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Fuzzy Analogues of Sets and Functions Can Be Uniquely Determined from the Corresponding Ordered Category: A Theorem

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Abstract: In modern mathematics, many concepts and ideas are described in terms of category theory. From this viewpoint, it is desirable to analyze what can be determined if, instead of the basic category of sets, we consider a similar category of fuzzy sets. In this paper, we describe a natural fuzzy analog of the category of sets and functions, and we show that, in this category, fuzzy relations (a natural fuzzy analogue of functions) can be determined in category terms – of course, modulo 1-1 mapping of the corresponding universe of discourse and 1-1 re-scaling of fuzzy degrees.

Keywords: fuzzy set; ordered category; category of fuzzy sets

1. Introduction

Category theory is one of the main tools of modern mathematics. Many mathematical theories can be naturally described in terms of a directed graph, where vertices are objects studied in this theory (e.g., sets is set theory, topological spaces in topology, linear spaces in linear algebra), and edges relate different objects: e.g., functions map one set into another, continuous mappings map one topological space into another one, linear mapping map one linear space into another ones, etc. The corresponding graph is known as a category; see, e.g., [1].

In precise terms, a category is a tuple (Ob, Mor, ·, id, ◦), where:

- Ob is the set whose elements are called objects,
- Mor is a set whose elements are called morphisms,
- : Mor → Ob × Ob is a mapping that assigns, to each morphism f ∈ Mor a pair of objects (a, b) ∈ Ob × Ob; this is denoted by f : a → b; the object a is called f’s domain, and b is called f’s range;
- id is a mapping that assigns, to each object a ∈ Ob, a morphism idₐ : a → a; and
- ◦ is a mapping that assigns, to each pair of morphisms f : a → b and g : b → c for which the range of f is equal to the domain of g, a new morphism g ◦ f : a → c so that for every f : a → b, we have idₐ ◦ f = f ◦ idₐ = f.

Because of its universal character, category theory plays an important role in modern mathematics [1]. Many new mathematical concepts are defined in category terms, and many original concepts are re-formulated in category terms – such a reformulation in very general terms often enables mathematicians to generalize their ideas and results to a more general context.

As we have mentioned, different areas of mathematics can be described in terms of different categories:
What we do in this paper. In this paper, as an answer to the above questions, we present an axiomatic description of fuzzy sets in the language of categories, with a proof of the soundness of this description.

2. Results
Towards a precise formulation of the problem. It is easy to see that if we have a 1-1 mapping \( \pi : U \rightarrow U \) of the Universe of discourse \( U \) onto itself (i.e., a bijection), then the corresponding transformation \( R(x,y) \rightarrow R(\pi(x), \pi(y)) \) is an automorphism of the corresponding category in the sense that it preserves the identity, composition, and order.
Similarly, if we have a 1-1 monotonically increasing mapping \( H : [0, 1] \to [0, 1] \), then the transformation \( R(x, y) \to H(R(x, y)) \) is also such an automorphism. Indeed, since we only consider order between degrees, monotonic transformation of degrees should not change anything.

It turns out that modulo this simple equivalence, we can uniquely determine all the elements \( x \in U \) and all the relations \( R(x, y) \) from the ordered category, i.e., in precise terms, that every automorphism is a composition of the automorphisms of the above two types. The proof of this result will be based on an explicit description of elements of \( U \) and relations \( R_f(x, y) \) in category terms.

Let us describe the problem in precise terms.

**Definition 1.** By an ordered category, we mean a category in which for every two objects \( a \) and \( b \), there is a partial order \( \leq \) on the set \( \text{Mor}(a, b) \) of all morphisms from \( a \) to \( b \).

**Definition 2.** Let \( U \) be a set; we will call it the Universe of discourse. By a \( U \)-fuzzy ordered category, we mean an ordered category in which:

- the only object is the set \( U \),
- morphisms are fuzzy relations, i.e., mappings \( R : U \times U \to [0, 1] \),
- the morphism \( \text{id} \) is defined as the mapping for which \( \text{id}(x, x) = 1 \) and \( \text{id}(x, y) = 0 \) for \( x \neq y \),
- the composition of morphisms is defined by the formula

\[
(g \circ f)(x, z) = \max_y \min(f(x, y), g(y, z)),
\]

and

- the order between the morphisms is the component-wise order: \( f \leq g \) means that \( f(x, y) \leq g(x, y) \) for all \( x \) and \( y \).

