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Does the Universe Really Expand Faster than the Speed of Light: Kinematic Analysis Based on Special Relativity and Copernican Principle

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Abstract

In the first approximation, the Universe's expansion is described by the Hubble's law $v = H \cdot R$, according to which the relative speed v of two objects in the expanding Universe grows linearly with the distance R between them. This law can be derived from the Copernican principle, according to which, cosmology-wise, there is no special location in the Universe, and thus, the expanding Universe should look the same from every starting point. The problem with the Hubble's formula is that for large distance, it leads to non-physical larger-than-speed-of-light velocities. Since the Universe's expansion is a consequence of Einstein's General Relativity Theory (GRT), this problem is usually handled by taking into account GRT's curved character of space-time. In this paper, we consider this problem from a purely kinematic viewpoint. We show that if we take into account special-relativistic effects when applying the Copernican principle, we get a modified version of the Hubble's law in which all the velocities are physically meaningful – in the sense that they never exceed the speed of light.

1 Introduction

Universe's expansion and Hubble's law: reminder. Since the 1920s, it is known that distant galaxies are moving away, with a speed v which is proportional to the distance R: $v = H \cdot R$. This empirical formula is known as the *Hubble's law*.

The empirical discovery of the Universe's expansion turned out to be in perfect accordance with Einstein's General Relativity theory, according to which the Universe cannot be stationary: it either expands or retracts. Moreover, the

expansion predicted by General Relativity is in very good accordance with the Hubble's law; see, e.g., [1].

Hubble's law follows from the Copernican principle. Later, it turned out that the Hubble's law can be derived from the so-called *Copernican principle*, according to which, from the cosmological viewpoint, there is no special location in the Universe, and thus, the expanding Universe should look the same from every starting point. This principle is named after Copernicus, who argued that, contrary to the then-prevalent opinion, there is nothing special about the location of Earth in space – and moreover, if we do not try to place Earth at the center of the Universe, our description of celestial mechanics becomes much clearer and simpler; see, e.g., [1].

The problem with the Hubble's law. From the physical viewpoint, the Hubble's law has a problem: for large distances R, the corresponding velocity v exceeds the speed of light c. This runs contrary to one of the main principles of special relativity, according to which physical velocities cannot exceed c (see, e.g., [1]).

How this problem is solved now. Since the Universe's expansion is a consequence of Einstein's General Relativity Theory (GRT), this problem is usually handled by taking into account GRT's curved character of space-time [1].

What we do in this paper. In this paper, we consider this problem from a purely kinematic viewpoint.

We show that if we take into account special-relativistic effects when applying the Copernican principle, we get a modified version of the Hubble's law in which all the velocities are physically meaningful – in the sense that they never exceed the speed of light.

The structure of the paper. We start, in Section 2, by reminding the readers how, in the non-relativistic case, the Copernican principle leads to the Hubble's law. Then, in Section 3, we show that a special-relativistic modification of this derivation leads to a physically meaningful special-relativistic modification of the Hubble's law.

2 How the Hubble's Law Is Derived from the Copernican Principle: A Brief Reminder

What we want to analyze. We want to find out how the relative velocity v of two galaxies depends on the distance R between them.

We can safely assume that the dependence $v(R)$ is continuous – even differentiable.

Copernican principle: reminder. With respect to the Universe's expansion, the Copernican principle states that the expansion should look the same from every starting point.

Consequences of this principle. The Copernican principle states that, for any real number $R > 0$, if we take a object A at a distance R from the Earth, then, from the viewpoint of this object, the Universe's expansion looks the same as from the Earth. In other words, an object B who is at a distance r from the object A along the line Earth A (and which is thus at the distance $R + r$ from the Earth) moves with velocity $v(r)$ relative to the object A.

Relative to the Earth, the object A moves with the velocity $v(R)$. When B moves with velocity $v(r)$ relative to the object A, and the object A moves relative to the Earth with the velocity $v(R)$, we can conclude, in the non-relativistic case, that B moves with the velocity

$$v(R) + v(r)$$

relative to the Earth.

On the other hand, since the object B is located at the distance $R+r$ from the Earth, it moves with the velocity

$$v(R+r)$$

relative to the Earth. By comparing the above two expressions for the B-relative-to-Earth velocity, we conclude that

$$v(R+r) = v(R) + v(r) \tag{1}$$

for all $R > 0$ and $r > 0$.

