Fuzzy Pareto Solution in Multi-criteria Group Decision Making with Intuitionistic Linguistic Preference Relation

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Fuzzy Pareto Solution in multi-criteria group decision making with intuitionistic linguistic preference relation

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Abstract: In this paper, we investigate the multi criteria group decision making with intuitionistic linguistic preference relation. The concept of Fuzzy Collective Solution (FCS) is used to evaluate and rank the candidate solution sets for modeling under linguistic assessments. Intuitionistic linguistic preference relation and associated aggregation procedures are then defined in a new concept of Fuzzy Pareto Solution. Numerical examples are presented to demonstrate computing procedures. The results affirm efficiency of the proposed method.

Keywords: Fuzzy Collective Solution; Fuzzy Pareto Solution; Group Decision Making; Intuitionistic Fuzzy Sets; Intuitionistic Linguistic Preference Relation.

1. Introduction

Group decision making (GDM) is useful to evaluate and judge the best solution among alternatives through a social group. GDM, in a fuzzy environment, contains four steps (Herrera, Herrera-Viedma & Verdegay, 1996): i) unifying evaluations from experts; ii) aggregating their opinions to a final score for each alternative represented by a linguistic label (Chen & Hwang, 1992; Cheng, 1999; Delgado, Verdegay & Vila, 1992; Fodor & Roubens, 2013; Herrera & Herrera-Viedma, 1997; Hwang & Lin, 2012); iii) ranking the labels; and iv) selecting the preferred alternative by group decision. It has been the fact that using linguistic labels makes experts’ judgment more reliable and informative than using numeric scores. Fuzzy Collective Solution (FCS) is widely used to
evaluate and rank the candidate solution for a model under linguistic assessments. The existing algorithms for FCS provided the tool for aggregation in GDM with intuitionistic preference relations.

Intuitionistic fuzzy set (IFS) is an extension of Zadeh’s fuzzy set (1965) characterized by a membership and a non-membership function (Atanassov, 1986, 1999). In many complex decision making problems, the decision can be represented by IFS for better modelling of imprecise and uncertain data. Recently, some researchers applied IFS to GDM. Gau and Buehrer (1993) introduced the vague set, which is an equivalence of IFS. Hong and Choi (2000) developed approximate techniques for multi-attribute decision making using minimum and maximum operators. Szmidt and Kacprzyk (2002) proposed the intuitionistic fuzzy core and consensus winner in group decision making with intuitionistic (individual and social) fuzzy preference relations. Xu and Yager (2006) developed the intuitionistic fuzzy weighted geometric operator, fuzzy ordered and fuzzy hybrid geometric operator extending the traditional and ordered weighted geometric operator in the environment where given arguments are represented by IFSs.

Our contribution for GDM with intuitionistic preference relation is the new approach using the Pareto solution of the optimization theory and some aggregation procedures based on FCS to solve the model. This paper is organized in 8 sections with the first section is the introduction. Section 2 briefly reviews group decision making model and Low-operator. Section 3 introduces the concept of Fuzzy Collective Solution. Some computing algorithms for FCS of the multi-criteria problem are given in the Section 4. In section 5, intuitionistic linguistic preference relations are defined. Section 6 is devoted to a new concept of Fuzzy Pareto Solution (FPS) and some aggregation procedures for the FPS in the problems with intuitionistic linguistic preference relations. Section 7 shows an example to illustrate the approach. The final section is conclusion of this paper.

2. A group decision making model

2.1. A GDM model under linguistic

In this paper, let us consider the following model under linguistic assessments. Assume \( X = \{x_1, x_2, \ldots, x_n\} \) is a finite set of alternatives and \( E = \{e_1, \ldots, e_m\} \) is a finite set of experts. We assume that there exists a distinguished person, say the manager, who assigns an important degree \( w(e_k) = w(k) \) for each expert such that \( 0 \leq w(k) \leq 1 \), and \( \sum_k w(k) = 1 \).

Let \( S = \{s_t, t = 1, \ldots, T\} \) be a finite and totally ordered linguistic-labels set. Assume that each expert \( e_k \in E \) provides his/her opinions on \( X \) by mean of a linguistic preference relation \( p_k : X \times X \rightarrow S \), where \( p_k(i, j) = p_k(x_i, x_j) \in S \), represents the linguistically assessed preference degree of alternative \( x_i \) over \( x_j \). For example, we consider the following linguistic labels \( S \) [12]:
Fuzzy Pareto Solution in multi-criteria group decision making

\[ S = \{ I, EU, VLC, SC, IM, MC, ML, EL, C \} \]

where they are trapezoidal fuzzy numbers on [0,1].

- I: Impossible (0, 0, 0, 0);
- EU: Extremely-unlikely (0.00, 0.01, 0.02, 0.07);
- VLC: Very-low-chance (0.17, 0.22, 0.36, 0.42);
- SC: Small-chance (0.17, 0.22, 0.36, 0.42);
- IM: It-may (0.32, 0.41, 0.58, 0.65);
- MC: Meaningful-chance (0.58, 0.63, 0.80, 0.86);
- ML: Most-likely (0.72, 0.78, 0.92, 0.97);
- EL: Extremely-likely (0.93, 0.98, 0.99, 1);
- C: Certain (1, 1, 1, 1);

2.2. A modification of LOWA operator

Yager (1998) defined the ordered weighted averaging (OWA) operator. The LOWA operator is based on OWA and the convex combination of linguistic labels (Llamazares, 2007). However, in many real cases, the weights of experts were not considered. Thus, we are concerning the aggregation problem in which the weights of experts are unknown.

Definition 1. Let \( a = \{ a_1, \ldots, a_m \} \) be a set of linguistic-labels to aggregate, and \( b \) is the associated ordered labels vector, i.e. \( b = \{ a_{im}, a_{i(m-1)}, \ldots, a_i \} \), such that \( a_{im} \geq a_{i(m-1)} \geq \ldots \geq a_i \).

The Low operator is defined as

\[
\text{Low}(a, w) = C\left( (w_{im}, a_{im}), (1-w_{im}, \text{Low}(a', w')) \right)
\]

where \( w = [w_1, \ldots, w_m] \), is a weight vector such that, \( w_i \in [0,1] \) and \( \sum w_i = 1 \),

\[
a' = [a_{im-1}, \ldots, a_1], \quad w' = [w'_{im-1}, \ldots, w'_1], \quad w'_j = \frac{w_j}{1-w_{im}}.
\]

\( C \) is the convex combination operator of 2 labels \( s_j, s_i, j \geq i \) with \( w_j > 0, w_i > 0, w_j + w_i = 1 \),

\[
C\left( (w_j, s_j), (w_i, s_i) \right) = s_k,
\]

where \( k = i + \text{round}(w_j, (j-i)) \), where round is the usual round operator.

2.3. Multi-criteria group decision making model

In our model, assume that there is a finite set of criteria: \( C = \{ C_1, C_2, \ldots, C_L \} \). Respect to each criterion \( C_l \), each expert \( e_k \in E \) provides his/her opinions on \( X \) by mean of a linguistic preference relation

\[
p_{il} : X \times X \rightarrow S, \quad \text{where} \quad p_{il}(i, j) = p_{il}(x_i, x_j) \in S
\]

is the linguistically assessed preference degree of alternative \( x_i \) over \( x_j \). Moreover, assume that there are important weights of criteria \( \{ \beta_l, l=1,2,\ldots, L \} \), such that \( 0 \leq \beta_l \leq 1 \), and \( \sum \beta_l = 1 \).

In our model we assume that \( p_{il} : X \times X \rightarrow S \) is soft-recipodal in the sense:

\( i \) \( p_{il}(i, i) = s_{\frac{i+1}{2}} \), for all \( i=1,\ldots,n \)
(ii) \( p_{kl}(i, j) \geq s_{T+1/2} \), then \( p_{kl}(j, i) \leq s_{T+1/2} \) \hspace{1cm} (6)

Let \( \{p_k, k = 1, ..., m\} \) be linguistic preference relations. For every \((i, j), i = 1, ..., n, j = 1, ..., n\), and for each \( s_t \in S \), we define

\[ W_{i,j}[s_t] = \sum_k \{w(k) : p_k(i, j) = s_t\}. \hspace{1cm} (7)\]

This value is the sum of individual importance degrees of experts, who coincide to assign the linguistic label \( s_t \) as preference value of the alternative \( x_i \) over alternative \( x_j \).

In the next section, we will define the concept of Fuzzy Collective Solution (FCS) and apply to evaluate and to rank the alternatives set.

3. **Fuzzy Collective Solution**

3.1. **Notions**

The concept of fuzzy collective solution is similar to the concept of aggregated dominance degrees of alternatives. For each pair \((x_i, x_j)\), the linguistic dominance degree is defined by

\[ E(x_i, x_j) = Low(S, U) \hspace{1cm} (8)\]

where \( U = [u_1, ..., u_T] \), \( u_t = W_{i,j}[s_t] \) for \( t = 1, ..., T \).

**Definition 2.** The **fuzzy collective solution** (FCS) is a fuzzy set on the alternatives set \( X \)

\[ FCS = \{\text{fcs}(x_1)/x_1, \text{fcs}(x_2)/x_2, ..., \text{fcs}(x_n)/x_n\} \hspace{1cm} (10)\]

where the membership degree of the alternative \( x_i \) is calculated as follows:

\[ \text{fcs}(x_i) = Low(S, V) \hspace{1cm} (11)\]

where \( V = [v_1, ..., v_T] \), for \( t = 1, ..., T \), \( v_t = \#\{j : E(x_i, x_j) = s_t\}/n - 1 \).

3.2. **Aggregated Fuzzy Collective Solution**

Now, we give a notion for aggregation of FCS in multi-criteria decision making problems. Suppose that for each criterion \( C_l \), we obtained the corresponding fuzzy collective solution:

\[ FCS_l = \{\text{fcs}_l(x_1)/x_1, ..., \text{fcs}_l(x_n)/x_n\} \hspace{1cm} l = 1, 2, ..., L. \]

**Definition 3.** The **aggregated fuzzy collective solution** (aFCS) is a fuzzy set on the alternatives set \( X \)

\[ aFCS = \{\text{afcs}(x_1)/x_1, \text{afcs}(x_2)/x_2, ..., \text{afcs}(x_n)/x_n\} \hspace{1cm} (13)\]
where the membership degree of the alternative $x_i$ is calculated as

$$afcs(x_i) = \text{Low}(S, U_{\beta})$$

where $U_{\beta} = \left[ u_{\beta_1}, ..., u_{\beta_T} \right]$, $U_{\beta_t} = \sum_i \{ \beta_t : \text{fcs}(x_i) = s_t \}$, $t=1,...,T$ (14)

Note that $aFCS$ is a type-2 fuzzy set on $X$.

4. Some computing procedures using FCS

In the following computing procedures, we use the linguistic preference relations \( \{ p_{kl}, k=1,...,m, l=1,...,L \} \), the experts’ important weights \( \{ w(k) : e_k \in E \} \), such that \( 0 \leq w(k) \leq 1 \) and \( \sum_k w(k) = 1 \), and the criteria’s important weights \( \{ \beta_l, l=1,...,L \} \), such that \( 0 \leq \beta_l \leq 1 \) and \( \sum_l \beta_l = 1 \). Now we consider some computing procedures using FCS for the model.

Algorithm 1:

Step 1.

1a. For each criterion $C_l$, using \( \{ p_{kl}, k=1,...,m \} \) and the weights \( \{ w(k) : e_k \in E \} \) calculate the linguistic dominance degrees

$$E_{li} = \left[ E_{li}(i, j) \right] = \left[ \text{Low}(E_{li}(i, j), U_{\beta_t}) \right]$$

where

$$E_{li}(i, j) = \text{Low}(S, U_{\beta_t})$$

$$U_{\beta_t} = \left[ u_{\beta_1}, ..., u_{\beta_T} \right]$$

(16)

and the important weights \( \{ \beta_l, l=1,...,L \} \), such that \( 0 \leq \beta_l \leq 1 \) and \( \sum_l \beta_l = 1 \). Now we consider some computing procedures using FCS for the model.

Step 2. Using the \( \{ FCS_l, l=1,...,L \} \) and the important weights \( \{ \beta_l, l=1,...,L \} \), calculate the aggregated fuzzy collective solution (aFCS)

$$aFCS = \left\{ afcs(x_1) / x_1, afcs(x_2) / x_2, ..., afcs(x_n) / x_n \right\}$$

(17)

as in the Definition 3.

Step 3. Classify the set of alternatives $X$ into subsets
\[ X_t = \{ x_i : \text{afcs}(x_i) = s_t \}, \quad \text{with} \quad t = 1, \ldots, T \]  
(21)

Rank the non-empty sets \( X_t \prec X_{t'} \), iff \( t < t' \), and therefore, the solution is the set \( X^* \), where \( t^* = \max \{ t : X_t \neq \emptyset, s_t \in S \} \).

**Algorithm 2:**

*Step 1.* For each criterion \( C_i \), calculate the linguistic dominance degrees as in (16).

\[ E_i = \left[ E_i(x_i, x_j) \right], \quad i, j = 1, \ldots, n, \quad l = 1, \ldots, L \]

*Step 2.*

1. Using \( \{ E_i, \ l = 1, \ldots, L \} \) and the important weight \( \{ \beta_l, \ l = 1, \ldots, L \} \), calculate the *totally social opinion relation*:

\[ Q = \left[ q(i, j) \right] = \left[ q(x_i, x_j) \right], \quad i, j = 1, \ldots, n \]

(22)

where \( q(x_i, x_j) = \text{Low}(S, U_q) \), \( U_q = \left[ u_{ql}, \ldots, u_{qn} \right] \)

(23)

where \( u_q = W_q^q(s_t) = \sum_{q=1}^{T} \beta_q : E_q(i, j) = s_t \), \( t = 1, \ldots, T \)

(24)

2. Calculate the fuzzy collective solution according to the totally social opinion relation \( Q \)

\[ \text{FCS}_Q = \left\{ \text{fcs}_q(x_i) / x_1, \text{fcs}_q(x_2) / x_2, \ldots, \text{fcs}_q(x_n) / x_n \right\} \]

(25)

**Algorithm 3:**

*Step 1.*

1. For each expert \( e_k \), using \( \{ p_{kl}, \ l = 1, \ldots, L \} \) and the important weights \( \{ \beta_l, \ l = 1, \ldots, L \} \), calculate the *relatively dominance degrees* \( F^k \) according to expert \( e_k \) as

\[ F^k = \left[ F^k(i, j) \right] = \left[ F^k(x_i, x_j) \right], \quad i, j = 1, \ldots, n \]

(27)

where \( F^k(x_i, x_j) = \text{Low}(S, U^k) \)

and \( U^k = [u_{kl}^k, \ldots, u_{kn}^k] \) where \( u_{kl}^k = \sum_{q=1}^{T} \beta_q : p_{kl}(i, j) = s_t \), \( t = 1, \ldots, T \)

(28)

1. Using \( F^k \), calculate the *fuzzy evaluation* \( FE^k \) according to the expert \( e_k \)
\[
FE^k = \left\{ \frac{fe^k(x_i)}{x_1}, \ldots, \frac{fe^k(x_i)}{x_n} \right\}
\]

with the membership degree of the alternative \( x_i \) is calculated as
\[
fe^k(x_i) = \text{Low}(S, V_k), \quad i=1, \ldots, n
\]
where \( V_k = [v_1, \ldots, v_n] \), \( v_i = \frac{\left| \left\{ j : F^k(x_i, x_j) = s_i, j \neq i \right\} \right|}{n-1}, i=1, \ldots, T \) (31)

**Step 2.** Using the fuzzy evaluations \( \{FE^k, k = 1, \ldots, m\} \) and the weights \( \{w(k) : e_k \in E\} \), calculate the aggregated fuzzy evaluation \( aFE \), which is a fuzzy set on the alternatives set \( X \):
\[
aFE = \left\{ \frac{afe(x_i)}{x_1}, \ldots, \frac{afe(x_i)}{x_n} \right\},
\]
as in Definition 3.