The \( U \)-fuzzy ordered category will be denoted by \( F_U \).

**Comment.** One can easily see that this is indeed a category, i.e., that the composition of morphisms is associative, and the composition of any morphism \( f \) with the identity morphism \( \text{id} \) is equal to \( f \):

\[
f \circ \text{id} = \text{id} \circ f = f.
\]

**Definition 3.** An automorphism of an ordered category is a pair consisting of bijections \( F : \text{Ob} \to \text{Ob} \) and \( G : \text{Mor} \to \text{Mor} \) for which:

- for all \( f, a, \) and \( b \), we have \( f : a \to b \) if and only if \( G(f) : F(a) \to F(b) \);
- for all \( f \) and \( g \), we have \( G(f \circ g) = G(f) \circ G(g) \),
- for all \( a \), we have \( G(\text{id}_a) = \text{id}_{F(a)} \), and
- for all \( f \) and \( g \), we have \( f \leq g \) if and only if \( G(f) \leq G(g) \).

**Comment.** This definition is a natural generalization of the standard definition of automorphism of categories (see, e.g., [6,15,18]) to ordered categories.

**Proposition.** Let \( \pi : U \to U \) be a bijection of \( U \), and let \( H : [0, 1] \to [0, 1] \) be an increasing bijection of the interval \([0, 1]\). Then, the mapping \( G_{\pi, H} \) that maps each morphism \( f(x, y) \) into a morphism \( (G_{\pi, H}(f))(x, y) = H(f(\pi(x), \pi(y))) \) is an automorphism of the category \( F_U \).

Our main result is that these are the only automorphisms of the category \( F_U \).

**Theorem.** For every set \( U \), every automorphism of the ordered category \( F_U \) has the form \( G_{\pi, H} \) for some bijection \( \pi : U \to U \) and for some monotonic bijection \( H : [0, 1] \to [0, 1] \).

**Comment.** This may not be very clear from the formulation of the result, but the proof will show that we can determine elements of the set \( U \) and values of the mappings \( f(x, y) \) in category terms, i.e., we can indeed define fuzzy relations – a natural fuzzy analogue of functions – in category terms.
3. Proofs

3.1. Proof of the Proposition

This proposition is easy to prove: a permutation $\pi$ does not change anything, and the increasing bijection does not change the order.

3.2. Proof of the Theorem

1°. First, we can describe the morphism $f_0$ for which $f_0(x,y) = 0$ for all $x$ and $y$ in ordered-category terms, as the only morphism $f$ for which $f \leq g$ for all morphisms $g$.

Indeed, clearly $f_0 \leq g$ for all $g$. Vice versa, if $f \leq g$ for all $g$, then, in particular, $f \leq f_0$, i.e., $f(x,y) \leq f_0(x,y) = 0$ for all $x$ and $y$, and since $f(x,y) \in [0,1]$, this means that indeed $f(x,y) = 0$ for all $x$ and $y$.

2°. Let us first characterize all the morphisms $f \neq f_0$ for which the set $\{g : g \leq f\}$ is linearly ordered. Since an automorphism preserves order, every automorphism maps such morphisms into morphisms with the same property.

Specifically, we will prove that a morphism has this property if and only if we have $f(x,y) > 0$ only for one pair $(x,y)$, and we have $f(x',y') = 0$ for all other pairs $(x',y')$.

Indeed, one can easily check that for such morphisms $f$, the only morphisms $g \leq f$ are the morphisms which also have $g(x',y') = 0$ for all pairs $(x',y') \neq (x,y)$. Such morphisms $g$ are uniquely described by the corresponding value $g(x,y)$. For every two such morphisms $g$ and $g'$, depending on whether $g(x,y) \leq g'(x,y)$ or $g'(x,y) \leq g(x,y)$, we have $g \leq g'$ or $g' \leq g$, i.e., the set $\{g : g \leq f\}$ is indeed linearly ordered.

