6-2017

Quantitative Justification for the Gravity Model in Economics

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Comments:

Technical Report: UTEP-CS-17-50

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Vladik Kreinovich and Songsak Sriboonchitta

Abstract The gravity model in economics describes the trade flow between two countries as a function of their Gross Domestic Products (GDPs) and the distance between them. This model is motivated by the *qualitative* similarity between the desired dependence and the dependence of the gravity force (or potential energy) between the two bodies on their masses and on the distance between them. In this paper, we provide a *quantitative* justification for this economic formula.

1 Gravity Model in Economics: A Brief Introduction

What is gravity model. It is known that, in general:

- neighboring countries trade more than distant ones, and
- countries with larger Gross Domestic Product (GDP) g have a higher volume of trade than countries with smaller GDP.

Thus, in general, the trade flow t_{ij} between the two countries i and j:

- increases when the GDPs g_i and g_j increase and
- decreases with the distance r_{ij} increases.

A qualitatively similar phenomenon occurs in physics: the gravity force f_{ij} between the two bodies:

- increases when their masses m_i and m_j increase and
- decreases with the distance between then increases.

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Similarly, the potential energy e_{ij} of the two bodies at distance r_{ij}:

- increases when the masses increase and
- decreases when the distance r_{ij} increases.

For the gravity force and for the potential energy, there are simple formulas:

$$f_{ij} = G \cdot \frac{m_i \cdot m_j}{r_{ij}^2}; \;\; e_{ij} = G \cdot \frac{m_i \cdot m_j}{r_{ij}},$$

for some constant G. Both these formulas are a particular case of a general formula

$$G \cdot \frac{m_i \cdot m_j}{r_{ij}^\alpha}:$$

for the force, we take $\alpha = 2$, and for the energy, we take $\alpha = 1$.

By using the analogy with the gravity formulas, researchers have proposed to use a similar formula to describe the dependence of the trade flow t_{ij} on the GDPs g_i and and on the distance r_{ij}:

$$t_{ij} = G \cdot \frac{g_i \cdot g_j}{r_{ij}^\alpha}.$$

This formula – known as the *gravity model* in economics – has indeed been successfully used to describe the trade flows between different countries; see, e.g., [2, 3, 4, 5, 6].

Remaining problem and what we do in this paper. While an analogy with gravity provides a *qualitative* explanation for the gravity model, it is desirable to have a *quantitative* explanation as well. Such an explanation is provided in this paper.

2 Analysis of the Problem

What we want. We would like to have a formula that estimates the trade flow between the two countries t_{ij} as a function of their GDPs g_i and g_j and of the distance r_{ij} between the two countries. In other words, we would like to come up with a function $F(a,b,c)$ for which

$$t_{ij} = F(g_i, g_j, t_{ij}). \tag{1}$$

To describe the corresponding function $F(a,b,c)$, let us describe the natural properties of such a function.

First natural property: additivity. At first glance, the notion of a country seems to be very clear and well defined. However, there are many examples where this notion is not that clear. Sometimes, a country becomes a loose confederation of practically independent states. In other cases, several countries form such a close trade union – from Benelux to European Union – that most trade is regulated by the super-national organs and not by individual countries.

In all such cases, we have several different entities $i_1,\ldots,i_k,\ldots,i_\ell$ located nearby forming a single super-entity. If we apply the formula (1) to each individual entity i_k, we get the expression

$$t_{i_k j} = F(g_{i_k}, g_j, r_{i_k j}).$$

Since all the entities i_k are located close to each other, we can assume that the distances $r_{i_k j}$ are all the same: $r_{i_k j} = r_{ij}$. Thus, the above expression takes the form $t_{i_k j} = F(g_{i_k}, g_j, r_{ij})$.

