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***Abstract*—It is known that from the purely mathematical viewpoint, fuzzy sets can be interpreted as equivalent classes of random sets. This interpretations helps to teach fuzzy techniques to statisticians and also enables us to apply results about random sets to fuzzy techniques. The problem with this interpretation is that it is too complicated: a random set is not an easy notion, and classes of random sets are even more complex. This complexity goes against the spirit of fuzzy sets, whose purpose was to be simple and intuitively clear. From this viewpoint, it is desirable to simplify this interpretation. In this paper, we show that the random-set interpretation of fuzzy techniques can indeed be simplified: namely, we can show that fuzzy sets can be interpreted not as classes, but as *strongly consistent* random sets (in some reasonable sense). This is not yet at the desired level of simplicity, but this new interpretation is much simpler than the original one and thus, constitutes an important step towards the desired simplicity.**

I. Fuzzy Sets and Random Sets: A Brief Reminder and Formulation of the Problem

Fuzzy sets and random sets: a brief reminder for the knowledgeable readers. It is known that from the purely mathematical viewpoint, fuzzy sets can be interpreted as equivalent classes of random sets; see, e.g., [3]. (For readers who are not fully familiar with this interpretation, its main ideas will be presented in the following text.)

This interpretations is useful:

- it helps to teach fuzzy techniques to statisticians and
- it also enables us to apply results about random sets to fuzzy techniques.

The problem with this interpretation is that it is too complicated: a random set is not an easy notion, and classes of random sets are even more complex. This complexity goes against the spirit of fuzzy sets, whose purpose was to be simple and intuitively clear.

From this viewpoint, it is desirable to simplify this interpretation. In this paper, we show that the random-set interpretation of fuzzy techniques can indeed be simplified: namely, we can show that fuzzy sets can be interpreted:

- not as classes, but
- as *strongly consistent* random sets (in some reasonable sense).

This is not yet at the desired level of simplicity, but this new interpretation is much simpler than the original one and thus, constitutes an important step towards the desired simplicity.

Fuzzy sets and random sets: an explanation for the general fuzzy-related readers. To explain the problem to a general fuzzy-related reader, a reader who is:

- familiar with the concepts of fuzzy sets, but
- may not be very familiar with the technical details of random sets,

let us describe, in some detail, what are random sets and how they are related to fuzzy sets.

Random sets naturally appear in describing our knowledge and about ability to predict future events. So, to adequately describe random sets, we need to start with a brief reminder of the general prediction problem.

Predictions are important. One of the main applications of science and engineering is to predict future events – and, in the case of engineering, to come up with designs and controls for which the resulting future situation is the most beneficial.

For example, science predicts the position of the Moon in a few months, while engineering not only predicts the position of the spaceship in a month, but also describes the best trajectory correction that would bring the future location as close to the target as possible.

Some scientists say – correctly – that the main objective of science is to explain the world. But what does this mean in practical terms? The usual way to prove that a new physical theory explains the world better is to show that it enables us to give more accurate predictions of future events. This is how Einstein's General Relativity became accepted – when experiments confirmed its prediction of how much the light ray passing near the Sun will be distorted by the Sun's gravitational field.

Perfect knowledge is rarely available: need for set uncertainty. From the prediction viewpoint, perfect knowledge means that we know exactly what will happen in the future. Such a knowledge is rarely available, because that require a full knowledge of all the factors that can affect the future state. Usually, we have only partial knowledge. Thus, instead of a single future state, we have a *set* of future states.

This can be explained, e.g., as follows. One way to predict the future state is to look for similar situations in the past and to see what happened later in these situations. In the case of partial knowledge, we may have several different similar situations in the past. These situations are not identical, they are different, but they differ in the values of the quantities that we do not know, they all fit nicely with whatever information is available. Since the current situation is similar to one of the past one, in this case, all we can do is predict that the future situation will be similar to one of the corresponding outcomes.