**Step 3.** Classify the set of alternatives \( X \) into subsets
\[
X_t = \left\{ x_i : afe(x_i) = s_t \right\}, \quad t = 1, \ldots, T
\]
and choose the solution as in the step 3 of the Algorithm 1.

**5. Intuitionistic Linguistic Preference Relation**

**5.1. Intuitionistic Fuzzy Sets**

**Definition 4** (Atanassov, 1986). Let \( Y \) be a universe of discourse. Then an **intuitionistic fuzzy set** (IFS)
\[
A = \left\{ ((\mu_A(y), \upsilon_A(y))) / y \in Y \right\}
\]
is characterized by a membership function \( \mu_A : Y \rightarrow [0, 1] \), and a non-membership function \( \upsilon_A : Y \rightarrow [0, 1] \), with the condition \( 0 \leq \mu_A(y) + \upsilon_A(y) \leq 1 \) for all \( y \in Y \), where the numbers \( \mu_A(y) \) and \( \upsilon_A(y) \) represent, respectively, the degree of membership and the degree of non-membership of the element \( y \) to the set \( A \).

**Definition 5** (Xu, 2007). Let \( X = \{x_1, x_2, \ldots, x_n\} \). An **intuitionistic preference relation** \( B \) on the set \( X \) is represented by a matrix:
\[
B = \left[ b_{ij} \right]_{n \times n}, \quad b_{ij} = \left\{ (\mu(x_i, x_j), \upsilon(x_i, x_j)) / (x_i, x_j) \right\}, \quad \text{for all } i, j = 1, 2, \ldots, n
\]
where $\mu_{ij} = \mu(x_i, x_j) \in [0,1]$ is the preference degree of $x_i$ in comparing with $x_j$ and 
$\nu_{ij} = \nu(x_i, x_j) \in [0,1]$ is the non-preference degree of $x_i$ in comparing with $x_j$.

In [33], Xu supposed that $\mu_{ij} = \mu(x_i, x_j)$ and $\nu_{ij} = \nu(x_i, x_j)$ satisfy the following condition:

$$0 \leq \mu_{ij} + \nu_{ij} \leq 1, \quad \mu_{ji} = \nu_{ji}, \quad \nu_{ii} = \mu_{ii} = 0.5,$$

for all $i, j = 1, 2, ..., n$. (36)

Next, we will define the intuitionistic linguistic preference relation and will present a new approach to solve the group decision problems with intuitionistic linguistic preference relations.

### 5.2. Intuitionistic linguistic preference relation

**Definition 6.** Let $X = \{x_1, x_2, ..., x_n\}$ be an alternatives set. Let $S$ be a finite and totally ordered linguistic labels set: $S = \{s_t, t=1,..., T\}$ where $T$ is an odd number. An intuitionistic linguistic preference relation $P$ on $X$ is a matrix

$$P = \left[ \begin{array}{cccc} p_{ij} \\ \hline n, n \\ \end{array} \right]$$

where $p_{ij} = \{(\mu(x_i, x_j), \nu(x_i, x_j))/n(x_i, x_j)\}$, for all $i, j = 1, 2, ..., n$. (37)

where $\mu_{ij} = \mu(x_i, x_j) \in S$ is the linguistic preference degree of $x_i$ in comparing with $x_j$ and $\nu_{ij} = \nu(x_i, x_j) \in S$ is the linguistic non-preference degree of $x_i$ in comparing with $x_j$.

**Definition 7.** (Soft-reciprocal condition) An intuitionistic linguistic preference relation $P$ on the set $X$ is said to be soft-reciprocal if

$$\mu_{ij} = \mu(x_i, x_j) \geq S_{\frac{T+1}{2}}, \quad \text{then} \quad \mu_{ji} = \mu(x_j, x_i) < S_{\frac{T+1}{2}} \quad \text{and} \quad \nu_{ij} = \nu(x_i, x_j) < S_{\frac{T+1}{2}},$$

for all $i, j = 1, 2, ..., n$. (38)

**Note 1.** This condition on preference relation is much weaker than other supposed conditions in the papers on decision making problems using preference relations. For example see (Xu, 2006).

### 6. Fuzzy Pareto Solution

In our approach, we need the Pareto Solution concept of Optimization Theory (Tuy, 1998).
6.1. Pareto Solution in Optimization Theory

Let \( D \neq \emptyset \). Let \( f : D \to \mathbb{R}^n \) be a real function, i.e. for each \( x \in D \),

\[
 f(x) = (f_1(x), f_2(x), \ldots, f_j(x), \ldots, f_n(x)).
\]  

(39)

**Definition 8 (Pareto Solution).** Consider the following optimization problem

\[
 f(x) \to \max
\]

subject to \( x \in D \).

A \( x^* \in D \) is a Pareto solution, if there not exists \( x' \in D \) such that

\[
 f(x^*) \leq f(x') \quad \text{and} \quad f(x^*) \neq f(x'),
\]

i.e.,

(40)

for any \( x' \in D \), if \( f_j(x^*) < f_j(x') \) for some \( j \), then there is \( k \neq j \) such that

\[
 f_k(x^*) < f_k(x').
\]

A generalization of this concept is the following.

6.2. Generalized Pareto Solution

Let \( D \neq \emptyset \), \( S_i \), \( i = 1, 2, \ldots, n \) be totally ordered sets, \( S = S_1 \times S_2 \times \ldots \times S_n \)

(41)

Let \( g : D \to S \) be a map from \( D \) to \( S \).

For each \( x \in D \),

\[
 g(x) = (g_1(x), g_2(x), \ldots, g_n(x)).
\]

(42)

**Definition 9.** (Generalized Pareto Solution)

A \( x^* \in D \) is a generalized Pareto solution if there not exists \( x' \in D \) such that

\[
 g(x^*) \leq g(x') \quad \text{and} \quad g(x^*) \neq g(x'),
\]

i.e.,

(43)

for any \( x' \in D \), if \( g_j(x^*) < g_j(x') \) for some \( j \), then there is \( k \neq j \) such that

\[
 g_k(x^*) < g_k(x').
\]

(44)

We will see that this new definition is a useful concept in the group decision making problems with intuitionistic linguistic preference relations.

6.3. Fuzzy Pareto Solution

**Definition 10.** (Fuzzy Pareto Solution)

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be an alternatives set. The evaluation information of the expert

\( e_k \) about \( X \) is a IFS on \( X \)

\[
 \{(p^k_{1M}(x_1), p^k_{1V}(x_1))/x_1, \ldots, (p^k_{nM}(x_n), p^k_{nV}(x_n))/x_n\}, \quad k = 1, 2, \ldots, m
\]

\( 1 \leq k \leq m \).
Let $S_1, S_2$ be totally ordered linguistic-label sets.

Suppose that there is an aggregated evaluation map $g : X \rightarrow S_1 \times S_2$

\[
g(x) = \{ (g_M(x_i), g_V(x_i)) / x_i, \ldots, (g_M(x_n), g_V(x_n)) / x_n \} \quad (45)
\]

For each $x_i \in X$, $i = 1, \ldots, n$

\[g(x_i) = (g_M(x_i), g_V(x_i)) \text{ is a linguistic IFS on } X,
\]

where $g_M(x_i) \in S_1$ is the first membership degree and $g_V(x_i) \in S_2$ is the second membership degree. A $x^* \in X$ is an Fuzzy Pareto Solution (FPS), if there not exists $x' \in D$ such that:

\[
\begin{align*}
g_M(x^*) &< g_M(x') \quad \text{and} \quad g_V(x') \leq g_V(x^*), \quad \text{or} \\
g_M(x^*) &\leq g_M(x') \quad \text{and} \quad g_V(x') < g_V(x^*).
\end{align*}
\]

In the case that $S_1 = S_2 = S$ is a linguistic labels set defined as in Section 1, FPS is a type-2 IFS.

6.4. Some aggregation procedures

We consider the following multi-criteria problem:

Given the alternatives set $X = \{x_1, x_2, \ldots, x_n\}$.

The criteria set $\{C_1, C_2, \ldots, C_L\}$.

The expert set $E = \{e_1, \ldots, e_m\}$ with their importance $\{w(k) : e_k \in E\}$ such that

\[0 \leq w(k) \leq 1,\]

and $\sum w(k) = 1$. Given criteria’s important weights $\{\beta_1, \beta_2, \ldots, \beta_L\}$ such that

\[0 \leq \beta_k \leq 1, \quad \text{and} \quad \sum \beta_k = 1.
\]

The linguistic labels set:

$S = \{s_1 = I, s_2 = EU, s_3 = VLC, s_4 = SC, s_5 = IM, s_6 = MC, s_7 = ML, s_8 = EL, s_9 = C\}$

We assume that each expert $e_k \in E$ provides his/her opinions on $X$ by mean of intuitionistic linguistic preference relations

\[P_{il} = (M_{il}, V_{il}), \quad k = 1, \ldots, m, \quad l = 1, \ldots, L \quad (48)
\]

satisfying the soft-reciprocal condition.

The linguistic preference relations are:

\[M_{il} = [\mu_{il}(x_i, x_j)], \quad \text{for } i, j = 1, \ldots, n, \quad \mu_{il}(x_i, x_j) \in S \quad (49)
\]

and the linguistic non-reference relations are

\[V_{il} = [\nu_{il}(x_i, x_j)], \quad \text{for } i, j = 1, \ldots, n, \quad \nu_{il}(x_i, x_j) \in S, \quad (50)
\]
We have to evaluate and to choose the solution sets based on the given intuitionistic linguistic preference relations. Using the algorithms in Section 4, we present three aggregation procedures for FPS.

**Aggregation procedure 1:**

Step 1.

1.1.a. For each criterion \( C_l \), using \( p_{ik} = (M_{ik}, V_{ik}) \), \( k = 1, \ldots, m \) and the weights \( \{ w(k) : e_k \in E \} \), calculate the linguistic dominance degrees:

\[
E_{M}^{l}(x_i, x_j) = \text{Low}(S, U_{M}), \quad i, j = 1, \ldots, n
\]

(51)

where \( U_{ML} \) is calculated as (18) with \( p_{il} = M_{il} \).

1.1.b. For each criterion \( C_l \), using the relation \( E_{M}^{l} \), calculate the FCS according to each criterion \( C_l \):

\[
FCS_{M}^{l} = \left\{ \frac{\text{fcs}_{M}^{l}(x_i)}{x_i}, \ldots, \frac{\text{fcs}_{M}^{l}(x_n)}{x_n} \right\}, \quad l = 1, \ldots, L
\]

(52)

as in Definition 2.

1.2.a. For each criterion \( C_l \), calculate the linguistic non-dominance degrees:

\[
E_{V}^{l}(x_i, x_j) = \text{Low}(S, U_{V}), \quad i, j = 1, \ldots, n
\]

(53)

where \( U_{VL} \) is calculated as (18) with \( p_{il} = V_{il} \).

1.2.b. For each criterion \( C_l \), using the relation \( E_{V}^{l} \), calculate the FCS according to each criterion \( C_l \):

\[
FCS_{V}^{l} = \left\{ \frac{\text{fcs}_{V}^{l}(x_i)}{x_i}, \ldots, \frac{\text{fcs}_{V}^{l}(x_n)}{x_n} \right\}, \quad l = 1, \ldots, L
\]

(54)

as in Definition 2.

Now we obtain Intuitionistic Fuzzy Evaluation

\[
IFE^{l} = \left\{ \left( \frac{\text{fcs}_{M}^{l}(x_i)}{x_i}, \frac{\text{fcs}_{V}^{l}(x_i)}{x_i} \right) / x_i, \ldots, \left( \frac{\text{fcs}_{M}^{l}(x_n)}{x_n}, \frac{\text{fcs}_{V}^{l}(x_n)}{x_n} \right) / x_n \right\}
\]

(55)

Step 2

Using the \( \{(FCS_{M}^{l}, FCS_{V}^{l}), l = 1, \ldots, L\} \) and the important weights \( \{\beta_1, \beta_2, \ldots, \beta_L\} \), calculate the aggregated fuzzy collective solutions (aFCS):

\[
aFCS_{M} = \left\{ \frac{afcs_{M}(x_i)}{x_i}, \ldots, \frac{afcs_{M}(x_n)}{x_n} \right\}
\]

and

(56)

\[
aFCS_{V} = \left\{ \frac{afcs_{V}(x_i)}{x_i}, \ldots, \frac{afcs_{V}(x_n)}{x_n} \right\}
\]

(57)

as in Definition 3.

Finally, we obtain Aggregated Intuitionistic Fuzzy Evaluation (aIFE):

\[
aIFE = \left\{ \frac{afcs_{M}(x_i)}{x_i}, \frac{afcs_{V}(x_i)}{x_i}, \ldots, \frac{afcs_{M}(x_n)}{x_n}, \frac{afcs_{V}(x_n)}{x_n} \right\}
\]

(58)
Using aIFE, we choose Aggregated Fuzzy Pareto Solution (aFPS) of the problem.

**Aggregation procedure 2:**

Step 1.
For each criterion $C_i$, using $p_{kl} = (M_{kl}, V_{kl})$, $k = 1, ..., m$ and the weights $\{w(k) : e_k \in E\}$, calculate the linguistic dominance degrees:

$$E_{M}^i(x_i, x_j) = \text{Low}(S, U_{M})$$ \quad i, j = 1, ..., n \tag{59}

where $U_{M}$ is calculated as (18) with $p_{kl} = M_{kl}$.

Calculate the linguistic non-dominance degrees:

$$E_{V}^i(x_i, x_j) = \text{Low}(S, U_{V})$$ \quad i, j = 1, ..., n \tag{60}

where $U_{V}$ is calculated as (18) with $p_{kl} = V_{kl}$.

Step 2.

2.a. Using the relations of linguistic dominance degrees $\{E_{M}^i, ..., E_{M}^{L}\}$ and the important weights $\beta = \{\beta_1, ..., \beta_L\}$, calculate the totally social opinion relation

$$Q_{M} = \left[q_{M}(x_i, x_j)\right]$$ \quad i, j = 1, ..., n \tag{61}

Where $q_{M}(x_i, x_j) = \text{Low}(S, U_{qM})$, $U_{qM} = [u_{qT}, ..., u_{q1}]$ \tag{62}

where $u_{qt} = W_{qT}^{i} (s_i) = \sum_{l=1}^{T} \beta_l : E_{M}^l (i, j) = s_i$, \quad t=1,...,T . \tag{63}

Using the relations of linguistic non-dominance degrees $\{E_{V}^i, ..., E_{V}^{L}\}$ and the important weights $\beta = \{\beta_1, ..., \beta_L\}$, calculate the totally social opinion relation

$$Q_{V} = \left[q_{V}(x_i, x_j)\right]$$ \quad i, j = 1, ..., n \tag{64}

Where $q_{V}(x_i, x_j) = \text{Low}(S, U_{qV})$, $U_{qV} = [u_{qT}, ..., u_{q1}]$ \tag{65}

where $u_{qt} = W_{qT}^{i} (s_i) = \sum_{l=1}^{T} \beta_l : E_{V}^l (i, j) = s_i$, \quad t=1,...,T . \tag{66}

2.b. Using $Q_M, Q_V$, calculate the fuzzy collective solution according to the totally social opinion relations

$$FCS_{Q_M} = \{fcs_{qM}(x_i) / x_i, ..., fcs_{qM}(x_n) / x_n\}$$ \quad and \tag{67}

$$FCS_{Q_V} = \{fcs_{qV}(x_i) / x_i, ..., fcs_{qV}(x_n) / x_n\}$$ \quad \tag{68}
as in Definition 2
From (67,68) we obtain Intuitionistic Fuzzy Evaluation

\[
IFE_Q = \left\{ (fcs_{a_l}(x_i), fcs_{a_l}(x_j))/x_i, ..., (fcs_{a_l}(x_n), fcs_{a_l}(x_n))/x_n \right\}
\]

Using \( IFE_Q \), we choose FPS of the problem.

