Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets? Fuzzy Sets And/Or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?

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Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets?  
Fuzzy Sets And/Or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?  

Christian Servin, Gerardo Muela, and Vladik Kreinovich

Abstract Fuzzy sets are naturally ordered by the subsethood relation \( A \subseteq B \). If we only know which set which fuzzy set is a subset of which – and have no access to the actual values of the corresponding membership functions – can we detect which fuzzy sets are crisp? In this paper, we show that this is indeed possible. We also show that if we start with interval-valued fuzzy sets, then we can similarly detect type-1 fuzzy sets and crisp sets.

1 Formulation of the Problem

Fuzzy sets: a brief reminder. A fuzzy set is usually defined as a function \( \mu : U \to [0, 1] \) from some set \( U \) (called Universe of discourse) to the interval \([0, 1]\); see, e.g., [1, 2, 3]. This function is also known as a membership function.

A fuzzy set \( A \) with a membership function \( \mu_A(x) \) is called a subset of a fuzzy set \( B \) with a membership function \( \mu_B(x) \) if \( \mu_A(x) \leq \mu_B(x) \) for all \( x \). The subsethood relation is an order in the sense that it is reflexive (\( A \subseteq A \)), asymmetric (\( A \subseteq B \) and \( B \subseteq A \) imply \( A = B \)), and transitive (\( A \subseteq B \) and \( B \subseteq C \) imply \( A \subseteq C \)).

Traditional (crisp) sets \( S \) can be viewed as particular cases of fuzzy sets, with their characteristic functions playing the role of membership functions: \( \mu_S(x) = 1 \) if \( x \in S \) and \( \mu_S(x) = 0 \) if \( x \notin S \).

A natural question: can we detect crisp sets based only on the subsethood ordering of fuzzy sets? If we have a class \( F \) of all fuzzy sets, and for each fuzzy
set $A$ and for each element $x \in U$, we know the value $\mu_A(x)$ of the corresponding membership function, then we can easily detect which of the fuzzy sets are crisp: a fuzzy set is crisp if for every $x \in U$, we have either $\mu_A(x) = 0$ or $\mu_A(x) = 1$.

Suppose now that we have a class $F$ of all fuzzy sets with the subsethood ordering $A \subseteq B$ – but we have no access to the actual values of the corresponding membership functions. Based only on this ordering relation $A \subseteq B$, can we then detect crisp sets?

**What if we only consider interval-valued fuzzy sets.** A similar question can be asked if we consider interval-valued fuzzy sets, for which the value of the membership function is a subinterval of the interval $[0, 1]: \mu(x) = [\mu(x), \overline{\mu}(x)] \subseteq [0, 1]$, and $A \subseteq B$ means that $\mu_A(x) \leq \mu_B(x)$ and $\overline{\mu}_A(x) \leq \overline{\mu}_B(x)$ for all $x$.

**What we do in this paper.** In this paper, we prove that in both cases – when we consider fuzzy sets and when we consider interval-valued fuzzy sets – we can indeed detect crisp sets and type-1 fuzzy sets based only on the subsethood relation $A \subseteq B$.

2 What If We Consider $[0, 1]$-Based Fuzzy Sets

**Our plan.** To describe crisp sets in terms of the subsethood relation $A \subseteq B$, we will follow the following four steps:

- first, we will prove that the empty set $\emptyset$ can be uniquely determined based on the subsethood relation;
- second, we will show that 1-element crisp sets, i.e., sets of the type $\{x_0\}$, can be thus determined,
- third, we will prove that 1-element fuzzy sets, i.e., fuzzy sets $A$ for which for some $x_0 \in U$, we have $\mu_A(x_0) > 0$ and $\mu_A(x) = 0$ for all $x \neq x_0$, can be determined based on the subsethood relation, and
- finally, we prove that crisp sets can be uniquely determined based on the subsethood relation.

**First step: how to detect an empty set?** An empty set $\emptyset$ is a fuzzy set for which $\mu_{\emptyset}(x) = 0$ for all $x \in U$. The detection of an empty set can be made based on the following simple result:

**Proposition 1.** A fuzzy set $A$ is an empty set if and only if $A \subseteq B$ for all fuzzy sets $B$.

**Proof.**

1°. Let us first prove that when $A = \emptyset$, then $A \subseteq B$ for all fuzzy sets $B$.

Indeed, for every fuzzy set $B$, we have $0 \leq \mu_B(x)$ for all $x$ and thus, $\mu_B(x) = 0 \leq \mu_B(x)$ for all $x$, i.e., we indeed have $\emptyset \subseteq B$.

