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From Fuzzy Universal Approximation to Fuzzy Universal Representation: It All Depends on the Continuum Hypothesis

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***Abstract*—It is known that fuzzy systems have a universal approximation property. A natural question is: can this property be extended to a universal *representation* property? Somewhat surprisingly, the answer to this question depends on whether the following *Continuum Hypothesis* holds: every infinite subset of the real line has**

- **either the same number of elements as the real line itself**
- **or as many elements as natural numbers.**

I. When Can We Go from Fuzzy Universal Approximation to Fuzzy Universal Representation: Formulation of the Problem

Need to translate expert statements into precise terms. In many practical situations:

- there is a correlation between two quantities x and y, and
- the only information that we have to describe this correlation are expert statements formulated in terms of imprecise (fuzzy) words from natural language, such as "small".

For example, an expert can say that:

- if x is small,
- then y is big,

and vice versa.

Fuzzy logic provides the desired translation. Fuzzy logic (see, e.g., [4], [5], [6]) is a technique that translates this knowledge into precise mathematical terms.

In this technique, each fuzzy term A is described by a function $A(x)$ assigning,

- to each possible value x of the corresponding quantity,
- a degree $A(x)$ to which this value has the appropriate property (e.g., is small).

Once we have rules $A_i(x) \Rightarrow B_i(y)$, the degree $d(x,y)$ to which each pair (x,y) is possible can be described as the degree to which:

- either the first rule is satisfied (i.e., $A_1(x)$ and $B_1(y)$)
- or the second rule is satisfied, etc.

One possible way to interpret "and" is to use product: namely, if we know:

- the degrees a to which the statement A is satisfied and
- the degree b to which a statement B is satisfied,

then it is reasonable to estimate the degree to which the conjunction $A \,\&\, B$ is satisfied as $a \cdot b$.

Similarly, a possible way to interpret "or" is to use sum: namely, if we know:

- the degrees a to which the statement A is satisfied and
- the degree b to which a statement B is satisfied,

then it is reasonable to estimate the degree to which the disjunction $A \vee B$ is satisfied as $a + b$ (t be more precise, $\min(a+b,1)$).

Under these interpretations of "and" and "or",

- the degree to which the i-th rule is satisfied for a given pair (x,y) can be estimated as the product $A_i(x) \cdot B_i(y)$, and
- the desired degree to which one of the rules is satisfied, i.e., to which:
  - either the first rule is satisfies,
  - or the second rule is satisfied, etc.,

  takes the form

$$d(x,y) = \sum_{i=1}^{n} A_i(x) \cdot B_i(y). \quad (1)$$

Universal approximation property. It is known that this expression has a universal approximation property:

- for every $\varepsilon > 0$,
- every continuous function on a box can be ε-approximated by such sums.

Is there a universal representation property? A natural question is:

When can we get an exact representation of every function?

II. Let us Formulate the Problem in Precise Terms

Towards a precise formulation of the problem: first attempt. The simplest way to interpret the above question is to ask whether there exists an integer n such that:

- *any* function of two variables
- can be represented in the form (1) with this particular n.

It turns our that this is *not* possible; see, e.g., [2].

Second attempt. Since we cannot have universal representation by using a fixed finite number of terms, a natural next idea is to have a representation in which:

- the number of terms is finite for every function $d(x, y)$ and for every pair (x, y), but
- this number of terms may be different for different functions $d(x, y)$ and different pairs (x, y).

Thus, we arrive at the following definition.

Definition 1. *We say that* there is a universal representation property *if every function $d(x, y)$ of two variables can be represented as the sum*

$$d(x, y) = \sum_{i=1}^{\infty} A_i(x) \cdot B_i(y), \tag{2}$$

so that for every pair (x, y), only finitely many terms in the sum are different from 0.

Resulting question. So, the resulting question is: is there a universal representation property?

III. Somewhat Surprising Answer: It All Depends on the Continuum Hypothesis

What we should intuitively expect. Intuitively, we expect:

- either a positive answer to the above question – i.e., a proof that the universal representation is possible,
- or a proof that such a universal representation is not possible.

The actual answer is different from our intuition. The actual answer is not what we would intuitively expect.

Proposition 1. [2] *The universal representation property is equivalent to the Continuum Hypothesis.*

Discussion. In a second, we will recall what is the Continuum Hypothesis, but first, let us explain what this result means: that whether we have a universal representation property depends on a somewhat obscure hypothesis from set theory.

What is the Continuum Hypothesis: reminder. In the late 19th century, Georg Cantor invented set theory, the theory which is now the foundations of mathematics. Among other interesting results, he proved that:

- while each infinite subset S of the set N of natural numbers is equivalent to N – in the sense that there is a 1-to-1 correspondence between N and S,
- the *continuum* – i.e., the set R of real numbers – is *not* equivalent to N (in the above sense).

Cantor conjectured that every infinite subset S of the continuum R is equivalent:

- either to N
- or to R.

This conjecture became known as *Continuum Hypothesis* (CH).

- Working mathematicians usually assume this hypothesis.
- However, specialists in foundations of mathematics were interested whether this hypothesis can be proven or disproven based on other – more intuitive – axioms.

This remained an open problem for a long time. The first breakthrough came from the famous logician Kurt Gödel, who proved that the negation of Continuum Hypothesis cannot be proven in set theory [3]. He proved it by showing that:

- if set theory is consistent, i.e., has a model,
- then, based on this model, we can build another model in which CH is true.

The question was settled in the 1960s, when Paul Cohen proved that Continuum Hypothesis is independent of set theory, i.e., we can neither prove not disprove it based on other axioms of set theory [1]. For this result, he was awarded the Fields Medal – the mathematical equivalent of the Nobel Prize.

Why this result is interesting. At first glance, the Continuum Hypothesis is

- an obscure statement of set theory,
- of little interest to working mathematicians
- (and probably of even less interest to applications of mathematics).

However, surprisingly,

- this *abstract* statement is equivalent to
- something much more *practical* and interesting: namely, the universal representation property for fuzzy systems.

Of course, one can argue that in practice, when everything is measured and implemented with some accuracy anyway, all we care about is the universal *approximation* property – but still, the universal representation property makes application sense: it shows that we can have an approximation in which:

- for every property $d(x, y)$ and for every pair (x, y),
- the number of non-zero terms (i.e., applicable expert rules) remains constant no matter how much accuracy we seek.

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