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For Multi-Interval-Valued Fuzzy Sets, Centroid Defuzzification Is Equivalent to Defuzzifying Its Interval Hull: A Theorem

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Abstract. In the traditional fuzzy logic, the expert's degree of certainty in a statement is described either by a number from the interval $[0, 1]$ or by a subinterval of such an interval. To adequately describe the opinion of several experts, researchers proposed to use a union of the corresponding sets – which is, in general, more complex than an interval. In this paper, we prove that for such set-valued fuzzy sets, centroid defuzzification is equivalent to defuzzifying its interval hull.
As a consequence of this result, we prove that the centroid defuzzification of a *general* type-2 fuzzy set can be reduced to the easier-to-compute case when for each x, the corresponding fuzzy degree of membership is *convex*.

1 Formulation of the Problem

Outline of this section. Our main objective is to come up with a centroid defuzzification formula for multi-interval-valued fuzzy sets. Before we start describing our results and algorithms, let us briefly recall why we need centroid defuzzification and why we need multi-interval-valued fuzzy sets. To explain this need:

- we will start with the regular fuzzy sets,
- then we explain the need for interval-valued fuzzy sets, and
- the need for multi-interval-valued fuzzy sets;
- finally, we explain the need for centroid defuzzification for all these types of fuzzy sets.

Need for interval-valued fuzzy sets: a brief reminder. In the traditional fuzzy logic, an expert describes his or her degree of confidence in different statements by a number from the interval $[0, 1]$. In particular, for statements like "x is small" corresponding to different values x, the corresponding degree $\mu(x)$ form a *membership function* describing the imprecise (fuzzy) concept like "small"; see, e.g., [1, 6].

In many practical situations, experts are not comfortable describing their degree of confidence by an exact number; they feel more comfortable describing their degree of confidence by an interval – e.g., by an interval $[0.7, 0.8]$. In particular, for statements like "x is small", the corresponding interval-valued degrees of confidence $\left[\underline{\mu}(x), \overline{\mu}(x)\right]$ form an *interval-valued* membership function.

The intuitive meaning of this membership function is that in principle, we can have many different number-valued membership functions $\mu(x)$ as long as $\mu(x) \in \left[\underline{\mu}(x), \overline{\mu}(x)\right]$ for every x.

Another case when an interval-valued membership function naturally appears is when we ask several experts. For the same value x, different experts give, in general, different degrees of confidence $\mu_1(x)$, $\ldots$, $\mu_n(x)$. When experts are equally good, there is no reason to select one of these values, it make more sense to consider the interval $\left[\min\limits_i \mu_i(x), \max\limits_i \mu_i(x)\right]$ spanned by these values. This smallest interval containing the values $\mu_1(x), \ldots, \mu_n(x)$ is also known as the *interval hull* of the corresponding finite set $\{\mu_1(x), \ldots, \mu_n(x)\}$.

Need for multi-interval-valued fuzzy sets. Once each expert provides his or her degree $\mu_i(x)$ or interval-valued degree $\left[\underline{\mu}_i(x), \overline{\mu}_i(x)\right]$, then, instead of taking the interval hull of all these degrees, we can get a more adequate description of the experts' opinions if we simply take the *union* of these values and intervals. Such unions are known as *multi-intervals.* If for each x, the experts' degrees of confidence in the corresponding statement "x is small" is described by a multi-interval $M(x)$, then we get a multi-interval-valued membership function $M(x)$; see, e.g., [7].

Centroid defuzzification for regular fuzzy sets. In control (or, more generally, decision) applications, when for each possible value x of control, we know the degree $\mu(x)$ to which this value is reasonable, we then need to decide which control value c to apply.

In fuzzy applications, we usually select the value c for which the weighted mean square deviation from this value is the smallest possible:

$$\int\limits_{\underline{L}}^{\overline{L}} \mu(x) \cdot (x - c)^2\, dx \to \min_c,$$

where $\left[\underline{L}, \overline{L}\right]$ is the range of possible values of x. Differentiating this objective function with respect to the unknown c and equating the derivative to 0, we conclude that

$$c(\mu) = \frac{\int\limits_{\underline{L}}^{\overline{L}} x \cdot \mu(x)\, dx}{\int\limits_{\underline{L}}^{\overline{L}} \mu(x)\, dx}.$$

This formula is known as *centroid defuzzification.*

Centroid defuzzification for interval-valued fuzzy sets. As we have mentioned, an interval-values fuzzy set $\left[\underline{\mu}(x), \overline{\mu}(x)\right]$ means that many different membership functions $\mu(x) \in \left[\underline{\mu}(x), \overline{\mu}(x)\right]$ are possible. For different possible membership functions $\mu(x)$, in general, we have different defuzzification results $c(\mu(x))$. It is therefore reasonable to find the set of all possible value of these results:

$$\{c(\mu) : \underline{\mu}(x) \le \mu(x) \le \overline{\mu}(x) \text{ for all } x\}.$$