From set uncertainty to probabilistic uncertainty. When we have many similar situations, we can determine not only which future states are possible, but also how frequent are different future states. For each of the possible future states $s_1, \ldots, s_n$, the observed frequency of this state serves as a natural estimate for the *probability* p_i of this state. Thus, in this case, we know the set of possible states $s_1, \ldots, s_n$, and we know the probabilities $p_1, \ldots, p_n$ of different possible states, probabilities adding up to 1: $\sum\limits_{i=1}^{n} p_i = 1$.

From probabilistic uncertainty to random set uncertainty. Past observations were also only partial. We did not get a full knowledge of a state, we get a partial knowledge. So, we have different observations $o_1, \ldots, o_n$ with probabilities p_i that add up to 1, but each observation o_i corresponds not to a single state, but the whole set S_i of possible states.

For example, then the measuring instrument records 1.0 in 40% of the cases, 1.1 in 20% of the cases, and 1.2 in the remaining 20% of the cases, and the accuracy of the measurement is 0.1, this means that:

- with probability 40%, we have values from the interval $S_1 = [0.9, 1.1]$;
- with probability 20%, we have values from the interval $S_2 = [1.0, 1.2]$; and
- with the remaining probability 20%, we have values from the interval $S_3 = [1.1, 1.3]$.

A situation in which we have several sets with different probabilities is known as a *random set* – similarly to how the situation when we have different numbers with different probabilities is known as a random number, and the situation when we have different vectors with different probabilities is known as a random vector.

We will consider finite sets. In practice, because of the limits of measurement accuracy, only finitely many different states are distinguishable. For example, even if we can measure lengths from 0 to 1 m with accuracy of 1 micron, we still have only a million possible values. Thus, in this paper, we will assume that our Universe of discourse U is finite.

Resulting definition of a random set. Once a finite set U is fixed, we can define a *random set* as a set of pairs (S_i, p_i), where $S_i \subseteq U$, $p_i > 0$, and $\sum\limits_{i=1}^{n} p_i = 1$.

Relation to fuzzy sets. A fuzzy set, for each possible state $x \in U$, describes the degree to which this state x is possible; see, e.g., [2], [4], [5]. We can gauge the degree of its possibility by describing the probability that x is possible with respect to the corresponding observation S_i – i.e., that $x \in S_i$. This probability is equal to $\sum\limits_{i: x \in S_i} p_i$. It is therefore reasonable to interpret the membership degree $\mu(x)$ as such a probability.

One can easily check that every membership function can indeed be thus interpreted. Indeed, let us start with any membership function, i.e., with the values $\mu(x_1), \ldots, \mu(x_n)$. Let us then sort these values in a decreasing order:

$$\mu(x_{(1)}) \geq \mu(x_{(2)}) \geq \ldots \geq \mu(x_{(n)}).$$

We can then define the following random set:

- we have $S_0 = \emptyset$ with probability $p_0 = \mu(x_{(1)})$;
- we have the set $S_1 = \{x_{(1)}\}$ with probability $p_1 = \mu(x_{(1)}) - \mu(x_{(2)})$;
- we have the set $S_2 = \{x_{(1)}, x_{(2)}\}$ with probability $p_2 = \mu(x_{(2)}) - \mu(x_{(3)})$;
- $\ldots$
- we have the set $S_k = \{x_{(1)}, \ldots, x_{(k)}\}$ with probability $p_k = \mu(x_{(k)}) - \mu(x_{(k+1)})$,
- $\ldots$,
- and, finally, we have the set $S_n = \{x_{(1)}, \ldots, x_{(n)}\}$ with probability $p_n = \mu(x_{(n)})$.

One can see that $\sum\limits_{i=1}^{n} p_i = 1$ and that for every k, we have $\sum\limits_{i: x_{(k)} \in S_i} p_i = \mu(x_{(k)})$.

For a *normalized* fuzzy set, for which $\max\limits_{k} \mu(x_k) = 1$, there is no need for a weird empty set.

There are other possible random sets that lead to the same fuzzy set μ. As a result, we can interpret a fuzzy set $\mu(x)$ as an equivalence class of random sets, namely, the class of all random sets for which, for every $x \in U$, we have $\sum\limits_{i: x \in S_i} p_i = \mu(x)$.

