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How to Teach Implication

Martha Osegueda Escobar
The University of Texas at El Paso, mcoseguedaescobar@miners.utep.edu

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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HOW TO TEACH IMPLICATION

Osegueda Escobar, Martha, Student
Kosheleva, Olga, PhD, Associate Professor
Kreinovich, Vladik, PhD, Professor
University of Texas at El Paso
mcoseguedaescobar@miners.utep.edu, olgak@utep.edu, vladik@utep.edu

Abstract: Logical implication is a somewhat counter-intuitive notion. For students, it is difficult to understand why a false statement implies everything. In this paper, we present a simple pedagogical way to make logical implication more intuitive.

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Logical implication is a difficult topic. One of the most important concepts of mathematical logic is the concept of implication A --> B ("if A then B"). Professional mathematicians are accustomed to the fact that out of four possible combinations of truth values of A and B: (T, T), (T, F), (F, T), and (F, F), implication is true for there of them. The only case when implication is not true is when

- A is true and
- B is false.

The corresponding formal definition of logical implication leads to many statements which are intuitively true: e.g.,

- if a, b, and c are the three sides of a right triangle,
- then $c^2 = a^2 + b^2$.

On the other hand, the same formal implication leads to many statements which are not intuitive at all. For example,

- it is true that if witches exist,
- then 0 = 1.

A natural student's reaction to this "implication" is that it cannot be true -- what do witches have to do with arithmetic?

Because of this non-intuitive character of implication, logical implication is a difficult topic to teach.

What we do in this paper. In this paper, we explain how to make the notion of logical implication more intuitive.

Main implicit assumption. The main implicit assumption behind our examples is that – similarly to "and" and "or" – the truth value of the implication A→ B is uniquely determined by the truth values of the two statements A and B.

Thus, to convince the students that, for example, False always implies False, it is sufficient to come up with one intuitively clear example when

- A is false,
- B is false, and
- the implication A → B is true.

So, what we will do is provide examples of such cases.

First set of examples: let us start with a simple implication which is definitely true. To make the notion of a logical implication more intuitive, let us start with the case in which implication is definitely true. Namely, no matter what values x, y, and z we take, it is definitely true that:

- if x = y,
- then zx = zy.

In this case,

- A is the equality x = y,
- B is the equality zx = zy, and
- the implication A → B is always true.

No one doubts this statement: to conclude that zx = zy, we simply multiply both sides of the equality x = y by the same number z.

Let us enumerate different particular cases of this statement.

Case when x = y = z = 1. Let us first consider the case when x = y = z = 1. In this case, x = y, so the equality A is true. Here, zx = zy, so the equality B is also true. Thus, here,

- A is true,
- B is true, and
- the implication A --> B is a particular case of a general statement which is always true.

Thus, we conclude that if A and B are both true, then the implication A → B is also true.

Case when x = 0 and y = z = 1. Let us now consider the case when x = 0 and y = z = 1. In this case, x =/= y, so the equality A is false. Similarly, since zx = 0 and zy =1, we have zx =/= zy, so the equality B is also false. Thus, here,

- A is false,
- B is false, and
- the implication A --> B is a particular case of a general statement which is always true.

Thus, we conclude that if A and B are both false, then the implication A → B is true.

Case when x = 0, y = 1, and z = 0. Let us consider the case when x = 0, y = 1, and z = 0. In this case, x =/= y, so the equality A is false. Here, zx = zy = 0, so the equality B is true. Thus, here,

- A is false,
- B is true, and
- the implication A --> B is a particular case of a general statement which is always true.

Thus, we conclude that if A is false and B is true, then the implication A → B is true.

Remaining case. Out of four possible cases (T,T), (T, F), (F, T), and (F, F), we showed that the implication is true in three of them: in the cases of (F, F), (F, T), and (T, T). The only remaining case is the case (T, F), when A is true and F is false.

If the implication was true in this case, this would mean that implication holds in all possible cases. But that would mean that this is a vacuous notion, the notion which is always trivially true. Intuitively, it is not true that every statement implies every other statement. There should be cases when implication is not true. Thus, in this remaining case (T, F), the implication should be false.

So, we have our explanation. So, in all four cases, we have an intuitive explanation of the corresponding row in the 4-row truth table of the implication – which is exactly what we wanted to achieve.

Second set of examples. Let us now move from mathematics to everyday life. Let us consider a simple implication, i.e., a simple if-then rule that everyone understands. For example, we can consider the following rule regarding cars that approach an intersection at which the traffic light is red. The red light means that the car should not continue going.

Thus,

- if a police officer sees the car going straight in spite of the red light,
- then he or she should issue a fine to the violating driver.

Here:

- A is “the car continues going straight in spite of the red light”, and
- B is “the police officer issues a fine”.

If this rule is satisfied, the police officer is doing his/her job. It could be that at some moment, the police officer is inattentive and misses a violating car. In this case, the officer has not followed the desired rule – and this officer will be reprimanded by his/her superior if the superior happens to be around.

So far, so good. Let us consider all possible situations.

Case when a car goes straight on red and gets a fine. Let us consider the simplest case:

- a car continue going straight on red light, and
- the police officer sees this violation and issues a fine.

Clearly, the officer is doing his/her job, so the rule A → B is satisfied. In this case:

- the statement A is true,
- the statement B is also true, and
- the implication A → B is true.

Thus, if A is true and B is true, the implication A → B should be true.

Case when a car goes straight and does not get a fine. If the car continues to go straight on red and the police officer does not issue a fine, this clearly means that the police officer is not doing his/her job. In this case:

- the statement A is true,
- the statement B is false, and
- the implication A → B is false.

Thus, if A is true and B is false, the implication A → B should be false.

Case when a car stops and does not get a fine. What if the car obediently stops at the red light and does not get a fine? In this case, if the police officer's superior is present, the superior will not find any fault with the police officer who is clearly doing his job. This means that in this case, there is no violation of the if-then (implication) rule that describes the desired behavior of the police officer – in other words, the if-then rule is true. Here:

- the statement A is false,
- the statement B is false, and
- the implication A → B is true.

Thus, if A is false and B is false, the implication A → B should be true.

Case when a car stops but still get a fine. Let us now consider the final case when the car stops at the red light, but – dues to some other violation – still get a fine. For example, we are at night, when the cars should use their lights, but in this particular case, one of the car's front lights is broken.

In this case, the police officer did not violate the original if-then rule and thus, this if-then rule is true. Here:

- the statement A is false,
- the statement B is true, and
- the implication A → B is true.

Thus, if A is false and B is true, the implication A → B should be true.

So, we again have our explanation. So, in this set of examples as well, in all four possible cases, we have an intuitive explanation of the corresponding row in the 4-row truth table of the implication – which is exactly what we wanted to achieve.