It is known (see, e.g., [3–5]) that this range is always an interval $[\underline{c}, \overline{c}]$, where

$$\underline{c} = \frac{\int\limits_{\underline{L}}^{x_-} x \cdot \overline{\mu}(x)\,dx + \int\limits_{x_-}^{\overline{L}} x \cdot \underline{\mu}(x)\,dx}{\int\limits_{\underline{L}}^{x_-} \overline{\mu}(x)\,dx + \int\limits_{x_-}^{\overline{L}} \underline{\mu}(x)\,dx}$$

and

$$\overline{c} = \frac{\int\limits_{\underline{L}}^{x_+} x \cdot \underline{\mu}(x)\,dx + \int\limits_{x_+}^{\overline{L}} x \cdot \overline{\mu}(x)\,dx}{\int\limits_{\underline{L}}^{x_+} \underline{\mu}(x)\,dx + \int\limits_{x_+}^{\overline{L}} \overline{\mu}(x)\,dx}$$

for appropriate values x^- and x^+. These formulas underlie the known algorithms for computing the range $[\underline{c}, \overline{c}]$.

Formulation of the problem. Now, we are ready to formulate our problem. What if instead of an interval-values fuzzy set, we have a multi-interval-valued fuzzy set $M(x)$? What will then be the set

$$\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\}?$$

2 Analysis of the Problem and the Main Result

Discussion. A multi-set is a union of finitely many one-points sets and (closed) intervals. Each of the united sets is closed, thus their union $M(x)$ is closed. In general, we will consider functions that assign, to each value x, a closed set $M(x) \subseteq [0, 1]$; see, e.g., [7].

Each such closed set contains its own infimum $\underline{M}(x) \stackrel{\text{def}}{=} \inf M(x)$ and supremum $\overline{M}(x) \stackrel{\text{def}}{=} \sup M(x)$. The interval hall of the set $M(x)$ is the interval $\left[\underline{M}(x), \overline{M}(x)\right]$.

We assume that this function $M(x)$ is defined for all the values x from some interval $\left[\underline{L}, \overline{L}\right]$. We also assume that the lower and upper bounds $\underline{M}(x)$ and $\overline{M}(x)$

are measurable functions. It turns out that under these conditions, the centroid defuzzification of the set-valued membership function $M(x)$ is equivalent to the centroid defuzzification of its interval hull $\left[\underline{M}(x), \overline{M}(x)\right]$. Let us formulate this result in precise terms.

Definition. *By a* set-valued membership function*, we mean a function M that assigns, to each real number from some interval $\left[\underline{L}, \overline{L}\right]$, a closed set $M(x) \subseteq [0,1]$ for which the functions*

$$\underline{M}(x) = \inf M(x) \text{ and } \overline{M}(x) = \sup M(x)$$

are measurable.

Proposition. *For the centroid defuzzification functional $c(\mu)$, we have*

$$\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\} = \{c(\mu) : \mu(x) \in [\inf M(x), \sup M(x)] \text{ for all } x\}.$$

Comment. Thus, the result of applying centroid defuzzification to the original set-valued fuzzy set $M(x)$ is equivalent to applying the same centroid defuzzification to its interval hull $\left[\underline{M}(x), \overline{M}(x)\right]$.

Proof.

1°. Let us first prove that out of all possible functions $\mu(x) \in M(x)$, the smallest and the largest possible values of $\mu(x)$ are attained when for each x, the value $\mu(x)$ is equal to either $\underline{M}(x)$ or to $\overline{M}(x)$.

It is sufficient to prove this result in the discrete case, when instead of the whole interval $\left[\underline{L}, \overline{L}\right]$, we have finitely many value $x_1, \ldots, x_n$: e.g., the values $x_i = \underline{L} + h \cdot (i-1)$, where $h = \dfrac{\overline{L} - \underline{L}}{n-1}$; the general case can be obtained when we take $n \to \infty$. In this discrete cases, instead of the whole membership function $\mu(x)$, we have n values $\mu(x_n)$, and the centroid defuzzification takes the form

$$c = \frac{\sum\limits_{i=1}^{n} x_i \cdot \mu(x_i) \cdot h}{\sum\limits_{i=1}^{n} \mu(x_i) \cdot h}.$$

Dividing both the numerator and the denominator of this expression by the common factor h, we get a simplified expression

$$c = \frac{\sum\limits_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum\limits_{i=1}^{n} \mu(x_i)}.$$

Let us show that for each j, this expression is either monotonically increasing or monotonically decreasing as a function of $\mu(x_j)$. A monotonic function attains its maximum and its minimum on an interval on the endpoints of this interval, thus, the minimum and maximum are attained when either $\mu(x_j) = \underline{M}(x_j)$ or $\mu(x_j) = \overline{M}(x_j)$.