Aggregation procedure 3:
Step 1. For each expert \( e_k \), for \( k=1,...,m \), use \( \beta = \{ \beta_1, ..., \beta_l \} \), calculate

\[
F^{\beta}_M(x_i, x_j) = \text{Low}(S,U^k_M)
\]

where

\[
U^k_M = [u_L, ..., u_U], \quad u_L = \sum \{ \beta_i : M_M(i, j) = s_i \} \quad t=1,...,T,
\]

and the fuzzy evaluation

\[
FE^k_M = \left\{ f^{k}_M(x_i)/x_i, ..., f^{k}_M(x_n)/x_n \right\}, \quad k=1,...,m
\]

where

\[
f^{k}_M(x_i) = \text{Low}(S,U^k_M),
\]

and

\[
V^k_M = [v_L, ..., v_U]
\]

as (31) with \( F^k = F^k_M \).

Analogously, calculate

\[
F^{\beta}_V(x_i, x_j) = \text{Low}(S,U^k_V),
\]

where

\[
U^k_V = [u_L, ..., u_U], \quad u_L = \sum \{ \beta_i : M_V(i, j) = s_i \} \quad t=1,...,T,
\]

and the fuzzy evaluation

\[
FE^k_V = \left\{ f^{k}_V(x_i)/x_i, ..., f^{k}_V(x_n)/x_n \right\}, \quad k=1,...,m
\]

where

\[
f^{k}_V(x_i) = \text{Low}(S,V^k_V), \quad V^k_V = [v_L, ..., v_U],
\]

and

\[
v = \left| \left\{ j : F^k_V(x_i, x_j) = s, j \neq i \right\} \right|/n-1, \quad t=1,...,T
\]

as (31) with \( F^k = F^k_V \).

Step 2. Using the fuzzy evaluation \( \left\{ (FE^k_M, FE^k_V), k=1,...,m \right\} \) and the weights \( \{ w(k) : e_k \in E \} \), calculate the aggregated fuzzy evaluation (aFE):

\[
aFE_M = \{ afe_M(x_i)/x_i, ..., afe_M(x_n)/x_n \} \quad \text{and} \quad aFE_V = \{ afe_V(x_i)/x_i, ..., afe_V(x_n)/x_n \}
\]

as in Definition 3.

Then we obtain Aggregated Intuitionistic Fuzzy Evaluation (aIFE):

\[
aIFE = \{ (afe_M(x_i), afe_V(x_j))/x_i, ..., (afe_M(x_n), afe_V(x_n))/x_n \}
\]

Using aIFE, we choose aFPS of the problem.
7. A numerical example

We consider the following multi criteria GDM problem with intuitionistic linguistic preference relations:

The alternatives set: \( X = \{ x_1, x_2, x_3, x_4 \} \).

The expert set: \( E = \{ e_1, e_2, e_3 \} \) with their importance \( w = \{0.2, 0.5, 0.3\} \).

The set of criteria: \( C = \{ C_1, C_2, C_3 \} \) with their importance weights \( \beta = \{0.35, 0.4, 0.25\} \).

The linguistic labels set:

The intuitionistic linguistic preference relations are relations

\[ p_{kl} = (M_{kl}, V_{kl}), \quad k, l = 1, 2, 3, \quad i, j, 1, 2, 3, 4 \]  \hspace{1cm} (79)

where

\[ M_{kl} = [\mu_{kl}(x_i, x_j)], \quad i, j, 1, 2, 3, 4 \]  \hspace{1cm} (80)

\[ V_{kl} = [\nu_{kl}(x_i, x_j)], \quad i, j, 1, 2, 3, 4 \]  \hspace{1cm} (81)

are given in the Appendix.

7.1. Computing with the aggregation procedure 1

Step 1.

Step 1.1. Computing for criterion \( C_1 \)

1.1.a. Use the linguistic preference relations

\[
\begin{bmatrix}
IM & ML & MC & EU \\
SC & IM & MC & EU \\
ML & SC & IM & EU \\
IM & ML & EL & IM
\end{bmatrix}
\quad \begin{bmatrix}
IM & MC & MC & EU \\
SC & IM & SC & EU \\
SC & MC & IM & EU \\
ML & IM & EL & IM
\end{bmatrix}
\quad \begin{bmatrix}
IM & MC & MC & EU \\
SC & IM & SC & EU \\
SC & MC & IM & EU \\
ML & IM & EL & IM
\end{bmatrix}
\]

Calculate the linguistic dominance degrees:

\[ E_M^1 (x_i, x_j) = \text{Low}(S, U_{M1}), \quad i, j = 1, 2, 3, 4 \] as in (51).

\[ E_M^2 (x_i, x_j) = \text{Low}((s_{i}, s_{j}), (0.2, 0.5, 0.3)) = C \{ (0.5, s_{j}), (0.5, \text{Low}((s_{i}), (1)) \}
\]

\[ = C \{ (0.5, s_{j}), (0.5, s_{i}) \} = s_{k(1,2)}, \quad k(1,2) = 6 + \text{round}((0.5)(7-6)) = 6 + 1 = 7 \quad \Rightarrow s_{k(1,2)} = s_{7} = ML. \]

\[ E_M^1 (x_i, x_j) = \text{Low}((s_{i}, s_{j}), (0.2, 0.5, 0.3)) = C \{ (0.2, s_{j}), (0.8, \text{Low}((s_{i}), (1)) \}
\]

\[ = C \{ (0.2, s_{j}), (0.8, s_{i}) \} = s_{k(1,3)}, \quad k(1,3) = 6 + \text{round}((0.2)(8-6)) = 6 + 0 = 6 \quad \Rightarrow s_{k(1,3)} = s_{6} = MC. \]
\( E_M^1 (x_2, x_1) = \text{Low}(s_2, s_2, s_2, (0.2, 0.5, 0.3)) = C \{ (0.2, s_2), (0.8, \text{Low}(s_2), (1)) \} \)
\( = C \{ (0.2, s_2), (0.8, s_2) \} = s_2 = EU. \)
\( E_M^1 (x_2, x_1) = \text{Low}(s_4, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, s_4) \} = s_4 = SC. \)
\( E_M^1 (x_2, x_3) = \text{Low}(s_5, s_5, s_5, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, s_5) \} = s_5 = IM. \)
\( E_M^1 (x_2, x_3) = \text{Low}(s_6, s_6, s_6, (0.2, 0.5, 0.3)) = C \{ (0.5, s_6), (0.5, \text{Low}(s_4), (1)) \} \)
\( = C \{ (0.5, s_6), (0.5, s_4) \} = s_{k(2,3)}, \)
\( k(2,3) = 4 + \text{round}((0.5) , (6 - 4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(2,3)} = s_5 = IM. \)
\( E_M^3 (x_2, x_4) = \text{Low}(s_3, s_3, s_2, (0.2, 0.5, 0.3)) = C \{ (0.2, s_3), (0.8, \text{Low}(s_2), (1)) \} \)
\( = C \{ (0.2, s_3), (0.8, s_2) \} = s_{k(2,4)}, \)
\( k(2,4) = 2 + \text{round}((0.2), (3 - 1)) = 2 + 0 = 2 \quad \Rightarrow s_{k(2,4)} = s_2 = EU. \)
\( E_M^4 (x_3, x_1) = \text{Low}(s_3, s_4, s_3, (0.2, 0.5, 0.3)) = C \{ (0.5, s_4), (0.5, \text{Low}(s_3), (3/5, 2/5)) \} \)
\( = C \{ (0.5, s_4), (0.5, s_3) \} = s_4 = SC. \)
\( E_M^4 (x_3, x_2) = \text{Low}(s_6, s_4, s_6, (0.2, 0.5, 0.3)) = C \{ (0.2, s_6), (0.8, \text{Low}(s_4), (3/8, 5/8)) \} \)
\( = C \{ (0.2, s_6), (0.8, s_4) \} = s_{k(3,2)} = s_5 = IM. \)

since
\( \text{Low}(s_6, s_4), (3/8, 5/8) = s_k, \quad k = 4 + \text{round}((3/8), (6 - 4)) = 4 + 1 = 5 \quad \Rightarrow s_k = s_5 \)
\( E_M^1 (x_3, x_3) = \text{Low}(s_5, s_5, s_3, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, s_3) \} = s_5 = IM. \)
\( E_M^1 (x_3, x_3) = \text{Low}(s_2, s_2, s_2, (0.2, 0.5, 0.3)) = C \{ (0.2, s_2), (0.8, \text{Low}(s_2), (1)) \} \)
\( = C \{ (0.2, s_2), (0.8, s_2) \} = s_3 = EU. \)
\( E_M^1 (x_4, x_1) = \text{Low}(s_8, s_7, s_7, (0.2, 0.5, 0.3)) = C \{ (0.2, s_8), (0.8, \text{Low}(s_7), (1)) \} \)
\( = C \{ (0.2, s_8), (0.8, s_7) \} = s_{k(4,1)}, \)
\( k(4,1) = 7 + \text{round}((0.2), (8 - 7)) = 7 \quad \Rightarrow s_{k(4,1)} = s_7 = ML. \)
\( E_M^1 (x_4, x_2) = \text{Low}(s_5, s_5, s_5, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}(s_5), (1)) \} \)
\( = C \{ (0.2, s_5), (0.8, s_5) \} = s_5 = IM. \)
\( E_M^1 (x_5, x_3) = \text{Low}(s_5, s_8, s_5, (0.2, 0.5, 0.3)) = C \{ (0.3, s_9), (0.7, \text{Low}(s_8), (5/8, 2/8)) \} \)
\( = C \{ (0.3, s_9), (0.7, s_8) \} = s_{k(4,1)}, \)

since
\( \text{Low}(s_8, s_7), (5/7, 2/7) = s_k, \quad k = 7 + \text{round}((5/7), (8 - 7)) = 7 + 1 = 8 \quad \Rightarrow s_k = s_8 \)
\( k(4,3) = 8 + \text{round}((0.3), (9 - 8)) = 8 \quad \Rightarrow s_{k(4,3)} = s_8 = EL. \)
We obtain the linguistic dominance degrees for criterion $C_1$:

$$E^i_a = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & IM & EU \\ SC & IM & IM & EU \\ ML & IM & EL & IM \end{bmatrix}.$$  \hspace{1cm} (82)

Calculate the FCS by Definition 2.

$$\text{fcs}^1_M(x_i) = \text{Low}((s_7, s_6, s_2), (1/3, 1/3, 1/3)) = C \{ (1/3, s_7), (2/3, \text{Low}(s_6, s_2), (0.5, 0.5)) \}
\text{csc}\{1/3, 1/3, 1/3\} = s_k,$$

since

$$\text{Low}(s_6, s_2), (0.5, 0.5) = s_k, k^\prime = 2 + \text{round}((0.5), (6 - 2)) = 2 + 2 = 4 \Rightarrow s_k = s_4.$$ 

\[ k = 4 + \text{round}((1/3), (7 - 4)) = 4 + 1 = 5 \Rightarrow s_k = s_5 = IM. \]

$$\text{fcs}^1_M(x_2) = \text{Low}((s_4, s_2, s_2), (1/3, 1/3, 1/3)) = C \{ (1/3, s_2), (2/3, \text{Low}(s_4, s_2), (1/2, 1/2)) \}
\text{csc}\{1/3, 1/3, 1/3\} = s_k,$$

since

$$\text{Low}(s_4, s_2), (0.5, 0.5) = s_k, k^\prime = 2 + \text{round}((0.5), (4 - 2)) = 2 + 1 = 3 \Rightarrow s_k = s_3.$$ 

\[ k = 3 + \text{round}((1/3), (5 - 3)) = 3 + 1 = 4 \Rightarrow s_k = s_4 = SC. \]

$$\text{fcs}^1_M(x_3) = \text{Low}((s_3, s_4, s_2), (1/3, 1/3, 1/3)) = C \{ (1/3, s_3), (2/3, \text{Low}(s_4, s_2), (1/2, 1/2)) \}
\text{csc}\{1/3, 1/3, 1/3\} = s_k = SC.$$ 

$$\text{fcs}^1_M(x_4) = \text{Low}((s_7, s_6, s_6), (1/3, 1/3, 1/3)) = C \{ (1/3, s_6, 2/3, \text{Low}(s_7, s_6), (0.5, 0.5)) \}
\text{csc}\{1/3, 1/3, 1/3\} = s_k,$$

since

$$\text{Low}(s_7, s_6), (0.5, 0.5) = s_k, k^\prime = 5 + \text{round}((0.5), (7 - 5)) = 5 + 1 = 6 \Rightarrow s_k = s_6.$$ 

\[ k = 6 + \text{round}((1/3), (8 - 6)) = 6 + 1 = 7 \Rightarrow s_k = s_7 = ML. \]

We obtain FCS for criterion $C_1$.

$$\text{FCS}^1_M = \{ IM / x_1, SC / x_2, SC / x_3, ML / x_4 \}. \hspace{1cm} (83)$$

1.1.1b. Use the linguistic non-preference relations

$$V_{ii} = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix}, \hspace{1cm} V_{ii} = \begin{bmatrix} SC & EU & VLC & SC \\ EU & SC & VLC & SC \\ VLC & MC & SC & VLC \\ EU & EU & VLC & SC \end{bmatrix}, \hspace{1cm} V_{ii} = \begin{bmatrix} SC & SC & SC & SC \\ EU & SC & SC & EU \\ VLC & MC & SC & VLC \\ EU & SC & SC & I \end{bmatrix}. \hspace{1cm} (84)$$
Calculate the linguistic dominance degrees using non-preference relations:

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}(S, U_{ij}), \quad i, j = 1, 2, 3, 4 \] as in (53).