Vice versa, let us prove that if a morphism has this property, then it has $f(x,y) > 0$ only for one pairs $(x,y)$. Indeed, if we have $f(x,y) > 0$ and $f(x',y') > 0$ for two different pairs $(x,y) \neq (x',y')$, then we would be able to construct two different morphisms $g \leq f$ and $g' \leq f$ for which $g \neq g'$ and $g' \neq g$. Namely, we take:

- $g(x,y) = f(x,y) > 0$ and $g(x'',y'') = 0$ for all pairs $(x'',y'') \neq (x,y)$, and
- $g'(x',y') = f(x',y') > 0$ and $g(x''',y''') = 0$ for all pairs $(x''',y''') \neq (x',y')$.

This contradicts our assumption that the set $\{g : g \leq f\}$ is linearly ordered.

3°. Let us now describe, in ordered-category terms, morphisms $f$ for which $f(x,x) > 0$ for some $a \in U$ and $f(x',y')$ for all other pairs $(x',y') \neq (x,x)$.

Indeed, out of all morphisms described in Part 2 of this proof, such morphisms can be determined by the additional condition that $f \circ f = f$. This condition is clearly satisfied for such morphisms, while for morphisms for which $f(x,y) > 0$ for some $b \neq a$, the composition $f \circ f$ is, as one can see, identically 0 and thus, different from $f$.

4°. One can see that two morphisms $f$ and $f'$ of the type described in Part 3 are connected by the relation $\leq$ (i.e., $f \leq f'$ or $f' \leq f$) if and only if they correspond to the same element $a \in U$.

Thus, we can describe elements of the set $U$ in ordered-category terms: as equivalent classes of morphisms of the type described in Part 3 with respect to the relation $(f \leq f') \lor (f' \leq f)$.

Hence, if we have an automorphism, elements are mapped into elements in a 1-1 way, i.e., indeed we have a bijection of the Universe of discourse.

5°. Let us now show that the degrees from the interval $[0,1]$ can also be described – modulo increasing bijections of this interval – in ordered-category terms.

5.1°. Indeed, for each element $a \in U$, different degrees $v \in [0,1]$ can be associated with different morphisms $f$ described in Part 3 of this proof, i.e., morphisms for which:

- $f(x,x) > 0$ for this element $a$ and
We want, for every $a \neq b$, to connect the values $v$ and $w$ corresponding to functions $f_{x,a}$ and $f_{y,b}$. This connection comes from the following auxiliary result:

$$w \leq v \iff \exists f_{x \to y} \exists f_{y \to x} (f_{x \to y} \circ f_{x,a} \cdot f_{y \to x} = f_{y,b}).$$

Indeed, by definition of a composition, the values of the composition $g \circ f$ cannot exceed the largest value of each of the composed relations $g$ and $f$. Thus, if $f_{x \to y} \circ f_{x,a} \cdot f_{y \to x} = f_{y,b}$, then the value $f_{y,b}(b,b) = w$ cannot exceed the maximum value $v$ of the function $f_{x,a}$; thus, $w \leq v$.

Vice versa, if $w \leq v$, then we can take the following morphisms $f_{x \to y}$ and $f_{y \to x}$:

1. $f_{x \to y}(x,y) = w$ and $f_{x \to y}(x',y') = 0$ for all other pairs $(x',y') \neq (x,y)$, and similarly,
2. $f_{y \to x}(y,x) = w$ and $f_{y \to x}(y',x') = 0$ for all other pairs $(x',y') \neq (y,x)$.

In this case, as one can easily check, we have $f_{x \to y} \circ f_{x,a} \cdot f_{y \to x} = f_{y,b}$.

5.3°. Now that we know how to describe the relation $w \leq v$ for functions $f_{x,a}$ and $f_{y,b}$ in ordered-category form, we can describe equality $v = w$ between the degrees $v$ and $w$ corresponding to morphisms $f_{x,a}$ and $f_{y,b}$ as $(v \leq w)$ & $(w \leq v)$, i.e., in view of Part 5.2, as:

$$(\exists f_{x \to y} \exists f_{y \to x} (f_{x \to y} \circ f_{x,a} \cdot f_{y \to x} = f_{y,b})) \& (\exists g_{y \to x} \exists g_{x \to y} (g_{y \to x} \circ f_{y,b} \cdot g_{x \to y} = f_{x,a})).$$

This enables us to identify degrees $v \in [0,1]$ in ordered-category terms – by identifying them with the functions $f_{x,a}$ and taking into account the above possibility to compare degrees at different elements $a$.