Comment. In general, a random set is nothing else but a mass distribution in the Dempster-Shafer approach. In this approach, the above relation between $\mu(x)$ and p_i can de described as saying that for every $x \in U$, the membership degree $\mu(x)$ is equal to the plausibility $\text{Pl}(\{x\})$ of the 1-element set $\{x\}$.

Current interpretation of fuzzy sets in terms of random sets: advantages and limitations. The above interpretation helps to teach fuzzy techniques to statisticians and also enables us to apply results about random sets to fuzzy techniques.

The main problem with this interpretation is that it is too complicated: a random set is not an easy notion, and classes of random sets are even more complex. This complexity goes against the spirit of fuzzy sets, whose purpose was to be simple and intuitively clear.

From this viewpoint, it is desirable to simplify this interpretation.

What we do in this paper. In this paper, we show that the random-set interpretation of fuzzy techniques can indeed be simplified: namely, we can show that fuzzy sets can be

interpreted not as classes, but as *strongly consistent* random sets (in some reasonable sense). This is not yet at the desired level of simplicity, but this new interpretation is much simpler than the original one and thus, constitutes an important step towards the desired simplicity.

II. Analysis of the Problem

Notion of consistency. In many cases, different alternative are inconsistent and thus, different sets S_i and S_j are disjoint: $S_i \cap S_j = \emptyset$. For example, if we have two different measurement results 1.0 and 1.3, both with accuracy 0.1, then the corresponding sets $S_i = [0.9, 1.1]$ and $S_j = [1.2, 1.4]$ are disjoint.

On the other hand, in many other cases, we have *consistency* in the sense that every two sets S_i and S_j have a non-empty intersection. This is true, e.g., for the random set that we used to represent a given fuzzy set. This is also true for the above example of three measurements 1.0, 1.1, and 1.2.

Let us require that the random set be consistent.

Consistency is not always preserved if we learn additional information. Sometimes, we learn an additional information, e.g., we learn that some alternative x is not possible. In this case, a previously consistent random set may stop being consistent.

For example, a random set with two sets $S_1 = \{x_1, x_2\}$ and $S_2 = \{x_2, x_3\}$ of equal probability $p_1 = p_2 = 0.5$ is clearly consistent: $S_1 \cap S_2 = \{x_2\} \neq \emptyset$.

However, if we learn that the alternative x_2 is not possible, this means that instead of the set S_1, we get a smaller set $S_1' = \{x_1\}$, and instead of the set S_2, we get a smaller set $S_2' = \{x_3\}$. So, the random set stops being consistent.

It is therefore reasonable to require that the random set is not only consistent by itself, but that it also remains consistent when learn additional information. We will call such random sets *strongly consistent*.

How does a random set change when we learn additional information. Let us analyze how a random set changes when we learn additional information. Suppose that we had the original Universe of discourse U and then we learn that only some of the original alternatives are possible.

Let $S \subseteq U$ denote the set of all possible alternatives. Then, if for some original set S_i, we have $S_i \cap S = \emptyset$, then this set is no longer possible. Only sets S_i for which $S_i \cap S \neq \emptyset$ remain; each such set becomes a smaller set $S_i' = S_i \cap S$. Since some sets S_i are no longer possible, the probability of the remaining sets changes according to the usual formula of conditional probability

$$p(S_i' \,|\, S_i \text{ is possible}) = \frac{p_i}{\sum\limits_{j: S_j \cap S \neq \emptyset} p_j}.$$

In Dempster-Shafer terms, the denominator is equal to the plausibility $\mathrm{Pl}(S)$ of the set S.

Some of the sets may become equal, so we will have to combine their probabilities. So, for each set $s \subseteq S$, we have

$$p'(s) = \frac{\sum\limits_{i: S_i \cap S = s} p_i}{\sum\limits_{i: S_i \cap S \neq \emptyset} p_i}.$$

Thus, we are ready for the following definition.