By adding all these expressions, we can come up with the trade flow between the whole super-entity i and the country j:

$$t_{ij} = \sum_{k=1}^{\ell} t_{i_k j} = \sum_{k=1}^{\ell} F(g_{i_k}, g_j, r_{ij}). \tag{2}$$

Alternatively, we can treat the super-entity as a single country with the overall GDP $g_i = \sum\limits_{k=1}^{\ell} g_{i_k}$. In this case, by applying the formula (1) to this super-entity, we get

$$t_{ij} = F(g_i, g_j, r_{ij}) = F\left(\sum_{k=1}^{\ell} g_{i_k}, g_j, r_{ij}\right). \tag{3}$$

It is reasonable to require that our estimate for the trade flow should not depend on whether we treat this loose confederation a single country or as several independent countries. By equating the estimates (2) and (3), we conclude that

$$F(g_{i_1}, g_j, r_{ij}) + \ldots + F(g_{i_\ell}, g_j, r_{ij}) = F(g_{i_1} + \ldots + g_{i_\ell}, g_j, r_{ij}).$$

In other words, we must have the following additivity property for all possible values a, ..., a', and b:

$$F(a,b,c) + \ldots + F(a',b,c) = F(a + \ldots + a', b, c). \tag{4}$$

A similar argument can be make if we consider the case when j is a loose confederation of states. In this case, the requirement that our estimate for the trade flow should not depend on whether we treat this loose confederation as a single country or as several independent countries leads to

$$F(g_i, g_{j_1}, r_{ij}) + \ldots + F(g_i, g_{j_\ell}, r_{ij}) = F(g_i, g_{j_1} + \ldots + g_{j_\ell}, r_{ij}),$$

i.e., to

$$F(a,b,c) + \ldots + F(a,b',c) = F(a, b + \ldots + b', c). \tag{5}$$

Second natural property: scale-invariance. The numerical value of the distance depends on what unit we use for measuring distance. For example, the distance in miles in different from the same distance in kilometers. If we replace the original

unit with a one which is λ times smaller, all numerical values of the distance multiply by λ, i.e., each original numerical value r_{ij} is replaced by a new numerical value

$$r'_{ij} = \lambda \cdot r_{ij}.$$

It is reasonable to require that the estimates for the trade flow should not depend on what unit we use. Of course, we cannot simply require that $F(g_i, g_j, r_{ij}) = F(g_i, g_j, \lambda \cdot r_{ij})$ – this would mean that the trade flow does not depend on the distance at all. This is OK, since the numerical value of the trade flow also depends on what units we use: we get different numbers if we use US dollars or Thai Bahts. It is therefore reasonable to require that when we change the unit for measuring r_{ij}, then after an appropriate change $t_{ij} \to t'_{ij} = \mu \cdot t_{ij}$ in the measuring unit for trade flow we get the same formula. In other words, we require that for every $\lambda > 0$, there exists a $\mu > 0$ for which

$$F(g_i, g_j, \lambda \cdot r_{ij}) = \mu \cdot F(g_i, g_j, r_{ij}).$$

In other words, we require that

$$F(a,b,\lambda \cdot c) = \mu \cdot F(a,b,c). \tag{6}$$

Third natural property: monotonicity. The final natural property is that as the distance increases, the trade flow should decrease. In other words, the function $F(a,b,c)$ should be a decreasing function of c.

Now, we are ready to formulate our main result.

3 Definitions and the Main Result

Definition 1.

- *A non-negative function $F(a,b,c)$ of three non-negative variables is called* additive *if the following two equalities hold for all possible values $a, \ldots, a', b, \ldots, b'$, and c:*

$$F(a,b,c) + \ldots + F(a',b,c) = F(a + \ldots + a', b, c);$$

$$F(a,b,c) + \ldots + F(a,b',c) = F(a, b + \ldots + b', c).$$

- *A function $F(a,b,c)$ is called* scale-invariant *if for every λ, there exists a μ for which, for all a, b, and c, we have*

$$F(a,b,\lambda \cdot c) = \mu \cdot F(a,b,c).$$

- *A function $F(a,b,c)$ is called a* trade function *if it is additive, scale-invariant, and increasing as a function of c.*

Proposition 1. *Every trade function has the form* $F(a,b,c) = G \cdot \dfrac{a \cdot b}{c^\alpha}$ *for some constants* G *and* α.