This formula implies the Hubble's law. Indeed, by applying the formula (1) multiple times, we conclude that

$$v(r_1 + \ldots + r_n) = v(r_1) + \ldots + v(r_n)$$

for all possible values $r_1, \ldots, r_n > 0$. In particular, for every natural number n, for $r_1 = \ldots = r_n = \dfrac{1}{n}$, we have $r_1 + \ldots + r_n = 1$ and thus,

$$v(1) = \underbrace{v\left(\frac{1}{n}\right) + ... + v\left(\frac{1}{n}\right)}_{n \text{ times}}.$$

Thus, $v(1) = n \cdot v\left(\dfrac{1}{n}\right)$, hence $v\left(\dfrac{1}{n}\right) = \dfrac{1}{n} \cdot v(1)$.

Similarly, for any natural number m, for $r_1 = \ldots = r_m = \dfrac{1}{n}$, we get

$$v\left(\frac{m}{n}\right) = \underbrace{v\left(\frac{1}{n}\right) + ... + v\left(\frac{1}{n}\right)}_{m \text{ times}},$$

thus

$$v\left(\frac{m}{n}\right) = m \cdot v\left(\frac{1}{n}\right) = \frac{m}{n} \cdot v(1).$$

So, for rational numbers $R = \frac{m}{n}$, we have $v(R) = H \cdot R$, where we denoted $H \stackrel{\text{def}}{=} v(1)$.

Since we assumed that the dependence $v(R)$ is continuous, and every real number can be approximated, with arbitrary accuracy, by rational numbers, we conclude that $v(R) = H \cdot R$ for all real values $R > 0$. This is exactly the Hubble's law.

3 What If We Take Special Relativity into Account

Let us recall the above situation. Let us consider the same situation: we have the Earth, we have an object A at distance R from the Earth, and we have an object B at the distance $R + r$ from the Earth along the same line as the object A. Relative to the Earth:

- the object A moves with velocity $v(R)$, and
- the object B moves with the velocity $v(R + r)$.

The expansion should look the same from the viewpoint of the object A as it looks from the viewpoint of the Earth.

Let us take relativistic effects into account. In the non-relativistic case, from the viewpoint of the object A, the object B was at the distance r. However, in the relativistic case, since the object A is moving with velocity $v(R)$ relative to Earth, the distance AB shrinks to $\widetilde{r} = r \cdot \sqrt{1 - \left(\frac{v(R)}{c}\right)^2}$; see, e.g., [1]. Therefore, from the viewpoint of the object A, B moves with velocity $v(\widetilde{r})$ relative to A.

We need to combine the A-relative-to-Earth and B-relative-to-A velocities into the B-relative-to-Earth velocity. In the non-relativistic case, we simply added the given velocities. In the relativistic case, we need to use the special-relativity formula for such a combination: $v = \dfrac{v_1 + v_2}{1 + \dfrac{v_1 \cdot v_2}{c^2}}$; see, e.g., [1]. In particular, for $v_1 = v(R)$ and $v_2 = v(\widetilde{r})$, we conclude that

$$v(R + r) = \frac{v(R) + v(\widetilde{r})}{1 + \dfrac{v(R) \cdot v(\widetilde{r})}{c^2}} = \frac{v(R) + v\left(r \cdot \sqrt{1 - \left(\dfrac{v(R)}{c}\right)^2}\right)}{1 + \dfrac{v(R) \cdot v\left(r \cdot \sqrt{1 - \left(\dfrac{v(R)}{c}\right)^2}\right)}{c^2}}.$$

This formula can be simplified if we consider an auxiliary function $u(R) \stackrel{\rm def}{=} \dfrac{v(R)}{c}$ instead of the desired function $v(R)$. For this auxiliary function, the above formula takes the following simplified form:

$$u(R+r) = \frac{u(R) + u\left(r\sqrt{1-(u(R))^2}\right)}{1 + u(R)\cdot u\left(r\sqrt{1-(u(R))^2}\right)}. \tag{2}$$

What can we derive from this equation? Since we assumed that the dependence $v(R)$ is differentiable, we can differentiate both sides of the equality (2) by r and take $r = 0$.