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i4}, s_{j4}, s_{d}), (0.2, 0.5, 0.3)) = C \{(0.2, s_d), (0.8, \text{Low}((s_{j4}), (1))\}
\]
\[ = C \{(0.2, s_d), (0.8, s_d)\} = s_4 = SC. \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i4}, s_{j4}, s_{d}), (0.2, 0.3, 0.5)) = C \{(0.2, s_d), (0.8, \text{Low}((s_{j4}, s_{24}), (3/8, 5/8))\}
\]
\[ = C \{(0.2, s_d), (0.8, s_3)\} = s_{k(1.2)}, \]

since

\[ \text{Low}((s_{j4}, s_{24}), (3/8, 5/8)) = s_{k}, \quad k' = 2 + \text{round}((3/8, (4 - 2)) = 2 + 1 = 3 \quad \Rightarrow s_{k} = s_3 \]
\[ k(1, 2) = 3 + \text{round}((0.2), (4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(1.2)} = s_3 = VLC \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i3}, s_{j3}, s_{d}), (0.3, 0.5, 0.2)) = C \{(0.3, s_3), (0.7, \text{Low}((s_{j3}, s_{34}), (2/7, 5/7))\}
\]
\[ = C \{(0.3, s_3), (0.7, s_3)\} = s_3 = VLC, \]

since

\[ \text{Low}((s_{j3}, s_{34}), (2/7, 5/7)) = s_{k'}, \quad k' = 3 + \text{round}((2/7, (4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k'} = s_3 \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i3}, s_{j3}, s_{d}), (0.2, 0.2, 0.3)) = C \{(0.5, s_4), (0.5, \text{Low}((s_{j4}, s_{34}), (3/5, 2/5))\}
\]
\[ = C \{(0.5, s_4), (0.5, s_4)\} = s_4 = SC, \]

since

\[ \text{Low}((s_{j4}, s_{34}), (3/5, 2/5)) = s_{k}, \quad k' = 3 + \text{round}((3/5, (4 - 3)) = 3 + 1 = 4 \quad \Rightarrow s_{k} = s_4 \]
\[ k(2, 4) = 3 + \text{round}((0.2), (4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(2.4)} = s_3 = VLC \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i2}, s_{j2}, s_{2}), (0.3, 0.5, 0.2)) = s_2 = EU \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i2}, s_{j2}, s_{2}), (0.2, 0.5, 0.3)) = C \{(0.2, s_4), (0.8, \text{Low}((s_{j2}), (1))\}
\]
\[ = C \{(0.2, s_4), (0.8, s_4)\} = s_4 = SC. \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i2}, s_{j3}, s_{4}), (0.2, 0.5, 0.3)) = C \{(0.3, s_4), (0.7, \text{Low}((s_{j3}), (1))\}
\]
\[ = C \{(0.3, s_4), (0.7, s_3)\} = s_{k(2,3)} = s_3 = VLC. \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i2}, s_{j2}, s_{2}), (0.3, 0.5, 0.2)) = s_2 = EU \]

\[ E_{ij}^{1}(x_i, x_j) = \text{Low}((s_{i3}, s_{j3}, s_{2}), (0.2, 0.2, 0.3)) = C \{(0.2, s_3), (0.8, \text{Low}((s_{j3}, s_{24}), (5/8, 3/8))\}
\]
\[ = C \{(0.2, s_3), (0.8, s_3)\} = s_{k(3,1)}, \]
We obtain the linguistic dominance degrees using non-preference relations for criterion $C_1$. 

\[
\text{Low}((s_3, s_2, (5/8, 3/8)) = s_k, \quad k = 2 + \text{round}((5/8, (3 - 2)) = 2 + 1 = 3 \quad \Rightarrow s_k = s_3
\]

\[
k(3, 1) = 3 + \text{round}((0.2, (5 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(3,1)} = s_3 = VLC.
\]

\[
E^1_V(x_3, x_2) = \text{Low}((s_4, s_6, s_4, (0.2, 0.5, 0.3)) = C \{ (0.5, s_6), (0.5 \text{Low}(s_4), (1)) \}
\]

\[
= C \{ (0.5, s_6), (0.5, s_4) \} = s_{k(3,2)},
\]

\[
k(3, 2) = 4 + \text{round}((0.5, (6 - 4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(3,2)} = s_5 = IM.
\]

\[
E^1_V(x_3, x_3) = \text{Low}((s_4, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, \text{Low}(s_3, s_4), (5/8, 3/8)) \}
\]

\[
= C \{ (0.2, s_4), (0.8, s_2) \} = s_{k(3,4)},
\]

since

\[
\text{Low}((s_3, s_1, (5/8, 3/8)) = s_k, \quad k = 1 + \text{round}((5/8, (3 - 1)) = 1 + 1 = 2 \quad \Rightarrow s_k = s_2
\]

\[
k(3, 4) = 2 + \text{round}((0.2, (4 - 2)) = 2 + 0 = 2 \quad \Rightarrow s_{k(3,4)} = s_2 = EU.
\]

\[
E^1_V(x_3, x_1) = \text{Low}((s_2, s_2, s_2, (0.3, 0.5, 0.2)) = s_2 = EU.
\]

\[
E^1_V(x_3, x_2) = \text{Low}((s_4, s_2, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, \text{Low}(s_4, s_2), (3/8, 5/8)) \}
\]

\[
= C \{ (0.2, s_4), (0.8, s_3) \} = s_{k(4,2)},
\]

since

\[
\text{Low}((s_4, s_3, (3/8, 5/8)) = s_k, \quad k = 2 + \text{round}((3/8, (4 - 2)) = 2 + 1 = 3 \quad \Rightarrow s_k = s_3
\]

\[
k(4, 2) = 3 + \text{round}((0.2, (4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(4,2)} = s_3 = VLC.
\]

\[
E^1_V(x_4, x_3) = \text{Low}((s_3, s_3, s_1, (0.2, 0.5, 0.3)) = C \{ (0.7, s_3), (0.3, s_1) \} = s_{k(4,3)},
\]

\[
k(4, 3) = 1 + \text{round}((0.7, (3 - 1)) = 1 + 1 = 2 \quad \Rightarrow s_{k(4,3)} = s_2 = EU.
\]

\[
E^1_V(x_4, x_2) = \text{Low}((s_4, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, \text{Low}(s_4, s_1), (1)) \}
\]

\[
= C \{ (0.2, s_4), (0.8, s_2) \} = s_4 = SC.
\]
Calculate the FCS as in (54).
\[ fcs^1_k(x_1) = \text{Low}((s_4, s_3), (1/3, 2/3)) = C \{ (1/3, s_4), (2/3, s_3) \} = s_k, \]
\[ k = 3 + \text{round}((1/3)(4-3)) = 3 + 0 = 3 \quad \Rightarrow s_k = s_3 = \text{VLC}. \]
\[ fcs^1_k(x_2) = \text{Low}((s_3, s_2), (1/3, 2/3)) = C \{ (1/3, s_3), (2/3, s_2) \} = s_k, \]
\[ k = 2 + \text{round}((1/3)(3-2)) = 2 + 0 = 2 \quad \Rightarrow s_k = s_2 = \text{EU}. \]
\[ fcs^1_k(x_3) = \text{Low}((s_3, s_5, s_2), (1/3, 1/3, 1/3)) = C \{ (1/3, s_3), (2/3, \text{Low}((s_3, s_2), (1/2, 1/2))) \}
\[ = C \{ (1/3, s_3), (2/3, s_2) \} = s_k, \]
since
\[ \text{Low}((s_3, s_2), (0.5, 0.5)) = s_k, k = 2 + \text{round}((0.5)(3-2)) = 2 + 1 = 3 \Rightarrow s_k = s_3 \]
\[ k = 3 + \text{round}((1/3)(5-3)) = 3 + 1 = 4 \quad \Rightarrow s_k = s_4 = \text{SC}. \]
\[ fcs^1_k(x_4) = \text{Low}((s_3, s_2), (1/3, 2/3)) = C \{ (1/3, s_3), (2/3, s_2) \} = s_k, \]
\[ k = 2 + \text{round}((1/3)(3-2)) = 2 + 0 = 2 \quad \Rightarrow s_k = s_2 = \text{EU}. \]

We obtain FCS using non-preference relations for the criterion C1.
\[ FCS^1_1 = \{ \text{VLC} / x_1, \text{EU} / x_2, \text{SC} / x_3, \text{EU} / x_4 \}. \quad (85) \]

Finally, we obtain IFE for the criterion C1.
\[ IFE^1 = \{ (\text{IM}, \text{VLC}) / x_1, (\text{SC}, \text{EU}) / x_2, (\text{SC}, \text{SC}) / x_3, (\text{ML}, \text{EU}) / x_4 \}. \quad (86) \]

and Fuzzy Pareto Solution (FPS) for criterion C1 is \( x_4 \).

Step 1.2. Computing for criterion C2.

1.2.a. Use the linguistic preference relations

\[
E^1 = \begin{bmatrix}
SC & VLC & VLC & SC \\
EU & SC & VLC & EU \\
VLC & IM & SC & EU \\
EU & VLC & EU & SC
\end{bmatrix}
\]

(84)
Calculate the linguistic dominance degrees:

$$E^2_M(x_i, x_j) = \text{Low}(S, U_{M_2})$$ \quad (i, j = 1, 2, 3, 4) (87)

$$U_{M_2} = [u_{12}, ..., u_{21}]$$, with $$u_{ij} = W_{ij}(s_t) = \sum_k w(k) : M_k^2(i, j) = s_t$$, \quad (88)

t = 1, ..., T.

$$E^2_M(x_1, x_2) = \text{Low}((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}((s_5, 1)) \} = s_5 = IM.$$

$$E^2_M(x_1, x_2) = \text{Low}((s_7, s_6, s_5), (0.5, 0.2, 0.3)) = C \{ (0.5, s_7), (0.5, \text{Low}((s_6, s_5), (2/5, 3/5)) \}$$

$$= C \{ (0.5, s_7), (0.5, s_5) \} = s_{k(1,2)},$$

since $$\text{Low}((s_6, s_5), (2/5, 3/5)) = s_k \Rightarrow s_k = s_5$$.

$$k(1, 2) = 5 + \text{round}((0.5), (7-5)) = 5 + 0 = 5 \Rightarrow s_k = s_5$$.

$$E^2_M(x_1, x_3) = \text{Low}((s_7, s_6, s_6), (0.2, 0.5, 0.3)) = C \{ (0.2, s_7), (0.8, \text{Low}((s_6, 1)) \}$$

$$= C \{ (0.2, s_7), (0.8, s_6) \} = s_{k(1,3)},$$

$$k(1, 3) = 6 + \text{round}((0.2), (7-6)) = 6 + 0 = 6 \Rightarrow s_k = s_6 = MC.$$
\( E^2_M (x_2, x_4) = \text{Low}(s_4, s_4, s_5, (0.2, 0.5, 0.3)) = C \{ (0.3, s_5), (0.7, \text{Low}(s_4), (1) \} \\
= C \{ (0.3, s_5), (0.7, s_4) \} = s_{k(2,4)}. \)

\( k(2, 4) = 4 + \text{round}(((0.3), (5 - 4))) = 4 + 0 = 4 \quad \Rightarrow s_{k(2,3)} = s_4 = SC. \)

\( E^2_M (x_3, x_1) = \text{Low}(s_3, s_3, s_3, (0.2, 0.5, 0.3)) = C \{ (s_3), (1) \} = s_3 = VLC. \)

\( E^2_M (x_3, x_2) = \text{Low}(s_4, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (s_4), (1) \} = s_4 = SC. \)

\( E^2_M (x_3, x_3) = \text{Low}(s_5, s_5, s_5, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}(s_5, 1) \} = s_5 = IM. \)

\( E^2_M (x_3, x_4) = \text{Low}(s_2, s_2, s_3, (0.2, 0.5, 0.3)) = C \{ (0.5, s_4), (0.5) \text{Low}(s_3, s_2)(3/5, 2/5) \} \\
= C \{ (0.5, s_4), (0.5, s_3) \} = s_{k(3,4)}. \)

since

\( \text{Low}(s_3, s_2), (3/5, 2/5) = s_{k'}, k' = 2 + \text{round}((3/5), (3 - 2)) = 2 + 1 = 3 \)

\( k(3, 4) = 3 + \text{round}((0.5), (4 - 3)) = 3 + 1 = 4 \quad \Rightarrow s_{k(3,4)} = s_4 = SC. \)

\( E^2_M (x_4, x_1) = \text{Low}(s_4, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (s_4), (1) \} = s_4 = SC. \)

\( E^2_M (x_4, x_2) = \text{Low}(s_5, s_6, s_4, (0.2, 0.5, 0.3)) = C \{ (0.5, s_6), (0.5) \text{Low}(s_5, s_4)(2/5, 3/5) \} \\
= C \{ (0.5, s_6), (0.5, s_4) \} = s_{k(4,2)}. \)

since

\( \text{Low}(s_5, s_4), (2/5, 3/5) = s_{k'}, k' = 4 + \text{round}((2/5), (5 - 4)) = 4 + 0 = 4 \)

\( k(4, 2) = 4 + \text{round}((0.5), (6 - 4)) = 1 + 1 = 5 \quad \Rightarrow s_{k(4,2)} = s_5 = IM. \)

\( E^2_M (x_4, x_3) = \text{Low}(s_6, s_6, s_5, (0.2, 0.5, 0.3)) = C \{ (0.7, s_6), (0.3, s_2) \} = s_6 = MC. \)

\( E^2_M (x_4, x_4) = \text{Low}(s_5, s_5, s_5, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}(s_5, 1) \} = s_5 = IM. \)

We obtain the linguistic dominance degrees for criterion \( C_2 \):

\[
E^2_M = \begin{bmatrix}
IM & MC & MC & MC \\
SC & IM & IM & SC \\
VLC & SC & IM & SC \\
SC & SC & MC & IM
\end{bmatrix}
\]

(89)
Calculate the FCS

\[ fcs^2(x_1) = Low((s_6, s_5, 1)) = s_6 = MC, \]

\[ fcs^2(x_2) = Low((s_3, s_4, (1/3, 2/3)) = C((1/3, s_3, 2/3, s_4)) = s_k, \]

\[ k = 4 + \text{round}(1/3, (5 - 4)) = 4 + 0 = 4 \Rightarrow s_k = s_4 = SC \]

\[ fcs^2(x_3) = Low((s_4, s_4, (2/3, 1/3)) = C((2/3, s_4, 1/3, s_3)) = s_k, \]

\[ k = 3 + \text{round}(1/3, (4 - 3)) = 3 + 1 = 4 \Rightarrow s_k = s_5 = IM \]

and the FCS \( FCS^2 = \{MC / x_1, SC / x_2, SC / x_3, IM / x_4\}. \) (90)

1.2.b. Use the linguistic non-preference relations

\[
V_{ij} = \begin{bmatrix}
SC & VLC & SC & EU \\
SC & SC & IM & VLC \\
MC & SC & SC & MC \\
MC & MC & VLC & SC
\end{bmatrix}, \quad V_{ij} = \begin{bmatrix}
SC & EU & VLC & VLC \\
IM & SC & VLC & SC \\
IM & SC & SC & VLC \\
SC & VLC & SC & SC
\end{bmatrix}, \quad V_{ij} = \begin{bmatrix}
SC & SC & VLC & EU \\
IM & SC & VLC & SC \\
SC & SC & SC & SC \\
MC & SC & SC & SC
\end{bmatrix}
\]

Calculate the linguistic non-dominance degrees:

\[
E^2_i(x_i, x_j) = Low(S, U_{V_i}) \quad \text{for } i, j = 1, \ldots, 4
\]

\[
U_{V_i} = \left[u_{21}, \ldots, u_{24}\right], \quad u_{2t} = W_{ij}(s_t) = \sum_k \{w(k) : V_{k2}(i, j) = s_t\}, \text{ for each } t = 1, \ldots, T.
\]