2°. Let us now prove that, vice versa, if for some fuzzy set $A$, we have $A \subseteq B$ for every possible fuzzy set $B$, then $A = \emptyset$. 

Indeed, in particular, the property $A \subseteq B$ is true for the case when $B$ is the empty set. In this case, from the fact that $\mu_A(x) \leq \mu_B(x) = \mu_0(x) = 0$, we conclude that $\mu_A(x) = 0$ for all $x$, i.e., that $A$ is indeed the empty set.

The proposition is proven.

**Second step: how to detect 1-element crisp sets based on the subsethood relation.** Let us prove the following auxiliary result.

**Proposition 2.** A non-empty fuzzy set $A$ is a one-element crisp set if and only if the following two conditions are satisfied:

- the class $\{B : B \subseteq A\}$ is linearly ordered and
- for no proper superset $A'$ of $A$, the class $\{B : B \subseteq A'\}$ is linearly ordered.

**Proof.**

1°. Let us first prove that every 1-element crisp set, i.e., every set of the type $A = \{x_0\}$, satisfies the above two properties.

1.1°. Let us prove the first property: that the class $\{B : B \subseteq A\}$ is linearly ordered.

Indeed, for the given set $A$, we have $\mu_A(x_0) = 1$ and $\mu_A(x) = 0$ for all $x \neq x_0$. So, if $B \subseteq A$, i.e., if $\mu_B(x) \leq \mu_A(x)$ for all $x$, this means that $\mu_B(x) = 0$ for all $x \neq x_0$.

Thus, for such sets $B$, the only non-zero value of the membership function may be attained when $x = x_0$.

So, if we have two sets $B \subseteq A$ and $B' \subseteq A$, then for these two sets, $\mu_B(x) = \mu_{B'}(x) = 0$ for all $x \neq x_0$. Thus:

- if $\mu_B(x_0) \leq \mu_{B'}(x_0)$, then, as one can easily check, we have $\mu_B(x) \leq \mu_{B'}(x)$ for all $x$, i.e., we have $B \subseteq B'$, and
- if $\mu_B(x_0) \leq \mu_{B'}(x_0)$, then, as one can easily check, we have $\mu_B(x) \leq \mu_{B'}(x)$ for all $x$, we have $B' \subseteq B$.

Thus, for every two fuzzy sets $B$ and $B'$ from the class $\{B : B \subseteq A\}$, we have either $B \subseteq B'$ or $B' \subseteq B$. So, this class is indeed linearly ordered.

1.2°. Let us now prove that no proper superset $A'$ of the 1-element set $A = \{x_0\}$ has the property that the class $\{B : B \subseteq A'\}$ is linearly ordered.

For the set $A = \{x_0\}$, we have $\mu_A(x_0) = 1$ and $\mu_A(x) = 0$ for all other $x$. If $A'$ is a superset of $A$, this means that $\mu_{A'}(x) = 1$. The fact that $A'$ is a proper superset means that $A' \neq A$, thus we have $\mu_{A'}(x') > 0$ for some $x' \neq x_0$. In this case, we can define the following fuzzy set $B$: $\mu_B(x') = \mu_{A'}(x')$ and $\mu_B(x) = 0$ for all $x \neq x_0$. Then, we have $B \subseteq A'$, $A \subseteq A'$, but $B \not\subseteq A$ (since $\mu_B(x') > 0$) and thus, $\mu_B(x') \leq \mu_A(x') = 0$ and $A \not\subseteq B$ (since $1 = \mu_A(x_0) \leq \mu_B(x_0) = 0$). Thus, the class $\{B : B \subseteq A'\}$ is indeed not linearly ordered.

2°. Let us prove that, vice versa, if a fuzzy set $A$ has the above two properties, then it is a one-element crisp set.

2.1°. Let us first prove, by contradiction, that we can only have one element $x$ for which $\mu_A(x) > 0$. Indeed, if $\mu_A(x_1) > 0$ and $\mu_A(x_2) > 0$ for some $x_1 \neq x_2$, then we can take the following fuzzy sets $B_1$ and $B_2$:
• \( \mu_{B_1}(x_1) = \mu_A(x_1) \) and \( \mu_{B_1}(x) = 0 \) for all other \( x \), and
• \( \mu_{B_2}(x_2) = \mu_A(x_2) \) and \( \mu_{B_2}(x) = 0 \) for all other \( x \).

Here, \( B_1 \subseteq A \) and \( B_2 \subseteq A \), but \( B_2 \not\subseteq B_1 \) and \( B_1 \not\subseteq B_2 \) – which contradicts to our assumption that the class \( \{ B : B \subseteq A \} \) is linearly ordered.