III. Definitions and the Main Result

Definition 1. *Let U be a finite set; we will call this set the* Universe of discourse*. By a* random set*, we mean a pair $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$, where $S_i \in U$, $p_i > 0$, and $\sum\limits_{i=1}^{n} p_i = 1$.*

Definition 2. *A* fuzzy set *μ is a function from U to $[0, 1]$. A fuzzy set $\mu(x)$ is called* normalized *if $\max\limits_x \mu(x) = 1$.*

Definition 3. *We say that a fuzzy set $\mu(x)$ is* consistent *with a random set $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ if for every $x \in U$, we have $\mu(x) = \sum\limits_{i: x \in S_i} p_i$.*

Definition 4. *By a* standard random set *$\mathcal{S}_\mu$* corresponding to the fuzzy set*, we mean that following random set: we sort the values $\mu(x)$ into the decreasing sequence $\mu(x_{(1)}) \geq \ldots \geq \mu(x_{(n)})$, and take $S_i = \{x_{(1)}, \ldots, x_{(i)}\}$ with $p_i = \mu(x_{(i)}) - \mu(x_{(i+1)})$ for $i < n$ and $p_n = \mu(x_{(n)})$.*

Comment. One can easily see that every fuzzy set $\mu(x)$ is consistent with the standard random set $\mathcal{S}_\mu$ corresponding to the fuzzy set.

Definition 5. *We say that a random set $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ is* consistent *if $S_i \cap S_j \neq \emptyset$ for all i and j.*

Definition 6.

- *We say that a set $S \subseteq U$ is* consistent *with a random set $\mathcal{S} = ((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ if $S \cap S_i \neq \emptyset$ for some i.*
- *If S is consistent with $\mathcal{S}$, we can define the* restriction *$\mathcal{S}_S$ of the random set to the set S as follows: it has non-empty sets s of the type $S_i \cap S$ with probabilities*

$$p'(s) = \frac{\sum\limits_{i: S_i \cap S = s} p_i}{\sum\limits_{i: S_i \cap S \neq \emptyset} p_i}.$$

- *We say that a random set is* strongly consistent *if all its restrictions are consistent.*

Comment. One can easily see that the standard random set corresponding to the fuzzy set is always strongly consistent. It turns out that the inverse is also true:

Proposition 1. *For every normalized fuzzy set $\mu(x)$, the only strongly consistent random set consistent with $\mu(x)$ is the standard random set $\mathcal{S}_\mu$ corresponding to $\mu(x)$.*

Discussion. One of the problems of the existing random-set interpretation of fuzzy sets is that in this interpretation,

each fuzzy set is associated with the whole class of random sets. Proposition 1 shows that if we restrict ourselves to strongly consistent random sets, then to each fuzzy set there corresponds a unique random set – instead of the whole class of random sets.

Thus, the new random-set-related interpretation of fuzzy sets – as strongly consistent random sets – is indeed simpler than then the existing one.

Proof of Proposition 1: main idea. Due to [1], it is sufficient to prove that if a random set is strongly consistent, then for every two sets S_i and S_j, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$.

We will prove this by contradiction. Let us assume that for some strongly consistent random set $\mathcal{S}$ and for some i and j, we have $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$, i.e., $S_i - S_j \neq \emptyset$ and $S_j - S_i \neq \emptyset$. Let us then consider the set $S = U - (S_i \cap S_j)$.

This set S is consistent with the original random set, since in this case, $S \cap S_i = S_i - S_j \neq \emptyset$ and $S \cap S_j = S_j - S_i \neq \emptyset$. Thus, in the restriction $\mathcal{S}_S$, we have elements $S'_i = S \cap S_i = S_i - S_j$ and $S'_j = S \cap S_j = S_j - S_i$. One can easily see, however, that the sets S'_i and S'_j do not have any common elements – which contradicts to our assumption that the original random set is strongly consistent and thus, that in each of its restrictions, every two sets have a non-empty intersection.

This contradiction proves that the sets S_i are indeed linearly ordered by inclusion and thus, by a result from [1], that this is indeed the standard random set associated with the given fuzzy set.

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