(91)

\[
E^2_i(x_1, x_2) = Low((s_3, s_4, s_4, 0.2, 0.5, 0.3)) = C\{0.2, s_4, (0.8, Low(s_4), 1)\}
\]

\[
= C\{0.2, s_4, (0.8, s_4)\} = s_4 = SC.
\]

\[
E^2_i(x_1, x_2) = Low((s_3, s_2, s_2, 0.2, 0.5, 0.3)) = C\{0.3, s_4, (0.7, Low(s_3, s_2), (2/7, 5/7))\}
\]

\[
= C\{0.3, s_4, (0.7, s_2)\} = s_{k(2,1)},
\]

since

\[
Low((s_3, s_2), (2/7, 5/7)) = s_k', \quad k' = 2 + \text{round}(2/7, (3 - 2)) = 2 + 0 = 2,
\]

\[
k(1, 2) = 2 + \text{round}(0.3, (4 - 2)) = 2 + 1 = 3 \Rightarrow s_{k(2,1)} = s_3 = VLC.
\]
\[ E^2_{v}(x_1, x_3) = \text{Low}((s_4, s_3, s_3), (0.2, 0.5, 0.3)) = C \{(0.3, s_4), (0.7) \text{Low}((s_2)(1))\} \]
\[ = C \{(0.3, s_4), (0.7, s_3)\} = s_{k(1,3)}. \]

\[ k(1, 3) = 3 + \text{round}((0.3)(4 - 3)) = 3 + 0 = 3 \Rightarrow s_{k(1,3)} = s_3 = \text{VLC}. \]

\[ E^2_{v}(x_1, x_4) = \text{Low}((s_2, s_3, s_2), (0.2, 0.5, 0.3)) = C \{(0.5, s_3), (0.5) \text{Low}((s_2)(1))\} \]
\[ = C \{(0.5, s_3), (0.5, s_2)\} = s_{k(1,4)}. \]

\[ k(1, 4) = 2 + \text{round}((0.5)(3 - 2)) = 2 + 1 = 3 \Rightarrow s_{k(1,4)} = s_3 = \text{VLC}. \]

\[ E^2_{v}(x_2, x_1) = \text{Low}((s_4, s_5, s_5), (0.2, 0.5, 0.3)) = C \{(0.5, s_5), (0.5) \text{Low}((s_4, s_4)(3/5, 2/5))\} \]
\[ = C \{(0.5, s_5), (0.5, s_5)\} = s_5 = \text{IM}, \]

since \[ \text{Low}((s_5, s_4), (3/5, 2/5)) = s_k, k' = 4 + \text{round}((3/5)(5 - 4)) = 4 + 1 = 5. \]

\[ E^2_{v}(x_2, x_2) = \text{Low}((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_4), (0.8, \text{Low}((s_4)(1))\} \]
\[ = C \{(0.2, s_4), (0.8, s_4)\} = s_4 = \text{SC}. \]

\[ E^2_{v}(x_2, x_3) = \text{Low}((s_5, s_3, s_3), (0.2, 0.5, 0.3)) = C \{(0.2, s_5), (0.8) \text{Low}((s_3)(1))\} \]
\[ = C \{(0.2, s_5), (0.8, s_3)\} = s_{k(2,3)}, \]

\[ k(2, 3) = 3 + \text{round}((0.2)(5 - 3)) = 3 + 0 = 3 \Rightarrow s_{k(2,3)} = s_3 = \text{VLC}. \]

\[ E^2_{v}(x_2, x_4) = \text{Low}((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{(0.5, s_4), (0.5) \text{Low}((s_4, s_3)(3/5, 2/5))\} \]
\[ = C \{(0.5, s_4), (0.5, s_4)\} = s_4 = \text{SC}, \]

since \[ \text{Low}((s_4, s_3), (3/5, 2/5)) = s_k, k' = 3 + \text{round}((3/5)(4 - 3)) = 3 + 1 = 4. \]

\[ E^2_{v}(x_3, x_1) = \text{Low}((s_6, s_5, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_6), (0.8) \text{Low}((s_5, s_4)(5/8, 3/8))\} \]
\[ = C \{(0.2, s_6), (0.8, s_5)\} = s_{k(3,1)}. \]
Since

\[ \text{Low}((s_5, s_4), (5/8, 3/8)) = s_{k'}, \quad k' = 4 + \text{round}((5/8)(5 - 4)) = 4 + 1 = 5, \]

\[ k(3,1) = 5 + \text{round}((0.2)(6 - 5)) = 5 + 0 = 5 \quad \Rightarrow s_{k(3,1)} = s_5 = IM. \]

\[ E^2_V(x_3, x_2) = \text{Low}((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_4), (0.8, \text{Low}(s_4), (1))\} \]

\[ = C \{(0.2, s_4), (0.8, s_4)\} = s_4 = SC. \]

\[ E^2_V(x_3, x_3) = \text{Low}((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_4), (0.8, \text{Low}(s_4), (1))\} \]

\[ = C \{(0.2, s_4), (0.8, s_4)\} = s_4 = SC. \]

\[ E^2_V(x_3, x_4) = \text{Low}((s_6, s_3, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_6), (0.8)\text{Low}(s_4, s_3)(3/8, 5/8)\} \]

\[ = C \{(0.2, s_6), (0.8, s_3)\} = s_{k(3,4)}. \]

since

\[ \text{Low}((s_4, s_3), (3/8, 5/8)) = s_{k'}, \quad k' = 3 + \text{round}((3/8)(4 - 3)) = 3 + 0 = 3, \]

\[ E^2_V(x_4, x_1) = \text{Low}((s_4, s_4, s_6), (0.2, 0.5, 0.3)) = C \{(0.3, s_6), (0.7)\text{Low}(s_4)(1)\} \]

\[ = C \{(0.3, s_6), (0.7, s_4)\} = s_{k(4,1)}. \]

\[ k(4,1) = 4 + \text{round}((0.3)(6 - 4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(4,1)} = s_5 = IM. \]

\[ E^2_V(x_4, x_2) = \text{Low}((s_4, s_3, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_4), (0.8)\text{Low}(s_4, s_3)(3/8, 5/8)\} \]

\[ = C \{(0.2, s_4), (0.8, s_3)\} = s_{k(4,2)}. \]

since

\[ \text{Low}((s_4, s_3), (3/8, 5/8)) = s_{k'}, \quad k' = 3 + \text{round}((3/8)(5 - 4)) = 3 + 0 = 3, \]

\[ k(4,2) = 3 + \text{round}((0.2)(4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(4,2)} = s_3 = VLC. \]

\[ E^2_V(x_4, x_3) = \text{Low}((s_3, s_3, s_4), (0.2, 0.5, 0.3)) = C \{(0.3, s_4), (0.7)\text{Low}(s_3)(1)\} \]

\[ = C \{(0.3, s_4), (0.7, s_3)\} = s_{k(4,3)}. \]

\[ k(4,3) = 3 + \text{round}((0.3)(4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(4,3)} = s_3 = VLC. \]
\[ E^2_V(x_4, x_4) = \text{Low}((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_4), (0.8, \text{Low}((s_4), (1)) \} \]
\[ = C \{(0.2, s_4), (0.8, s_4)\} = s_4 = SC. \]

We obtain the linguistic non-dominance degrees using \( \{V_{12}, V_{22}, V_{32}\} \) for criterion \( C_2 \):

\[
E^2_V = \begin{bmatrix}
SC & VLC & VLC & VLC \\
IM & SC & VLC & SC \\
IM & SC & SC & SC \\
IM & VLC & VLC & SC \\
\end{bmatrix}.
\] (93)

Calculate FCS
\[
fcs^2_V(x_1) = \text{Low}((s_3), (1)) = s_3 = VLC
\]
\[
fcs^2_V(x_2) = \text{Low}((s_5, s_4, s_3), (1/3, 2/3)) = C \{(1/3, s_5), (2/3, \text{Low}((s_4, s_3), (0.5, 0.5)) \}
\]
\[ = C \{(1/3, s_5), (2/3, s_4)\} = s_k,
\] since
\[
\text{Low}((s_4, s_3), (0.5, 0.5)) = s_k, k = 3 + \text{round}((0.5), (4 - 3)) = 3 + 1 = 4 \Rightarrow s_k = s_4
\]

\[ k = 4 + \text{round}((1/3), (5 - 4)) = 4 + 0 = 4 \Rightarrow s_k = s_4 = SC. \]

\[
fcs^2_V(x_3) = \text{Low}((s_5, s_4), (1/3, 2/3)) = C \{(1/3, s_5), (2/3, s_4)\} = s_k,
\] since
\[
\text{Low}((s_4, s_3), (0.5, 0.5)) = s_k, k = 3 + \text{round}((0.5), (4 - 3)) = 3 + 1 = 4 \Rightarrow s_k = s_4
\]

\[ k = 4 + \text{round}((1/3), (5 - 4)) = 4 + 0 = 4 \Rightarrow s_k = s_4 = SC. \]

We obtain FCS using non-preference relations for criterion \( C_2 \)
\[
FCS^2_V = \{VLC/ x_1, SC/ x_2, SC/ x_3, SC/ x_4 \}.
\] (94)

Finally, we obtain IFE for criterion \( C_2 \)
\( \text{IFE}^2 = \{(MC, VLC) / x_1, (SC, SC) / x_2, (SC, SC) / x_3, (IM, SC) / x_4\} \). \hspace{1cm} (95)

and FPS for criterion \( C_2 \) is \( x_1 \).

Step 1.3. Computing for criterion \( C_3 \).

1.3.a. Use the linguistic preference relations

\[
M_0 = \begin{bmatrix}
IM & MC & ML & IM \\
SC & IM & MC & SC \\
VLC & SC & IM & IM \\
SC & MC & SC & IM 
\end{bmatrix}, \\
M_1 = \begin{bmatrix}
IM & ML & IM & ML \\
SC & IM & IM & SC \\
VLC & IM & IM & SC \\
SC & MC & SC & IM 
\end{bmatrix}, \\
M_2 = \begin{bmatrix}
IM & ML & SC & MC \\
SC & IM & MC & IM \\
VLC & SC & IM & IM \\
SC & IM & IM & SC 
\end{bmatrix},
\]

Calculate the linguistic dominance degrees

\[ E_3^M (x_i, x_j) = \text{Low}(S, U_{M_3}) , \quad i, j = 1, 2, 3, 4 \] \hspace{1cm} (96)

\[ U_{M_3} = [u_{31}, \ldots, u_{31}], \quad \text{with} \quad u_{3t} = W_{ij} (s_j) = \sum_k w(k) : M_{k3} (i, j) = s_j \}, \quad t=1, \ldots, T. \] \hspace{1cm} (97)

\[ E_3^M (x_1, x_1) = \text{Low}((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C \{(0.2, s_5), (0.8, \text{Low}((s_5, 1)) = s_5 = \text{IM}. \]

\[ E_3^M (x_1, x_2) = \text{Low}((s_6, s_7, s_7), (0.2, 0.5, 0.3)) = C \{(0.5, s_7), (0.5, \text{Low}((s_7, s_6), (3/5, 2/5))) \}
\]

since

\[ \text{Low}((s_7, s_6), (3/5, 2/5)) = s_k^*, k^* = 6 + \text{round}((3/5)(7-6)) = 6 + 1 = 7 \Rightarrow s_k^* = s_7 \]

\[ E_3^M (x_1, x_3) = \text{Low}((s_7, s_5, s_4), (0.2, 0.5, 0.3)) = C \{(0.2, s_7), (0.8, \text{Low}((s_5, s_4)(5/8, 3/8)) \}
\]

since

\[ \text{Low}((s_5, s_4), (5/8, 3/5)) = s_k^*, k^* = 4 + \text{round}((5/8)(5-4)) = 4 + 1 = 5 \Rightarrow s_k^* = s_5, \]

\[ k(1,3) = 5 + \text{round}((0.2)(7-5)) = 5 + 0 = 5 \Rightarrow s_k(1,3) = s_5 = \text{IM} \]
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\[ E^3_M (x_1, x_4) = \text{Low}((s_5, s_7, s_7), (0.2, 0.5, 0.3)) = C \{ (0.5, s_7), (0.5, \text{Low}((s_7, s_5), (3/5, 2/5)) \} = C \{ (0.5, s_7), (0.5, s_6) \} = s_{k(1,4)}, \]

since

\[ \text{Low}((s_7, s_5), (3/5, 2/5)) = s_k', k' = 5 + \text{round}((3/5, (7 - 5)) = 5 + 1 = 6 \Rightarrow s_k' = s_6, \]

\[ k(1,4) = 6 + \text{round}((0.5, (7 - 6)) = 6 + 1 = 7 \Rightarrow s_{k(1,4)} = s_7 = \text{ML}. \]

\[ E^3_M (x_2, x_1) = \text{Low}((s_4, s_3, s_4), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, \text{Low}((s_4, s_3), (3/8, 5/8)) \} = C \{ (0.2, s_4), (0.8, s_3) \} = s_{k(2,1)}, \]

since

\[ \text{Low}((s_4, s_3), (3/8, 5/8)) = s_k', k' = 3 + \text{round}((3/8, (4 - 3)) = 3 + 0 = 3 \Rightarrow s_k' = s_3, \]

\[ k(2,1) = 3 + \text{round}((0.2, (4 - 3)) = 3 + 0 = 3 \Rightarrow s_{k(2,1)} = s_3 = \text{VLC}. \]

\[ E^3_M (x_2, x_2) = \text{Low}((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}((s_5, s_5), (3/8, 5/8)) \} = s_5 = \text{IM}. \]

\[ E^3_M (x_2, x_3) = \text{Low}((s_6, s_5, s_6), (0.2, 0.5, 0.3)) = C \{ (0.2, s_6), (0.8, \text{Low}((s_6, s_5), (3/8, 5/8)) \} = C \{ (0.2, s_6), (0.8, s_5) \} = s_{k(2,3)}, \]

since

\[ \text{Low}((s_6, s_5), (3/8, 5/8)) = s_k', k' = 5 + \text{round}((3/8, (6 - 5)) = 5 + 0 = 5 \Rightarrow s_k' = s_5 \]

\[ k(2,3) = 5 + \text{round}((0.2, (6 - 5)) = 5 + 0 = 5 \Rightarrow s_{k(2,3)} = s_5 = \text{IM}. \]

\[ E^3_M (x_2, x_4) = \text{Low}((s_4, s_5, s_5), (0.2, 0.5, 0.3)) = C \{ (0.5, s_5), (0.5, \text{Low}((s_5, s_4), (3/5, 2/5)) \} = C \{ (0.5, s_5), (0.5, s_5) \} = s_5 = \text{IM}. \]
since

\[ \text{Low}(s_5, s_4, (3/5, 2/5)) = s'_k, \quad k' = 4 + \text{round}((3/5),(5 - 4)) = 4 + 1 = 5 \Rightarrow s'_{k'} = s_5. \]

\[ E_M^3(x_3, x_4) = \text{Low}(s_3, s_4, s_5, (0.2, 0.5, 0.3)) = C \{ (0.3, s_5), (0.7, \text{Low}(s_4, s_3)(5/7, 2/7)) \} \]
\[ = C \{ (0.3, s_5), (0.7, s_4) \} = s_{k(3, 1)}. \]

since

\[ \text{Low}(s_4, s_3, (5/7, 2/7)) = s'_k, \quad k' = 3 + \text{round}((5/7),(4 - 3)) = 3 + 1 = 4 \Rightarrow s'_{k'} = s_4. \]

\[ k(3, 1) = 4 + \text{round}((0.3),(5 - 4)) = 4 + 0 = 4 \quad \Rightarrow s_{k(3, 1)} = s_4 = SC. \]
\[ E_M^3(x_3, x_2) = \text{Low}(s_4, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, \text{Low}(s_4, 1)) \} = s_4 = SC. \]
\[ E_M^3(x_3, x_3) = \text{Low}(s_5, s_5, s_5, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}(s_5, 1)) \} = s_5 = IM. \]
\[ E_M^2(x_3, x_4) = \text{Low}(s_5, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}(s_4, 1)) \} \]
\[ = C \{ (0.2, s_5), (0.8, s_4) \} = s_{k(3, 4)}, \]

\[ k(3, 4) = 4 + \text{round}((0.2),(5 - 4)) = 4 + 0 = 4 \quad \Rightarrow s_{k(3, 4)} = s_4 = SC. \]
\[ E_M^3(x_4, x_3) = \text{Low}(s_3, s_3, s_3, (0.2, 0.5, 0.3)) = C \{ (0.2, s_3), (0.8, \text{Low}(s_3, 1)) \} \]
\[ = C \{ (0.2, s_3), (0.8, s_3) \} = s_{k(4, 1)}, \]

\[ k(4, 1) = 3 + \text{round}((0.2),(4 - 3)) = 3 + 0 = 3 \quad \Rightarrow s_{k(4, 1)} = s_3 = VLC. \]
\[ E_M^3(x_4, x_4) = \text{Low}(s_3, s_4, s_4, (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, \text{Low}(s_4, 1)) \} \]
\[ = C \{ (0.2, s_3), (0.8, s_4) \} = s_{k(4, 2)}, \]

\[ k(4, 2) = 4 + \text{round}((0.2),(6 - 4)) = 4 + 0 = 4 \quad \Rightarrow s_{k(4, 2)} = s_4 = SC. \]
\[ E_M^3(x_4, x_3) = \text{Low}(s_4, s_4, s_5, (0.2, 0.5, 0.3)) = C \{ (0.3, s_5), (0.7, \text{Low}(s_4, 1)) \} \]
\[ = C \{ (0.3, s_5), (0.7, s_4) \} = s_{k(4, 3)}, \]
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\[ k(4,3) = 4 + \text{round}((0.3).(5 - 4)) = 4 + 0 = 4 \quad \Rightarrow s_k(4,3) = s_4 = SC. \]
\[ E_M^3 (x_4, x_4) = \text{Low}((s_5, s_5, s_5), (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8, \text{Low}(s_5, 1) \} = s_5 = IM. \]