2.2°. Due to Part 2.1, we have \( \mu_A(x_0) > 0 \) for at most one element \( x_0 \); for all \( x \neq x_0 \), we have \( \mu_A(x) = 0 \). Let us prove, by contradiction, that \( \mu_A(x_0) = 1 \), i.e., that \( A \) is indeed a one-element crisp set.

Indeed, if \( \mu_A(x_0) < 1 \), then we can consider the following proper superset \( A' \supseteq A \): \( \mu_{A'}(x_0) = (1 + \mu_A(x_0))/2 < 1 \) and \( \mu_{A'}(x) = 0 \) for all other \( x \). Similarly to Part 1.1 of this proof, we can prove that for this superset \( A' \), the class \( \{ B : B \subseteq A' \} \) is linearly ordered – which contradicts to our assumption that such a proper superset does not exist.

The proposition is proven.

Third step: how to detect 1-element fuzzy sets based on the subsethood relation.

We say that a fuzzy set is a 1-element set if for some \( x_0 \in X \), we have \( \mu_A(x_0) > 0 \) and \( \mu_A(x) = 0 \) for all \( x \neq x_0 \). Let us prove the following auxiliary result.

**Proposition 3.** A non-empty fuzzy set \( A \) is a one-element fuzzy set if and only if the class \( \{ B : B \subseteq A \} \) is linearly ordered.

**Proof.**

1°. Arguments similar to Part 1.1 of the proof of Proposition 2 show that if \( A \) is a one-element fuzzy set, then the class \( \{ B : B \subseteq A \} \) is linearly ordered.

2°. Vice versa, if \( A \) is not an empty set and not a one-element fuzzy set, this means that there exist at least two values \( x_1 \neq x_2 \) for which \( \mu_A(x_1) > 0 \) and \( \mu_A(x_2) > 0 \). We can then take the following fuzzy sets \( B_1 \) and \( B_2 \):

• \( \mu_{B_1}(x_1) = \mu_A(x_1) \) and \( \mu_{B_1}(x) = 0 \) for all \( x \neq x_1 \), and
• \( \mu_{B_2}(x_2) = \mu_A(x_2) \) and \( \mu_{B_2}(x) = 0 \) for all \( x \neq x_2 \).

Then \( B_1 \subseteq A \) and \( B_2 \subseteq A \), but \( B_1 \not\subseteq B_2 \) and \( B_2 \not\subseteq B_1 \). Thus, the class \( \{ B : B \subseteq A \} \) is not linearly ordered.

The proposition is proven.

Final result: how to detect crisp sets based on the subsethood relation. Let us prove the following auxiliary result.

**Theorem 1.** A fuzzy set \( A \) is crisp if and only if every one-element fuzzy subset \( B \subseteq A \) can be embedded in a one-element crisp subset of \( A \).

**Comment.** In other words,

\[
A \text{ is crisp} \iff \forall B (B \text{ is a one-element fuzzy subset of } A \Rightarrow \exists C ((B \subseteq C \subseteq A) \land (C \text{ is a 1-element crisp set})))
\]
Proof.

1°. Let $A$ be a crisp set, and let $B \subseteq A$ be a 1-element fuzzy set. By definition, this means that for some $x_0$, we have $\mu_B(x_0) > 0$ and $\mu_B(x) = 0$ for all other $x$.

Since the set $A$ is crisp, the only possible values of $\mu_A(x_0)$ are 0 and 1. From $\mu_B(x_0) \leq \mu_A(x_0)$, we conclude that $\mu_A(x_0) > 0$ and thus, that $\mu_A(x_0) = 1$. So, $x_0 \in A$ and hence $B \subseteq \{x_0\} \subseteq A$.

2°. Vice versa, if $A$ is not a crisp set, this means that for some element $x_0$, we have $0 < \mu_A(x_0) < 1$. In this case, we can take the following 1-element fuzzy set $B \subseteq A$: $\mu_B(x_0) = \mu_A(x_0)$ and $\mu_B(x) = 0$ for all $x \neq x_0$. Here, $B \subseteq A$, but the only 1-element crisp set $C$ containing $B$ is the set $C = \{x_0\}$, and this 1-element crisp set is not a subset of the original set $A$: $C \not\subseteq A$.

The theorem is proven.

3 What If We Consider Interval-Valued Fuzzy Sets

First step: how to detect an empty set. An empty set $\emptyset$ is an interval-valued fuzzy set for which $\mu_{\emptyset}(x) = [0,0]$ for all $x \in U$. The detection of an empty set can be made based on the following result:

Proposition 4. An interval-valued fuzzy set $A$ is an empty set if and only if $A \subseteq B$ for all interval-valued fuzzy sets $B$.