Hence, if we have an automorphism, degrees are mapped into degrees in a 1-1 and order-preserving way, i.e., indeed we have a monotoniction bijection $H : [0,1] \to [0,1]$.

6°. To complete the proof, we need to show how, for each morphism $f$ and for every two elements $a$ and $b$, we can describe the value $f(x,y)$ in ordered-category terms. This will complete the proof that the given automorphism has the form $G_{\pi,H}$ for the mappings $\pi$ and $H$ as identified in Sections 4 and 5 of this proof.

6.1°. Let us first prove the following auxiliary result:

$$\exists f_{y \to x} (f_{y \to x} \circ f_{y,1} \circ f \circ f_{x,1} = f_{x,a}) \iff v \leq f(x,y).$$

Indeed, by definition of a composition, the composition $c \overset{\text{def}}{=} f \circ f_{x,1}$ has the following form:

1. $c(x,y') = f(x,y')$ for all $y'$ and
2. $c(x,y') = 0$ for all $y'$ and for all $x' \neq a$.

Similarly, the composition $c' \overset{\text{def}}{=} f_{y,1} \circ f \circ f_{y,1} = f_{y,1} \circ c$ has the following form:

1. $c'(x,y) = f(x,y)$, and
2. $c'(x',y') = 0$ for all other pairs $(x',y') \neq (x,y)$.
As we have argued in Part 5 of this proof, the value of a composition function cannot exceed the maximum value of each of the composed morphisms. Thus, for the composition \( f_{y \to x} \circ f_{y,1} \circ f \circ f_{x,1} = f_{y \to x} \circ c' \), the maximum value cannot exceed the maximum value \( f(x, y) \) of the morphism \( c' \). Thus, if \( f_{y \to x} \circ c' = f_{x,y} \), the maximum value \( v \) of the morphism \( f_{x,y} \) cannot exceed \( f(x, y) \): \( v \leq f(x, y) \).

Vice versa, for every \( v \leq f(x, y) \), we can construct a morphism \( f_{y \to x} \) for which \( f_{y \to x} \circ c' = f_{x,y} \):

namely, we can take:

- \( f_{y \to x}(y, x) = v \), and
- \( f_{y \to x}(x', y') = 0 \) for all pairs \((x', y') \neq (y, x)\).

One can easily check that in this case indeed \( f_{y \to x} \circ c' = f_{x,y} \).

6.2\(^{c}\). For each morphism \( f \) and for every two elements \( a \) and \( b \), we can identify the degree \( f(x, y) \) as the largest degree \( v \) for which the inequality \( v \leq f(x, y) \) holds.

Since, according to Part 6.1 of this proof, the inequality \( v \leq f(x, y) \) can be described in ordered-category terms, we can thus conclude that the degree \( f(x, y) \) can also be described in ordered-category terms.

The proposition is proven.

4. Conclusions

Many concepts of modern mathematics, starting from the basic notions of sets and functions, are described in terms of category theory. Many other mathematical concepts can be reformulated in category terms. Due to the general nature of category theory, such a reformulation often helps to extend notions and results from one area to different areas of mathematics.

Because of this potential advantage, it is reasonable to ask whether similar fuzzy notions can also be described in category terms. In this paper, we show that fuzzy relations – i.e., fuzzy analogues of functions – can indeed be described in category terms. Specifically, we show that, in the corresponding fuzzy category, we can describe both:

- elements of the original universe of discourse (modulo a 1-1 permutation), and
- fuzzy degrees (modulo a 1-1 monotonic mapping from the interval \([0, 1]\) onto itself).

This result shows the soundness of our axiomatic description of fuzzy sets in the language of categories.

At this moment, what we have is a very theoretical paper. However, we hope that, similarly to how the reformulation of crisp notions in category terms can help generalize the corresponding results, our reformulation will help extend fuzzy results to more general situations – and thus, will facilitate future applications.

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References


14. Kreinovich, V., Ceberio, M., Brefort, Q. In category of sets and relations, it is possible to describe functions in purely category terms, Eurasian Mathematical Journal, 2015, 6(2), 90–94.