We obtain the linguistic dominance degrees

\begin{equation}
E_M^3 = \begin{bmatrix}
IM & ML & IM & ML \\
VLC & IM & IM & IM \\
SC & SC & IM & SC \\
VLC & IM & SC & IM \\
\end{bmatrix} .
\end{equation}

Calculate FCS by Definition 2.

\[ fcs_M^3 (x_2) = \text{Low}((s_5, s_3), (2/3, 1/3)) = C \{ (2/3, s_7), (1/3, s_3) \} = s_k \]
\[ k = 3 + \text{round}((2/3).(5 - 2)) = 3 + 1 = 4 \quad \Rightarrow s_k = s_4 = SC. \]
\[ fcs_M^3 (x_3) = \text{Low}((s_4), (1)) = s_4 = SC. \]
\[ fcs_M^3 (x_4) = \text{Low}((s_4, s_4, s_3), (1/3, 1/3, 1/3)) = C \{ (1/3, s_3), (2/3, \text{Low}(s_4, s_3), (0.5, 0.5)) \} = C \{ (1/3, s_3), (2/3, s_4) \} = s_k, \]
\[ \text{since} \quad \text{Low}((s_4, s_3), (0.5, 0.5)) = s_{k'} \quad k' = 3 + \text{round}((0.5).(4 - 3)) = 3 + 1 = 4 \quad \Rightarrow s_{k'} = s_4, \]
\[ k = 4 + \text{round}((1/3).(5 - 4)) = 4 + 0 = 4 \quad \Rightarrow s_k = s_4 = SC. \]

We obtain
\[ FCS_M^3 = \{ MC / x_1, SC / x_2, SC / x_3, SC / x_4 \} . \]

1.3.b. Use the linguistic non-preference relations

\[ V_{sc} = \begin{bmatrix}
SC & SC & VLC & SC \\
SC & SC & SC & SC \\
IM & VLC & SC & SC \\
IM & SC & SC & SC \\
\end{bmatrix} \]
\[ V_{sc} = \begin{bmatrix}
SC & VLC & SC & VLC \\
IM & SC & VLC & SC \\
IM & IM & SC & SC \\
MC & VLC & VLC & SC \\
\end{bmatrix} \]
\[ V_{sc} = \begin{bmatrix}
SC & VLC & SC & VLC \\
IM & SC & SC & IM \\
SC & SC & SC & IM \\
SC & SC & SC & SC \\
\end{bmatrix} \]

Calculate the linguistic non-dominance degrees:

\[ E_V^3 (x_i, x_j) = \text{Low}(S, U_{V^3}) , \quad i,j = 1,...,4 . \]
\[ U_{V3} = [u_{31}, \ldots, u_{31}], \text{ with } \quad u_{3t} = W_{ij} (s_t) = \sum_k \{ w(k) : V_{k3}(i, j) = s_t \}, \]

\[ E^3_v(x_1, x_1) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, Low((s_4), (1)) \} \]
\[ = C \{ (0.2, s_4), (0.8, s_4) \} = s_4 = SC. \]

\[ E^3_v(x_1, x_2) = Low((s_4, s_3, s_3), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8)Low((s_3)(1)) \} \]
\[ = C \{ (0.2, s_4), (0.8, s_3) \} = s_4 = SC. \]

\[ k(1, 2) = 3 + round((0.2),(4-3)) = 3 + 0 = 3 \Rightarrow s_{k(2,1)} = s_3 = VLC. \]

\[ E^3_v(x_1, x_3) = Low((s_3, s_3, s_4), (0.2, 0.5, 0.3)) = C \{ (0.5, s_4), (0.5)Low((s_4, s_3)(3/5, 2/5)) \} \]
\[ = C \{ (0.5, s_4), (0.5, s_4) \} = s_4 = SC, \]

since
\[ Low((s_4, s_3), (3/5, 2/5)) = s_k, k = 3 + round((3/5),(4-3)) = 3 + 1 = 4 \Rightarrow s_k = s_4. \]

\[ E^3_v(x_1, x_4) = Low((s_4, s_3, s_3), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8)Low((s_3)(1)) \} \]
\[ = C \{ (0.2, s_4), (0.8, s_3) \} = s_{k(1,4)}. \]

\[ k(1, 4) = 3 + round((0.2),(4-3)) = 3 + 0 = 3 \Rightarrow s_{k(1,4)} = s_3 = VLC. \]

\[ E^3_v(x_2, x_1) = Low((s_4, s_5, s_5), (0.2, 0.5, 0.3)) = C \{ (0.5, s_5), (0.5)Low((s_4, s_3)(3/5, 2/5)) \} \]
\[ = C \{ (0.5, s_5), (0.5, s_5) \} = s_5 = IM, \]

since
\[ Low((s_5, s_4), (3/5, 2/5)) = s_k, k = 4 + round((3/5),(5-4)) = 4 + 1 = 5. \]

\[ E^3_v(x_2, x_2) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, Low((s_4), (1)) \} \]
\[ = C \{ (0.2, s_4), (0.8, s_4) \} = s_4 = SC. \]
$E^3_{v}(x_2, x_3) = Low((s_4, s_3, s_4), (0.2, 0.5, 0.3)) = C \{ (0.5, s_4), (0.5, s_3) \} = s_{k(2,3)}$

$k(2,3) = 3 + \text{round}((0.5),(4 - 3)) = 3 + 1 = 4, \Rightarrow s_{k(2,3)} = s_4 = SC$.

$E^3_{v}(x_2, x_4) = Low((s_4, s_3, s_3), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8)Low((s_3)(1)) \} = C \{ (0.2, s_4), (0.8, s_3) \} = s_{k(2,4)}$.

$k(2,4) = 3 + \text{round}((0.2),(4 - 3)) = 3 + 0 = 3 \Rightarrow s_{k(2,4)} = s_3 = VLC$.

$E^3_{v}(x_3, x_1) = Low((s_5, s_5, s_4), (0.2, 0.5, 0.3)) = C \{ (0.2, s_5), (0.8)Low((s_5, s_4)(5/8, 3/8)) \}$

$= C \{ (0.2, s_5), (0.8, s_5) \} = s_{k(3,1)}$.

since

$Low((s_5, s_4), (5/8, 3/8)) = s_k, k' = 4 + \text{round}((5/8),(5 - 4)) = 4 + 1 = 5$.

$k(3,1) = 5 + \text{round}((0.2),(5 - 5)) = 5 + 0 = 5 \Rightarrow s_{k(3,1)} = s_5 = IM$.

$E^3_{v}(x_3, x_2) = Low((s_3, s_5, s_4), (0.2, 0.5, 0.3)) = C \{ (0.2, s_3), (0.8, Low((s_5, s_4), (5/8, 3/8)) \}$

$= C \{ (0.2, s_3), (0.8, s_5) \} = s_{k(3,2)}$.

since

$Low((s_5, s_4), (5/8, 3/8)) = s_k, k' = 4 + \text{round}((5/8),(5 - 4)) = 4 + 1 = 5$.

$k(3,2) = 4 + \text{round}((0.8),(5 - 4)) = 4 + 1 = 5 \Rightarrow s_{k(3,2)} = s_5 = IM$.

$E^3_{v}(x_3, x_3) = Low((s_4, s_4, s_4), (0.2, 0.5, 0.3)) = C \{ (0.2, s_4), (0.8, Low((s_4, s_4), (1)) \}$

$= C \{ (0.2, s_4), (0.8, s_4) \} = s_4 = SC$.

$E^3_{v}(x_3, x_4) = Low((s_4, s_4, s_5), (0.2, 0.5, 0.3)) = C \{ (0.3, s_5), (0.8)Low((s_4)(1)) \}$

$= C \{ (0.2, s_5), (0.8, s_4) \} = s_{k(3,4)}$.

$k(3,4) = 4 + \text{round}((0.2),(5 - 4)) = 4 + 0 = 4 \Rightarrow s_{k(3,4)} = s_4 = SC$.

$E^3_{v}(x_4, x_1) = Low((s_5, s_6, s_4), (0.2, 0.5, 0.3)) = C \{ (0.5, s_6), (0.5)Low((s_5, s_4)(2/5, 3/5)) \}$

$= C \{ (0.5, s_6), (0.5, s_4) \} = s_{k(4,1)}$. 


We obtain the linguistic non-dominance degree

\[
E_{V}^{3} = \begin{bmatrix}
SC & VLC & SC & VLC \\
IM & SC & VLC & VLC \\
IM & SC & SC & SC \\
IM & VLC & VLC & SC
\end{bmatrix}.
\]  
(102)

Calculate FCS by Definition 2.

\[
fcs_{V}^{3}(x_{i}) = Low((s_{x_{i}}), (1/3, 2/3)) = C \{ (1/3, s_{x_{i}}), (2/3, s_{x_{i}}) \} = s_{k},
\]

\[k = 3 + round((1/3). (4 - 3)) = 3 + 0 = 3 \quad \Rightarrow \quad s_{k} = s_{3} = SC \Rightarrow \quad fcs_{V}^{3}(x_{i}) = SC.
\]
\[ fcs^3_v(x_2) = \text{Low}((s_5, s_4, s_3), (1/3, 2/3)) = C \{(1/3, s_5), (2/3, \text{Low}((s_4, s_3), (0.5, 0.5))\}\} \\
= C \{(1/3, s_5), (2/3, s_4)\} = s_k, \]

since \( \text{Low}((s_4, s_3), (0.5, 0.5)) = s_k, k' = 3 + \text{round}((0.5),(4 - 3)) = 3 + 1 = 4 \Rightarrow s_k = s_4 \)

\[ k = 4 + \text{round}((1/3),(5 - 4)) = 4 + 0 = 4 \Rightarrow s_k = s_4 = SC. \]

\[ fcs^3_v(x_3) = \text{Low}((s_5, s_4), (2/3, 1/3)) = C \{(2/3, s_5), (1/3, s_4)\} = s_k, \]

\[ k = 4 + \text{round}((2/3),(5 - 4)) = 4 + 1 = 5 \Rightarrow s_k = s_5 = IM \Rightarrow fcs^3_v(x) = IM. \]

\[ fcs^3_v(x_4) = \text{Low}((s_5, s_3), (1/3, 2/3)) = C \{(1/3, s_5), (2/3, s_3)\} = s_k, \]

\[ k = 3 + \text{round}((1/3),(5 - 4)) = 3 + 0 = 3 \Rightarrow s_k = s_3 = VLC \Rightarrow fcs^3_v(x_4) = VLC. \]

We obtain 
\[ FCS^3_v = \{\text{VLC} / x_1, \text{SC} / x_2, \text{IM} / x_3, \text{VLC} / x_4\} \quad (103) \]

Finally, we obtain IFE for criterion C³

\[ IFE^3 = \{(MC, VLC) / x_1, (SC, SC) / x_2, (SC, IM) / x_3, (SC, VLC) / x_4\} \quad (104) \]

and the FPS for criterion C³ is \( x_1 \).

\text{Step2. Use \{(FCS^l_M, FCS^l_v), l = 1, 2, 3\} and the criteria’s important weights \( \beta = \{0.35, 0.4, 0.25\}\), calculate the aggregated fuzzy collective solutions (aFCS) as in Definition 3.}

\[ aFCS = \{afcs(x_i) / x_1, ..., afcs(x_n) / x_n\}, \quad (105) \]

where \( afcs_M(x_i) = \text{Low}(S, U_M \beta), \quad i = 1, ..., n \quad (106) \)

where \( U_M \beta = [u_M \beta_1, ..., u_M \beta_T] \), \( t = 1, ..., T \), \( U_M \beta_t = \sum \{\beta_t : fcs^l_M(x_i) = s_t\} \quad (107) \)

Calculate afcs_M(x_i). Because \( fcs^1_M(x_1) = s_5, fcs^2_M(x_1) = s_6, fcs^3_M(x_1) = s_6, \)
\[ afcs_M(\chi) = \text{Low}(s_6, s_5, (0.65, 0.35)) = C \{ (0.65, s_6), (0.35, s_5) \} = s_k, \]

\[ k = 5 + \text{round}((0.65)(6 - 5)) = 5 + 1 = 6 \Rightarrow s_k = s_6 = MC \Rightarrow afcs_M(\chi) = MC. \]

Calculate \[ afcs_M(\chi_2). \] Because

\[ fcs_M^1(\chi_2) = s_4, \quad fcs_M^2(\chi_2) = s_4, \quad fcs_M^3(\chi_2) = s_4, \]

\[ afcs_M(\chi_2) = \text{Low}(s_4), (1) = s_4 = SC. \]

Calculate \[ afcs_M(\chi_3). \] Because

\[ fcs_M^1(\chi_3) = s_4, \quad fcs_M^2(\chi_3) = s_4, \quad fcs_M^3(\chi_3) = s_4, \]

\[ afcs_M(\chi_3) = \text{Low}(s_4), (1) = s_4 = SC. \]

Calculate \[ afcs_M(\chi_4). \] Because

\[ fcs_M^1(\chi_4) = s_7, \quad fcs_M^2(\chi_4) = s_5, \quad fcs_M^3(\chi_4) = s_4, \]

\[ afcs_M(\chi_4) = \text{Low}(s_7, s_5, s_4, (0.35, 0.4, 0.25)) = C \{ (0.35, s_7), (0.65, \text{Low}(s_5, s_4), (0.4/0.65, 0.25/0.65)) \} = s_k, \]

since \[ \text{Low}(s_5, s_4), (0.4/0.65, 0.25/0.65)) = s_k, k = 4 + \text{round}((0.4/0.65)(5 - 4)) = 4 + 1 = 5 \Rightarrow s_k = s_5, \]

\[ k = 5 + \text{round}((0.35)(7 - 5)) = 5 + 1 = 6 \Rightarrow s_k = s_6 = MC \Rightarrow afcs_M(\chi_4) = MC. \]

Finally, we obtain

\[ aFCS_m = \{ MC / \chi_1, SC / \chi_2, SC / \chi_3, MC / \chi_4 \}. \] (108)

Calculate

\[ aFCS_v = \{ afcs_v(\chi_1) / \chi_1, ..., afcs_v(\chi_4) / \chi_4 \}, \] (109)

where \[ afcs_v(\chi_i) = \text{Low}(S, U_{\beta t}). \] (110)

where \[ U_{\beta t} = \left[ \mu_{\beta t}, ..., \mu_{\beta T} \right] \], \[ t = 1, ..., T, \]

\[ U_{\beta t} = \sum_i \left\{ \beta_i : afcs_v(\chi_i) = s_1 \right\}. \] (111)

Calculate \[ afcs_v(\chi_1). \] Because \[ fcs_v^1(\chi_1) = s_3, \quad fcs_v^2(\chi_1) = s_3, \quad fcs_v^3(\chi_1) = s_3, \]

\[ afcs_v(\chi_1) = \text{Low}(s_3), (1)) = s_3 = VLC. \]

Calculate \[ afcs_v(\chi_2). \] Because \[ fcs_v^1(\chi_2) = s_4, \quad fcs_v^2(\chi_2) = s_4, \quad fcs_v^3(\chi_2) = s_3, \]

\[ afcs_M(\chi_2) = \text{Low}(s_4, s_4), (0.65, 0.35) = s_k. \]
We obtain 
\[ \{/ / / / \} = \{/ / / / \}. \]

Finally, we obtain Aggregated Intuitionistic Fuzzy Evaluation (aIFE)
\[ aIFE = \{(MC, VLC) / x_1, (SC, VLC) / x_2, (SC, SC) / x_3, (MC, VLC) / x_4\}. \]

The Aggregated Fuzzy Pareto Solution (aFPS) are \( \{x_1, x_4\} \).