Discussion. Thus, we have indeed justified the gravity model.

Proof of Proposition 1.

1°. Let us first use the additivity property.

For every b and c, we can consider an auxiliary function $f_{bc}(a) \stackrel{\text{def}}{=} F(a,b,c)$. In terms of this function, the first additivity property takes the form

$$f_{bc}(a + \ldots + a') = f_{bc}(a) + \ldots + f_{bc}(a').$$

Functions of one variable that satisfy this property are known as *additive*. It is known – see, e.g., [1] – that every non-negative additive function has the form $f(a) = k \cdot a$. Thus, $F(a,b,c) = f_{bc}(a)$ is equal to

$$F(a,b,c) = a \cdot k(b,c)$$

for some function $k(b,c)$.

Substituting this expression into the second additivity requirement, we conclude that

$$a \cdot k(b + \ldots + b', c) = a \cdot k(b,c) + \ldots + a \cdot k(b',c).$$

Dividing both sides of this equality by a, we conclude that

$$k(b + \ldots + b', c) = k(b,c) + \ldots + k(b',c).$$

Thus, the function $k_c(b) \stackrel{\text{def}}{=} k(b,c)$ is also additive. Hence, $k(b,c) = k_c(b) = b \cdot q(c)$ for some constant $q(c)$ depending on c. Substituting this expression for $k(b,c)$ into the formula describing $F(a,b,c)$ in terms of $k(b,c)$, we conclude that $F(a,b,c) = a \cdot b \cdot c(q)$.

Hence, to complete the proof, it is sufficient to find the function $q(c)$.

2°. For $a = b = 1$, we have $F(a,b,c) = q(c)$. Thus, for these a and b, the fact that $F(a,b,c)$ is a decreasing function of c implies that $q(c)$ is also an decreasing function of c.

3°. To find the function $q(c)$, let us now use scale invariance

$$F(a,b,\lambda \cdot c) = \mu(\lambda) \cdot F(a,b,c).$$

Substituting $F(a,b,c) = a \cdot b \cdot q(c)$ into this equality and dividing both sides by $a \cdot b$, we conclude that $q(\lambda \cdot c) = \mu(\lambda) \cdot q(c)$.

For every λ_1 and λ_2, we have

$$q((\lambda_1 \cdot \lambda_2) \cdot c) = \mu(\lambda_1 \cdot \lambda_2) \cdot q(c).$$

On the other hand, we also have $q(\lambda_2 \cdot c) = \mu(\lambda_2) \cdot q(c)$ and thus,

$$q(\lambda_1 \cdot (\lambda_2 \cdot c)) = \mu(\lambda_1) \cdot q(\lambda_2 \cdot c) = \mu(\lambda_1) \cdot \mu(\lambda_2) \cdot q(c).$$

By equating these two expressions for the same quantity $q(\lambda_1 \cdot \lambda_2 \cdot c)$, we conclude that

$$\mu(\lambda_1 \cdot \lambda_2) \cdot q(c) = \mu(\lambda_1) \cdot \mu(\lambda_2) \cdot q(c).$$

Dividing both sides by $q(c)$, we get

$$\mu(\lambda_1 \cdot \lambda_2) = \mu(\lambda_1) \cdot \mu(\lambda_2).$$

Functions $\mu(\lambda)$ with this property are known as *multiplicative*.

Here, for every c, we have $\mu(\lambda) = \dfrac{q(\lambda \cdot c)}{q(c)}$. In particular, for $c = 1$, we get $\mu(\lambda) = \dfrac{q(\lambda)}{q(1)}$. Since $q(c)$ is an increasing function, we conclude that $\mu(\lambda)$ is also an increasing function.

It is known [1] that every monotonic multiplicative function has the form $\mu(\lambda) = \lambda^{-\alpha}$ for some $\alpha > 0$. From $q(\lambda) = \mu(\lambda) \cdot q(1)$, we can conclude that $q(c) = G \cdot c^{-\alpha}$, where we denoted $G \stackrel{\text{def}}{=} q(1)$.