Proof is similar to proof of Proposition 1.

Second step: how to detect special 1-element interval-valued fuzzy sets based on the subsets relation. Let’s introduce an auxiliary notion. We say that an interval-valued fuzzy set $A$ is special if for some element $x_0$, we have $\mu_A(x_0) = [0,a]$ for some number $a > 0$ and $\mu_A(x) = 0$ for all $x \neq x_0$.

Proposition 5. A non-empty interval-valued fuzzy set $A$ is special if and only if the class $\{B : B \subseteq A\}$ is linearly ordered.

Proof.

1°. For special sets (in the sense of the above definition), the fact that the class $\{B : B \subseteq A\}$ is linearly ordered can be proven similarly to Part 1.1 of the proof of Proposition 2.

2°. Let us now prove that, vice versa, if for some non-empty interval-valued fuzzy set $A$, the class $\{B : B \subseteq A\}$ is linearly ordered, then the set $A$ is special.

2.1°. Since $A$ is non-empty, there exists an element $x_0$ for which $\mu_A(x_0) \neq [0,0]$. Let us prove, by contradiction, that for every other element $x \neq x_0$, we have $\mu_A(x) = [0,0]$.

Indeed, if we had $\mu_A(x_1) \neq [0,0]$ for some $x_1 \neq x_0$, then we would be able to take the following two sets $B_0$ and $B_1$:

- $\mu_{B_0}(x_0) = \mu_A(x_0)$ and $\mu_{B_0}(x) = [0,0]$ for all $x \neq x_0$, and
In this case, $B_0 \subseteq A$ and $B_1 \subseteq A$, but $B_0 \not\subseteq B_1$ and $B_1 \not\subseteq B_0$. This contradicts our assumption that the class $\{B : B \subseteq A\}$ is linearly ordered.

2.2°. To complete the proof of the proposition, we need to prove that the value $\mu_\mathcal{A}(x_0) = [\mu_\mathcal{A}^{-1}(x_0), \mathcal{P}_\mathcal{A}(x_0)]$ has the form $[0,a]$ for some $a > 0$, i.e., that $\mu_\mathcal{A}^{-1}(x_0) = 0$

We will prove it by contradiction. Suppose that, vice versa, $\mu_\mathcal{A}^{-1}(x_0) > 0$. In this case, we can take the following sets $B_1$ and $B_2$:

- $\mu_{B_1}(x_0) = [0.5 \cdot \mu_\mathcal{A}^{-1}(x_0), 0.5 \cdot \mu_\mathcal{A}^{-1}(x_0)]$ and $\mu_{B_1}(x) = 0$ for all $x \neq x_0$, and
- $\mu_{B_2}(x_0) = [0, \mu_\mathcal{A}^{-1}(x_0)]$ and $\mu_{B_2}(x) = 0$ for all $x \neq x_0$.

Then, $B_1 \subseteq A$ and $B_2 \subseteq A$, but $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. This contradicts our assumption that the class $\{B : B \subseteq A\}$ is linearly ordered.

The proposition is proven.

**Third step: how to detect 1-element type-1 fuzzy sets based on the subsethood relation.** We say that an interval-valued fuzzy set is a 1-element type-1 fuzzy set if there exists an element $x_0$ for which $\mu_\mathcal{A}(x_0) = [a,a]$ for some $a > 0$ and $\mu_\mathcal{A}(x) = [0,0]$ for all $x \neq x_0$.

**Proposition 6.** A non-empty interval-valued fuzzy set $\mathcal{A}$ is a 1-element type-1 set if and only if it satisfies the following three properties:

- the set $\mathcal{A}$ is not special (in the sense of the above definition),
- there exists a special set $\mathcal{B} \subseteq \mathcal{A}$ for which the class $\{C : \mathcal{B} \subseteq C \subseteq \mathcal{A}\}$ is linearly ordered, and
- for no proper superset $\mathcal{A}'$ of $\mathcal{A}$, the class $\{C : \mathcal{B} \subseteq C \subseteq \mathcal{A}\}$ is linearly ordered.

**Proof** is similar to the proof of Proposition 2.

**Final result.** Since we have subsethood, we also have union: the union of $\mathcal{A}_\alpha$ is the $\subseteq$-smallest set that contains all $\alpha$. We can thus define type-1 fuzzy sets as unions of 1-element type-1 fuzzy sets. Once we can detect type-1 fuzzy sets, we can use techniques from the previous section to detect crisp sets. Thus, we can indeed detect type-1 fuzzy sets and crisp sets based only on subsethood relation between interval-valued fuzzy sets.

**References**