7.2. Computing with the aggregation procedure 2

Step 1.

For each criterion \( C^l \), \( l=1,2,3 \) calculate the linguistic dominance degrees as in (51, 53) we obtain:
\[
E^1_M = \begin{bmatrix}
IM & ML & MC & EU \\
SC & IM & IM & EU \\
SC & IM & IM & EU \\
ML & IM & EL & IM \\
\end{bmatrix},
E^2_M = \begin{bmatrix}
IM & ML & MC & MC \\
SC & IM & IM & SC \\
VLC & IM & IM & SC \\
SC & SC & MC & IM \\
\end{bmatrix},
E^3_M = \begin{bmatrix}
IM & ML & IM & ML \\
VLC & IM & IM & IM \\
SC & SC & IM & SC \\
VLC & IM & SC & IM \\
\end{bmatrix}
\]

and
\[
E^1_V = \begin{bmatrix}
SC & VLC & VLC & SC \\
EU & SC & VLC & EU \\
VLC & IM & SC & EU \\
EU & VLC & EU & SC \\
\end{bmatrix},
E^2_V = \begin{bmatrix}
SC & VLC & VLC & VLC \\
IM & SC & VLC & SC \\
IM & SC & SC & SC \\
IM & SC & VLC & SC \\
\end{bmatrix},
E^3_V = \begin{bmatrix}
SC & ML & IM & ML \\
VLC & SC & IM & IM \\
SC & SC & SC & SC \\
VLC & IM & SC & SC \\
\end{bmatrix}
\]

Step 2.

2.a1. Using these relations of linguistic relative dominance degrees \( \{E^1_M, E^2_M, E^3_M\} \) and the important weights \( \beta = \{0.35, 0.4, 0.25\} \), calculate the totally social opinion relation:
\[ Q_M = \left[ q_M(x_i, x_j) \right], \quad i = 1, \ldots, 4, \quad j = 1, \ldots, 4. \]  
\[ (114) \]

where

\[ q_M(x_i, x_j) = \text{Low}(S, U_{qM}), \quad U_{qM} = \left[ u_{qj}, \ldots, u_{ql} \right]. \]  
\[ (115) \]

where

\[ u_{ql} = W_{ij}^q(s_t) = \sum_l \{ \beta_{ijl} : E_{M}^{q}(i, j) = s_t \}, t = 1, \ldots, T. \]  
\[ (116) \]

\[ q_M(x_1, x_2) = \text{Low}(s_5, s_7, s_5), (0.35, 0.4, 0.25)) = \text{Low}(s_5), (1) = s_5 = IM. \]

\[ q_M(x_1, x_2) = \text{Low}(s_7, s_6, s_7), (0.35, 0.4, 0.25)) = \text{Low}(s_7, 0.6, (s_6, 0.4)) = s_{k(1,2)}, \]

\[ k(1, 2) = 6 + \text{round}((0.6)(7 - 6)) = 6 + 1 = 7 \quad \Rightarrow s_{k(1,2)} = s_7 = ML \Rightarrow q_M(x_1, x_2) = ML. \]

\[ q_M(x_1, x_3) = \text{Low}(s_6, s_6, s_7), (0.2, 0.5, 0.3)) = \text{C} \{ (0.7, s_6), (0.3, s_3) \} = s_{k(1,3)}, \]

\[ k(1, 3) = 5 + \text{round}((0.7)(6 - 5)) = 5 + 1 = 6 \quad \Rightarrow s_{k(1,3)} = s_6 = MC. \]

\[ q_M(x_1, x_4) = \text{Low}(s_2, s_6, s_7), (0.35, 0.4, 0.25)) = \text{C} \{ (0.25, s_7), (0.75, \text{Low}(s_6, s_2), (40/75, 35/75)) \}
\[ \quad \quad \quad = \text{C} \{ (0.25, s_7), (0.75, s_4) \} = s_{k(1,4)}, \]

since

\[ \text{Low}(s_6, s_2), (40/75, 25/75)) = s_k, k' = 2 + \text{round}((40/75)((6 - 2)) = 2 + 2 = 4 \quad \Rightarrow s_{k'} = s_4, \]

\[ k(1, 4) = 4 + \text{round}((0.25)(7 - 4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(1,4)} = s_5 = IM. \]

\[ q_M(x_2, x_1) = \text{Low}(s_4, s_4, s_3), (0.35, 0.4, 0.25)) = \text{C} \{ (0.75, s_4), (0.25, s_3) \} = s_{k(2,1)}, \]

\[ k(2, 1) = 3 + \text{round}((0.75)(4 - 3)) = 3 + 1 = 4 \quad \Rightarrow s_{k(2,1)} = s_4 = SC. \]

\[ q_M(x_2, x_2) = \text{Low}(s_5, s_5, s_5), (0.35, 0.4, 0.25)) = \text{Low}(s_5), (1) = s_5 = IM. \]

\[ q_M(x_2, x_3) = \text{Low}(s_5, s_5, s_3), (0.35, 0.4, 0.25)) = \text{Low}(s_5), (1) = s_5 = IM. \]

\[ q_M(x_2, x_4) = \text{Low}(s_2, s_4, s_5), (0.35, 0.4, 0.25)) = \text{C} \{ (0.25, s_5), (0.75, \text{Low}(s_4, s_2), (40/75, 35/75)) \}
\[ \quad \quad \quad = \text{C} \{ (0.25, s_5), (0.75, s_3) \} = s_{k(2,4)}, \]

since

\[ \text{Low}(s_4, s_2), (40/75, 25/75)) = s_k, k' = 2 + \text{round}((40/75)(4 - 2)) = 2 + 1 = 3 \quad \Rightarrow s_{k'} = s_3, \]

\[ k(2, 4) = 3 + \text{round}((0.25)(5 - 3)) = 3 + 1 = 4 \quad \Rightarrow s_{k(2,4)} = s_4 = SC. \]

\[ q_M(x_3, x_1) = \text{Low}(s_4, s_3, s_4), (0.35, 0.4, 0.25)) = \text{C} \{ (0.6, s_4), (0.4, s_3) \} = s_{k(3,1)}, \]
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\[ k(3,1) = 3 + \text{round}((0.6)(4-3)) = 3 + 1 = 4 \quad \Rightarrow s_{k(3,1)} = s_4 = SC. \]

\[ q_M (x_3, x_2) = \text{Low}((s_5, s_4, s_4), (0.35, 0.4, 0.25)) = C \{ (0.35, s_5), (0.65, s_4) \} = s_{k(3,2)}, \]

\[ k(3,2) = 4 + \text{round}((0.35)(5-4)) = 4 + 0 = 4 \quad \Rightarrow s_{k(3,2)} = s_4 = SC. \]

\[ q_M (x_3, x_3) = \text{Low}((s_5, s_5, s_5), (0.35, 0.4, 0.25)) = \text{Low}((s_5), (1)) = s_5 = IM. \]

\[ q_M (x_3, x_4) = \text{Low}((s_2, s_4, s_4), (0.35, 0.4, 0.25)) = C \{ (0.65, s_4), (0.35, s_2) \} = s_{k(3,4)}, \]

\[ k(3,4) = 2 + \text{round}((0.65)(4-2)) = 2 + 1 = 3 \quad \Rightarrow s_{k(3,4)} = s_3 = VLC. \]

\[ q_M (x_4, x_1) = \text{Low}((s_7, s_4, s_3), (0.35, 0.4, 0.25)) = C \{ (0.25, s_7), (0.75, s_4, s_3, (40/75, 35/75)) \} = C \{ (0.25, s_7), (0.75, s_4) \} = s_{k(4,1)}, \]

since

\[ \text{Low}((s_4, s_3), (40/75, 25/75)) = s_4, k' = 3 + \text{round}((40/75)(4-3)) = 3 + 1 = 4 \Rightarrow s_{k'} = s_4, \]

\[ k(4,1) = 4 + \text{round}((0.25)(7-4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(4,1)} = s_5 = IM. \]

\[ q_M (x_4, x_2) = \text{Low}((s_5, s_4, s_5), (0.35, 0.4, 0.25)) = C \{ (0.6, s_5), (0.4, s_4) \} = s_{k(4,2)}, \]

\[ k(4,2) = 4 + \text{round}((0.6)(5-4)) = 4 + 1 = 5 \quad \Rightarrow s_{k(4,2)} = s_5 = IM. \]

\[ q_M (x_4, x_3) = \text{Low}((s_8, s_6, s_4), (0.35, 0.4, 0.25)) = C \{ (0.35, s_8), (0.65, s_6, s_4), (40/65, 25/65) \} = C \{ (0.35, s_8), (0.65, s_5) \} = s_{k(4,3)}, \]

\[ \text{Low}((s_6, s_4), (40/65, 25/65)) = s_5, k' = 4 + \text{round}((40/75)(6-4)) = 4 + 1 = 5 \Rightarrow s_{k'} = s_5, \]

\[ k(4,3) = 5 + \text{round}((0.35)(8-5)) = 5 + 1 = 6 \Rightarrow s_{k(4,3)} = s_6 = MC. \]

\[ q_M (x_4, x_4) = \text{Low}((s_5, s_5, s_5), (0.35, 0.4, 0.25)) = \text{Low}((s_5), (1)) = s_5 = IM. \]

We obtain the totally social opinion relation

\[ Q_M = \begin{bmatrix}
IM & ML & MC & IM \\
SC & IM & IM & SC \\
SC & IM & IM & VLC \\
IM & IM & MC & IM
\end{bmatrix}. \quad (117) \]

2.a2. Using these relations of linguistic non-dominance degrees \( \{ E^1_v, E^2_v, E^3_v \} \) and the important weights \( \beta = \{0.35, 0.4, 0.25\} \), calculate the totally social opinion relation
\[ Q_v = \left[ q_v(x_i, x_j) \right], i = 1, \ldots, 4, j = 1, \ldots, 4. \]  \hfill (118)

where

\[ q_v(x_i, x_j) = \text{Low}(S, U_{q_v}), \quad U_{q_m} = \left[ u_{q1}, \ldots, u_{q4} \right] \]  \hfill (119)

where

\[ u_{q_t} = W_t^q(s_t) = \sum_{i=1}^{T} \{ \beta_t : E_v^t(i, j) = s_t \}. \quad t = 1, \ldots, T. \]  \hfill (120)

\[ q_v(x_1, x_1) = \text{Low}(s_4, s_4, s_4, (0.35, 0.4, 0.25)) = \text{Low}(s_4, (1)) = s_4 = SC. \]

\[ q_v(x_1, x_2) = \text{Low}(s_3, s_3, s_3, (0.35, 0.4, 0.25)) = \text{Low}(s_3, (1)) = s_3 = VLC. \]

\[ q_v(x_1, x_3) = \text{Low}(s_3, s_3, s_4, (0.35, 0.4, 0.25)) = C\{ (0.25, s_4), (0.75, s_3) \} = s_{k(1,3)}, \]

\[ k(1,3) = 3 + \text{round}((0.25),(4-3)) = 3 + 0 = 3 \Rightarrow s_{k(1,3)} = s_3 = VLC. \]

\[ q_v(x_1, x_4) = \text{Low}(s_4, s_3, s_3, (0.35, 0.4, 0.25)) = C\{ (0.35, s_4), (0.65, s_3) \} = s_{k(1,4)}, \]

\[ s_{k(1,4)} = 3 + \text{round}((0.35),(4-3)) = 3 + 0 = 3 \Rightarrow s_{k(1,4)} = s_3 = VLC. \]

\[ q_v(x_2, x_1) = \text{Low}(s_2, s_3, s_3, (0.35, 0.4, 0.25)) = C\{ (0.75, s_3), (0.25, s_2) \} = s_{k(2,1)}, \]

\[ k(2,1) = 2 + \text{round}((0.75),(5-2)) = 2 + 2 = 4, \Rightarrow s_{k(2,1)} = s_4 = SC. \]

\[ q_v(x_2, x_2) = \text{Low}(s_4, s_4, s_4, (0.35, 0.4, 0.25)) = \text{Low}(s_4, (1)) = s_4 = SC. \]

\[ q_v(x_2, x_3) = \text{Low}(s_3, s_3, s_3, (0.35, 0.4, 0.25)) = \text{Low}(s_3, (1)) = s_3 = VLC. \]

\[ q_v(x_2, x_4) = \text{Low}(s_2, s_4, s_3, (0.35, 0.4, 0.25)) = C\{ (0.4, s_4), (0.6, \text{Low}(s_3, s_2), (25/60,35/60)) \} = C\{ (0.25, s_3), (0.6, s_2) \} = s_{k(2,4)}, \]

since

\[ \text{Low}(s_3, s_2), (25/60,35/60) = s_{k'}, k' = 2 + \text{round}((25/60),(3-2)) = 2 + 0 = 2 \Rightarrow s_{k'} = s_2, \]

\[ k(2,4) = 2 + \text{round}((0.25),(5-2)) = 2 + 1 = 3 \Rightarrow s_{k(2,4)} = s_3 = VLC. \]

\[ q_v(x_3, x_1) = \text{Low}(s_3, s_5, s_3, (0.35, 0.4, 0.25)) = C\{ (0.65, s_5), (0.35, s_3) \} = s_{k(3,1)}, \]

\[ k(3,1) = 3 + \text{round}((0.65),(5-3)) = 3 + 1 = 4, \Rightarrow s_{k(3,1)} = s_4 = SC. \]

\[ q_v(x_3, x_2) = \text{Low}(s_5, s_4, s_4, (0.35, 0.4, 0.25)) = C\{ (0.35, s_5), (0.65, s_4) \} = s_{k(3,2)}, \]

\[ k(3,2) = 4 + \text{round}((0.35),(5-4)) = 4 + 0 = 4 \Rightarrow s_{k(3,2)} = s_4 = SC. \]

\[ q_v(x_3, x_3) = \text{Low}(s_4, s_4, s_4, (0.35, 0.4, 0.25)) = \text{Low}(s_4, (1)) = s_4 = SC. \]
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\[ q_V(x_3, x_4) = \text{Low}(s_2, s_4, s_4, (0.35, 0.4, 0.25)) = C \{(0.65, s_4), (0.35, s_2)\} = s_{k(3,4)}, \]

\[ k(3, 4) = 2 + \text{round}((0.65).(4 - 2)) = 2 + 1 = 3, \Rightarrow s_{k(3,4)} = s_3 = VLC. \]

\[ q_V(x_4, x_1) = \text{Low}(s_2, s_5, s_3, (0.35, 0.4, 0.25)) = C \{(0.65, s_3), (0.35, s_2)\} = s_{k(4,1)}, \]

\[ k(4,1) = 2 + \text{round}((0.65).(5 - 2)) = 2 + 2 = 4, \Rightarrow s_{k(4,1)} = s_4 = SC. \]

\[ q_V(x_4, x_2) = \text{Low}(s_3, s_3, s_3, (0.35, 0.4, 0.25)) = \text{Low}(s_3, (1)) = s_3 = VLC. \]

\[ q_V(x_4, x_3) = \text{Low}(s_2, s_3, s_3, (0.35, 0.4, 0.25)) = C \{(0.65, s_3), (0.35, s_2)\} = s_{k(4,3)}, \]

\[ k(4,3) = 2 + \text{round}((0.65).(3 - 2)) = 2 + 1 = 3, \Rightarrow s_{k(4,3)} = s_3 = VLC. \]

\[ q_V(x_4, x_4) = \text{Low}(s_4, s_4, s_4, (0.35, 0.4, 0.25)) = \text{Low}(s_4, (1)) = s_4 = SC. \]

