

and compute the value $x_{ij} = b_i \cdot c_j$ for all pairs $i \neq j$.

These values are the exact maximum entropy solution to the interbank lending problem.

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