In the left-hand side, we get the derivative $u'(R)$. In the right-hand side, we can use the usual formula for the derivative of the ratio:

$$(f/g)'(r) = \frac{f'(r)\cdot g(r) - f(r)\cdot g'(r)}{(g(r))^2},$$

thus

$$(f/g)'(0) = \frac{f'(0)\cdot g(0) - f(0)\cdot g'(0)}{(g(0))^2},$$

For $f(r) = u(R) + u\left(r\sqrt{1-(u(R))^2}\right)$, we have $f(0) = v(R)$ and

$$f'(r) = u'\left(r\sqrt{1-(u(R))^2}\right)\cdot\sqrt{1-u(R)^2}.$$

So, for $r = 0$, we have

$$f'(0) = u'(0)\cdot\sqrt{1-(u(R))^2}.$$

Similarly, for $g(r) = 1 + u(R)\cdot u\left(r\sqrt{1-(u(R))^2}\right)$, we have $g(0) = 1$ and

$$g'(r) = u(R)\cdot u'\left(r\sqrt{1-(u(R))^2}\right)\cdot\sqrt{1-(u(R))^2}.$$

So, for $r = 0$, we have

$$g'(0) = u(R)\cdot u'(0)\cdot\sqrt{1-(u(R))^2}.$$

Let us denote $u'(0)$ by h. Then, by equating the derivatives of both sides of the formula (2), we conclude that

$$u'(R) = \frac{\left[h\cdot\sqrt{1-(u(R))^2}\right]\cdot 1 - u(R)\cdot\left[u(R)\cdot h\cdot\sqrt{1-(u(R))^2}\right]}{1^2} =$$

$$\left[h\cdot\sqrt{1-(u(R))^2}\right] - \left[(u(R))^2\cdot h\cdot\sqrt{1-(u(R))^2}\right],$$

hence

$$\frac{du}{dR} = u'(R) = h \cdot \sqrt{1 - (u(R))^2} \cdot \left(1 - (u(R))^2\right).$$

By moving all the terms related to u to the left-hand side and all the terms related to R to the right-hand side, we get

$$\frac{du}{\sqrt{1-u^2} \cdot (1-u^2)} = h \cdot dR.$$

By integrating both sides, we get

$$\int \frac{du}{\sqrt{1-u^2} \cdot (1-u^2)} = \int h \cdot dR = h \cdot R + C,$$

for some integration constant C.

To find the expression for the integral in the left-hand side, we can substitute $u = \sin(\theta)$, then $du = \cos(\theta) \cdot d\theta$, and the integral takes the form

$$\int \frac{\cos(\theta)\, d\theta}{\sqrt{1-\sin^2(\theta)} \cdot \left(1 - \sin^2(\theta)\right)} = \int \frac{\cos(\theta)\, d\theta}{\sqrt{\cos^2(\theta)} \cdot cos^2(\theta)} = \int \frac{d\theta}{cos^2(\theta)}.$$

This integral is known – it is equal to $\tan(\theta)$, hence $\tan(\theta) = h \cdot R + C$. For $R = 0$, we have $v(0) = \sin(\theta)$, hence $\theta = 0$, $\tan(\theta) = 0$, and thus, $C = 0$ and $\tan(\theta) = h \cdot R$. Here,

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} = \frac{u}{\sqrt{1-u^2}},$$

so

$$\frac{u}{\sqrt{1-u^2}} = h \cdot R.$$

By squaring both sides and multiplying both sides by the resulting denominator, we get

$$u^2 = (1-u^2) \cdot h^2 \cdot R^2 = h^2 \cdot R^2 - u^2 \cdot h^2 \cdot R^2.$$

By moving the terms containing u^2 to the left-hand side, we get

$$u^2 \cdot (1 + h^2 \cdot R^2) = h^2 \cdot R^2,$$

hence

$$u^2 = \frac{h^2 \cdot R^2}{1 + h^2 \cdot R^2},$$

therefore

$$u(R) = \frac{h \cdot r}{\sqrt{1 + h^2 \cdot R^2}}.$$

So, for $v(R) = c \cdot u(R)$, we get

$$v(R) = \frac{c \cdot h \cdot r}{\sqrt{1 + h^2 \cdot R^2}}.$$

If we denote $H \stackrel{\rm def}{=} c \cdot h$, so that $h = \dfrac{H}{c}$, we get the following formula.

Resulting formula.

$$v(R) = \frac{H \cdot R}{\sqrt{1 + \left(\dfrac{H \cdot R}{c}\right)^2}}.$$

For this formula, as one can easily see, the velocity never exceeds the speed of light.

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