We obtain the totally social opinion relation using intuitionistic linguistic non-preference relations:

\[
Q_V = \begin{bmatrix}
SC & SC & SC & SC \\
VLC & SC & SC & SC \\
SC & SC & SC & VLC \\
VLC & SC & VLC & SC
\end{bmatrix}.
\]

(121)

2.b. Using \(Q_M, Q_V\) calculate the fuzzy collective solution, calculate

\[
FCS_{QM} = \{fcs_{QM}(x_1)/x_1, fcs_{QM}(x_2)/x_2, fcs_{QM}(x_3)/x_3, fcs_{QM}(x_4)/x_4\},
\]

(122)

\[
fcs_{QM}(x_1) = \text{Low}(s_2, s_5, s_5, (1/3, 1/3, 1/3)) = C\{(1/3, s_7), (2/3, \text{Low}(s_6, s_5), (0.5, 0.5))\} = C\{(1/3, s_7), (2/3, s_6)\} = k(1),
\]

since

\[
\text{Low}(s_6, s_5, (0.5, 0.5)) = C((0.5, s_6), (0.5, s_5)) = s_k, k' = 5 + \text{round}((0.5).(6 - 5)) = 5 + 1 = 6,
\]

\[ k(1) = 6 + \text{round}((1/3).(7 - 6)) = 6 + 0 = 6 \Rightarrow s_{k(1)} = s_6 = MC. \]

\[
fcs_{QM}(x_2) = \text{Low}(s_5, s_4, (1/3, 2/3)) = C((1/3, s_5), (2/3, s_4)) = s_{k(2)},
\]

\[ k(2) = 4 + \text{round}((1/3).(5 - 4)) = 4 + 0 = 4 \Rightarrow s_{k(2)} = s_4 = SC. \]

\[
fcs_{QM}(x_3) = \text{Low}(s_5, s_4, s_3, (1/3, 1/3, 1/3)) = C\{(1/3, s_5), (2/3, \text{Low}(s_4, s_3), (0.5, 0.5))\} = C\{(1/3, s_5), (2/3, s_4)\} = s_k, k = 4 + \text{round}((1/3).(5 - 4)) = 4 + 0 = 4, s_k = s_4 = SC.
\]

\[
fcs_{QM}(x_4) = \text{Low}(s_6, s_5, (1/3, 2/3)) = C\{(1/3, s_6), (2/3, s_5)\} = s_k, k = 5 + \text{round}((1/3).(6 - 5)) = 5 + 0 = 5 \Rightarrow s_k = s_5 = IM.
\]
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and \( FCS_{Q_M} = \{MC / x_1, SC / x_2, SC / x_3, IM / x_4\} \).

Calculate the
\[
FCS_{Q_V} = \{fc_{Q_V}(x_1) / x_1, fc_{Q_V}(x_2) / x_2, fc_{Q_V}(x_3) / x_3, fc_{Q_V}(x_4) / x_4\},
\]
\[
f_{fc_{Q_V}}(x_1) = Low((s_3), (1)) = s_3 = VLC.
\]
\[
f_{fc_{Q_V}}(x_2) = Low((s_4, s_3), (2/3, 1/3)) = C((2/3, s_4), (1/3, s_3)) = s_k(2),
k(2) = 3 + round((2/3), (4 - 3)) = 3 + 1 = 4, \quad \Rightarrow s_k(2) = s_4 = SC.
\]
\[
f_{fc_{Q_V}}(x_3) = Low((s_4, s_3), (2/3, 1/3)) = C((2/3, s_4), (1/3, s_3)) = s_k(3),
k(3) = 3 + round((2/3), (4 - 3)) = 3 + 1 = 4, \quad \Rightarrow s_k(3) = s_4 = SC.
\]
\[
f_{fc_{Q_V}}(x_4) = Low((s_4, s_3), (1/3, 2/3)) = C((1/3, s_4), (2/3, s_3)) = s_k(4),
k(4) = 3 + round((1/3), (4 - 3)) = 3 + 0 = 3, \quad \Rightarrow s_k(4) = s_3 = VLC.
\]
We obtain \( FCS_{Q_V} = \{VLC / x_1, SC / x_2, SC / x_3, VLC / x_4\} \).

Finally, we obtain Intuitionistic Fuzzy Evaluation:
\[
IFE_{Q} = \{(MC, VLC) / x_1, (SC, SC) / x_2, (SC, SC) / x_3, (IM, VLC) / x_4\}.
\]

From \( IFE_{Q} \) we choose the FPS of the problem. The Fuzzy Pareto Solution of the
problem is \( \{x_1\} \).

7.3. Computing with the Aggregation procedure 3

Step 1.1. Computing for expert \( e_1 \)

Use
\[
M_{11} = \begin{bmatrix}
IM & MC & ML & EU \\
SC & IM & SC & VLC \\
VLC & MC & IM & EU \\
EL & IM & ML & IM
\end{bmatrix},
M_{12} = \begin{bmatrix}
IM & MC & ML & IM \\
VLC & IM & SC & SC \\
VLC & SC & IM & EU \\
SC & IM & MC & IM
\end{bmatrix},
M_{13} = \begin{bmatrix}
IM & MC & ML & IM \\
SC & IM & MC & SC \\
VLC & SC & IM & IM \\
SC & MC & SC & IM
\end{bmatrix}
\]

with the weights \( \beta = (0.35, 0.4, 0.25) \), calculate \( F_M^j(x_i, x_j) = Low(S, U^j_M) \), \( i, j=1,2,3,4 \)

using (70, 71), we obtain the relative dominance degrees according to \( e_1 \)
Fuzzy Pareto Solution in multi-criteria group decision making

\[ F'_m = \begin{bmatrix} IM & MC & ML & SC \\ SC & IM & SC & SC \\ VLC & IM & IM & VLC \\ IM & IM & MC & IM \end{bmatrix} \]

and using (72, 73) calculate the fuzzy evaluation \( F'_{m} \)

\[ F'_{m} = \{ MC / x_1, SC / x_2, SC / x_3, IM / x_4 \} . \]

Use

\[ V_i = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix} \]

\[ V_i = \begin{bmatrix} SC & VLC & SC & EU \\ SC & SC & IM & VLC \\ MC & SC & SC & MC \\ IM & VLC & SC & SC \end{bmatrix} \]

\[ V_i = \begin{bmatrix} SC & SC & VLC & SC \\ IM & SC & SC & IM \\ SC & SC & VLC & SC \end{bmatrix} \]

with the weights \( \beta = (0.35, 0.4, 0.25) \). Calculate \( F^k_v (x_i, x_j) = Low(S, U^k_v) \), i.e., \( \beta = (0.35, 0.4, 0.25) \). Using (74, 75), we obtain the relative non-dominance degree according to \( e_i \)

\[ F_i^k = \begin{bmatrix} SC & SC & VLC & VLC \\ VLC & SC & SC & VLC \\ IM & SC & SC & IM \\ SC & SC & VLC & SC \end{bmatrix} \]

and using (76) calculate the fuzzy evaluation \( F'_{v} \)

\[ F'_{v} = \{ VLC / x_1, VLC / x_2, IM / x_3, SC / x_4 \} . \]

Finally, we obtain IFE according to \( e_1 \)

\[ \{ (MC, VLC) / x_1, (SC, VLC) / x_2, (SC, IM) / x_3, (IM, SC) / x_4 \} . \]

and the FPS according to \( e_1 \) is \( x_1 \).

Step 1.2. Computing for expert \( e_2 \)

Use

\[ M_{21} = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ ML & IM & EL & IM \end{bmatrix} \]

\[ M_{22} = \begin{bmatrix} IM & ML & MC & MC \\ SC & IM & IM & SC \\ VLC & SC & IM & SC \\ SC & MC & MC & IM \end{bmatrix} \]

\[ M_{23} = \begin{bmatrix} IM & ML & IM & ML \\ IM & ML & IM & IM \\ SC & MC & SC & SC \\ VLC & SC & SC & IM \end{bmatrix} \]

using (70, 71), we obtain the relative dominance degrees according to \( e_2 \).
and the fuzzy evaluation $F^2_E$ is

$$F^2_E = \{MC / x_1, SC / x_2, SC / x_3, IM / x_4\}.$$ 

Using (74, 75), we obtain the relative non-dominance degree according to $e_2$

$$F^2 = \begin{bmatrix}
SC & EU & VLC & SC \\
EU & SC & VLC & EU \\
VLC & MC & SC & VLC \\
EU & EU & VLC & SC \\
\end{bmatrix},$$

and the fuzzy evaluation $F^2_E$ is

$$F^2_E = \{VLC / x_1, VLC / x_2, SC / x_3, VLC / x_4\}.$$ 

Finally, we obtain IFE according to $e_2$

$$\{(MC, VLC) / x_1, (SC, VLC) / x_2, (SC, SC) / x_3, (IM, VLC) / x_4\}$$

and using (76), the FPS according to $e_2$ is $x_1$.

**Step 1.3. Computing for expert $e_3$**

Use

$$M_{31} = \begin{bmatrix}
IM & MC & MC & EU \\
SC & IM & SC & EU \\
ML & IM & EL & IM \\
\end{bmatrix},$$

$$M_{32} = \begin{bmatrix}
IM & IM & MC & MC \\
SC & IM & MC & IM \\
VLC & SC & IM & VLC \\
\end{bmatrix},$$

$$M_{33} = \begin{bmatrix}
IM & ML & SC & MC \\
SC & IM & MC & IM \\
VLC & SC & IM & IM \\
\end{bmatrix}.$$
Fuzzy Pareto Solution in multi-criteria group decision making

$$\mathbf{F}_M^3 = \begin{bmatrix}
\text{IM} & \text{MC} & \text{MC} & \text{IM} \\
\text{SC} & \text{IM} & \text{IM} & \text{SC} \\
\text{SC} & \text{IM} & \text{IM} & \text{VLC} \\
\text{IM} & \text{SC} & \text{MC} & \text{IM}
\end{bmatrix},$$

and the fuzzy evaluation $\mathbf{FE}_M^3$

$$\mathbf{FE}_M^3 = \{\text{MC} / x_1, \text{SC} / x_2, \text{SC} / x_3, \text{IM} / x_4\}.$$  

Use

\[
V_{m1} = \begin{bmatrix}
\text{SC} & \text{SC} & \text{SC} & \text{SC} \\
\text{EU} & \text{SC} & \text{SC} & \text{EU} \\
\text{EU} & \text{SC} & \text{I} & \text{SC} \\
\text{EU} & \text{SC} & \text{I} & \text{SC}
\end{bmatrix}, \quad V_{m2} = \begin{bmatrix}
\text{SC} & \text{SC} & \text{VLC} & \text{SC} \\
\text{IM} & \text{SC} & \text{VLC} & \text{SC} \\
\text{SC} & \text{SC} & \text{SC} & \text{SC} \\
\text{MC} & \text{SC} & \text{SC} & \text{SC}
\end{bmatrix}, \quad V_{m3} = \begin{bmatrix}
\text{SC} & \text{VLC} & \text{SC} & \text{VLC} \\
\text{IM} & \text{SC} & \text{SC} & \text{SC} \\
\text{SC} & \text{SC} & \text{SC} & \text{SC}
\end{bmatrix}.
\]

Using (74, 75), we obtain the relative non-dominance degree according to $e_3$

$$\mathbf{F}_V^3 = \begin{bmatrix}
\text{SC} & \text{SC} & \text{SC} & \text{VLC} \\
\text{SC} & \text{SC} & \text{SC} & \text{VLC} \\
\text{VLC} & \text{SC} & \text{SC} & \text{VLC} \\
\text{SC} & \text{SC} & \text{VLC} & \text{SC}
\end{bmatrix},$$

and the fuzzy evaluation $\mathbf{FE}_V^3$ is

$$\mathbf{FE}_V^3 = \{\text{SC} / x_1, \text{SC} / x_2, \text{VLC} / x_3, \text{SC} / x_4\}.$$  

Finally, we obtain IFE according to $e_3$

$$\left\{(\text{MC, SC}) / x_1, (\text{SC, SC}) / x_2, (\text{SC, VLC}) / x_3, (\text{IM, SC}) / x_4\right\},$$

and the FPS according to $e_3$ is $\left\{x_1, x_3\right\}.$

**Step 2.** Use the fuzzy evaluation $\left\{(\mathbf{FE}_M^k, \mathbf{FE}_V^k), k = 1, 2, 3\right\}$ and the weights $\{w(k); e_k \in E\}$, calculate the aggregated fuzzy evaluation (aFE).

$$a\mathbf{FE}_M = \{(\text{MC} / x_1, \text{SC} / x_2, \text{SC} / x_3, \text{IM} / x_4)\},$$

$$a\mathbf{FE}_V = \{\text{VLC} / x_1, \text{VLC} / x_2, \text{SC} / x_3, \text{SC} / x_4\}.$$  

We obtain the aggregated intuitionistic fuzzy evaluation (aIFE).

$$a\mathbf{IFE} = \{(\text{MC, VLC}) / x_1, (\text{SC, VLC}) / x_2, (\text{SC, SC}) / x_3, (\text{IM, SC}) / x_4\}.$$  

Finally, the aFPS of the problem is $x_1.$
8. Conclusions

This paper focuses on multi criteria group decision making problem under linguistic assessments. The linguistic preference relations model forms a useful tool in representing decision makers’ choices. Some aggregation operators and computing processes using Fuzzy Collective Solution were given in Sections 3 and 4. Then, we proposed a new approach based on the new concept of Fuzzy Pareto Solution for the group decision making based on intuitionistic linguistic preference relations. Some aggregation procedures for the FPS and a computing example were also proposed. In the future, we will further investigate various aggregation procedures in the situations with type-2 intuitionistic fuzzy preference information.

### APPENDIX

\[ M_{11} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & SC & VLC \\ VLC & MC & IM & EU \\ EL & IM & ML & IM \end{bmatrix}, \quad M_{12} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & SC & VLC \\ VLC & SC & IM & EU \\ SC & IM & MC & IM \end{bmatrix}, \quad M_{13} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & SC & VLC \\ VLC & SC & IM & EU \\ SC & IM & MC & IM \end{bmatrix}, \quad M_{21} = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ SC & ML & EL & IM \end{bmatrix}, \quad M_{22} = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ SC & ML & EL & IM \end{bmatrix}, \quad M_{23} = \begin{bmatrix} IM & ML & MC & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ SC & ML & EL & IM \end{bmatrix}, \quad M_{31} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ SC & MC & SC & IM \end{bmatrix}, \quad M_{32} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ SC & MC & SC & IM \end{bmatrix}, \quad M_{33} = \begin{bmatrix} IM & MC & ML & EU \\ SC & IM & MC & EU \\ SC & SC & IM & EU \\ SC & MC & SC & IM \end{bmatrix} \]

\[
V_{11} = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix}, \quad V_{12} = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix}, \quad V_{13} = \begin{bmatrix} SC & SC & VLC & VLC \\ EU & SC & VLC & EU \\ IM & SC & SC & SC \\ EU & SC & VLC & SC \end{bmatrix}, \quad V_{21} = \begin{bmatrix} SC & SC & VLC & VLC \\ SC & MC & SC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \end{bmatrix}, \quad V_{22} = \begin{bmatrix} SC & SC & VLC & VLC \\ SC & MC & SC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \end{bmatrix}, \quad V_{23} = \begin{bmatrix} SC & SC & VLC & VLC \\ SC & MC & SC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \end{bmatrix}, \quad V_{31} = \begin{bmatrix} SC & SC & VLC & VLC \\ SC & MC & SC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \end{bmatrix}, \quad V_{32} = \begin{bmatrix} SC & SC & VLC & VLC \\ SC & MC & SC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \end{bmatrix}, \quad V_{33} = \begin{bmatrix} SC & SC & VLC & VLC \\ SC & MC & SC & VLC \\ IM & SC & VLC & SC \\ IM & SC & SC & SC \end{bmatrix} \]

References