The proposition is proven.

4 Where Do We Go From Here

Trade flow may depend on other characteristics. In the previous text, we assumed that the trade flow depends only on the GDPs and on the distance. In reality, the trade flow may also other depend on other characteristics, such as the country's population p_i. Indeed, intuitively, the larger the population, the more it consumes, so the larger its trade flow with other countries.

Similar to GDP, population is an additive property, in the sense that if two countries merge together, their population adds up. So, a natural question is: how can we describe the dependence of the trade flow on two or more additive characteristics?

Let us describe this problem in precise terms. Let us consider the case when each country is described by several additive characteristics, i.e., that g_i is now a vector consisting of several components $g_i = (g_{1i}, \ldots, g_{mi})$. We are interested in finding the dependence $t_{ij} = F(g_i, g_j, r_{ij})$.

Let us describe the reasonable properties of this dependence.

Additivity and monotonicity. Similarly to the GDP-only case, we can conclude that

$$F(g_{i_1} + \ldots + g_{i_\ell}, g_j, r_{ij}) = F(g_{i_1}, g_j, r_{ij}) + \ldots + F(g_{i_\ell}, g_j, r_{ij})$$

and

$$F(g_i, g_{j_1} + \ldots + g_{j_\ell}, r_{ij}) = F(g_i, g_{j_1}, r_{ij}) + \ldots + F(g_i, g_{j_\ell}, r_{ij}).$$

Also, similarly to the GDP-only case, it makes sense to require that the function $F(a,b,c)$ is a decreasing function of c.

Definition 2. *Let $m > 1$.*

- *A non-negative function $F(a,b,c)$ of three non-negative variables $a,b \in \mathbb{R}^m$ and $c \in \mathbb{R}$ is called* additive *if the following two equalities hold for all possible values $a,\ldots,a',b,\ldots,b'$, and c:*

$$F(a,b,c)+\ldots+F(a',b,c) = F(a+\ldots+a',b,c);$$

$$F(a,b,c)+\ldots+F(a,b',c) = F(a,b+\ldots+b',c).$$

- *A function $F(a,b,c)$ is called* scale-invariant *if for every λ, there exists a μ for which, for all a, b, and c, we have*

$$F(a,b,\lambda\cdot c) = \mu\cdot F(a,b,c).$$

- *A function $F(a,b,c)$ is called a* trade function *if it is additive, scale-invariant, and increasing as a function of c.*

Proposition 2. *Every trade function has the form*

$$F(g_i,g_j,r_{ij}) = \frac{\sum\limits_{\beta}\sum\limits_{\gamma} G_{\beta\gamma}\cdot g_{\beta i}\cdot g_{\gamma j}}{r_{ij}^{\alpha}}$$

for some constants $G_{\beta\gamma}$ and α.

Example. For the case of GDP g_i and population p_i, we have

$$t_{ij} = \frac{G_{gg}\cdot g_i\cdot g_j + G_{gp}\cdot g_i\cdot p_j + G_{pg}\cdot p_i\cdot g_j + G_{pp}\cdot p_i\cdot p_j}{r_{ij}^{\alpha}}.$$

An interesting property of this example is that, in contrast to the GDP-only case, when we always had $t_{ij} = t_{ji}$, we can have "asymmetric" trade flows for which $t_{ij} \neq t_{ji}$.

Proof of Proposition 2 is similar to the proof of Proposition 1: first additivity requirement implies that $F(a,b,c)$ is linear in a, second – that it is linear in b, so it is bilinear in a and b. Now, scale-invariance implies that all the coefficients of this bilinear dependence be proportional to $r_{ij}^{-\alpha}$ for some $\alpha > 0$.

Discussion. It would be nice to test these formulas on real data.

Acknowledgments

We acknowledge the partial support of the Center of Excellence in Econometrics, Faculty of Economics, Chiang Mai University, Thailand. This work was also sup-

ported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

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