Let us prove the desired monotonicity. Indeed, the above expression for c can be described in the following equivalent form:

$$c = \frac{\sum\limits_{i \neq j} x_i \cdot \mu(x_i) + x_j \cdot \mu(x_j)}{\sum\limits_{i \neq j} \mu(x_i) + \mu(x_j)}.$$

If we subtract x_j from the right-hand side (and bring the difference to the common denominator) and then add x_j to the result, we get the following equivalent expression:

$$c = x_j + \frac{\sum\limits_{i \neq j} (x_i - x_j) \cdot \mu(x_i)}{\sum\limits_{i \neq j} \mu(x_i) + \mu(x_j)}.$$

The denominator is an increasing function of $\mu(x_i)$, and the numerator does not depend on $\mu(x_i)$ at all. Thus:

– if the numerator is positive, the expression is a decreasing function of $\mu(x_i)$, and
– if the numerator is negative, then the expression is an increasing function of $\mu(x_i)$.

The statement is proven.

2°. We have shown that the maximum and minimum of $c(\mu)$ – when for each x, we have $\mu(x) \in M(x)$ – is equal either to the smallest possible value $\underline{M}(x)$ or to the largest possible value $\overline{M}(x)$. Thus, the maximum and minimum of $c(\mu)$ over all $\mu(x) \in M(x)$ are equal to, correspondingly, the maximum and the minimum of $c(\mu)$ over all $\mu(x) \in \{\underline{M}(x), \overline{M}(x)\}$:

$$\overline{c} = \max\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\} =$$

$$\max\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}$$

and

$$\underline{c} = \min\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\} =$$

$$\min\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}.$$

As we have mentioned earlier, a similar property holds for interval-valued fuzzy sets, when instead of the restriction $\mu(x) \in M(x)$, we impose an interval restriction $\mu(x) \in [\underline{M}(x), \overline{M}(x)]$: here also,

$$\max\left\{c(\mu) : \mu(x) \in \left[\underline{M}(x), \overline{M}(x)\right] \text{ for all } x\right\} =$$

$$\max\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}$$

and

$$\min\left\{c(\mu) : \mu(x) \in \left[\underline{M}(x), \overline{M}(x)\right] \text{ for all } x\right\} =$$

$$\min\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}.$$

Thus, the maximum and minimum of $c(\mu)$ under the *set* condition $\mu(x) \in M(x)$ are equal to the maximum and minimum of $c(\mu)$ under the *interval* condition $\mu(x) \in \left[\underline{M}(x), \overline{M}(x)\right]$.

So, to complete our proof, we need to show that in both cases, every real number in between $\underline{c}$ and $\overline{c}$ belongs to the desired range, i.e., has the form $c(\mu)$ for an appropriate membership function $\mu(x)$, i.e., a membership function for which we have either $\mu(x) \in M(x)$ (in the set case) or $\mu(x) \in [\underline{M}(x), \overline{M}(x)]$ (in the interval case).

We will show that for every $c \in [\underline{c}, \overline{c}]$, we can select a function $\mu(x)$ for which $\mu(x) \in \{\underline{M}(x), \overline{M}(x)\}$ – this would guarantee both that $\mu(x) \in M(x)$ and that $\mu(x) \in [\underline{M}(x), \overline{M}(x)]$.

To prove the existence of such a function, let us start with the functions $\mu_-(x) \in \{\underline{M}(x), \overline{M}(x)\}$ and $\mu_+(x) \in \{\underline{M}(x), \overline{M}(x)\}$ for which $c(\mu_-) = \underline{c}$ and $c(\mu_+) = \overline{c}$. For each value $\ell \in [\,\underline{L}, \overline{L}\,]$, we can now consider an auxiliary function $\mu_\ell(x)$ which is:

- equal to $\mu_+(x)$ for $x \le \ell$ and
- equal to $\mu_-(x)$ for $x > \ell$.

For each x, the value of $\mu_\ell(x)$ is equal to either the value $\mu_-(x)$ or to the value $\mu_+(x)$. Since both of these values are from the set $\{\underline{M}(x), \overline{M}(x)\}$, the value $\mu_\ell(x)$ also belongs to this set for all x.

Since we assumed that the functions $\underline{M}(x)$ and $\overline{M}(x)$ are measurable, we can conclude that the value $c(\mu_\ell)$ is a continuous function of ℓ.

When $\ell = \underline{L}$, the function $\mu_\ell(x)$ coincides with $\mu_-(x)$, and for $\ell = \overline{L}$, it coincides with $\mu_+(x)$. Thus, as ℓ changes from $\underline{L}$ to $\overline{L}$, the value of $c(\mu_\ell)$ continuously changes from $c(\mu_-) = \underline{c}$ to $c(\mu_+) = \overline{c}$. A continuous function attains all intermediate values, so for each $c \in [\underline{c}, \overline{c}]$, there indeed exists a value ℓ for which $c(\mu_\ell) = c$, for the corresponding function $\mu_\ell(x) \in \{\underline{M}(x), \overline{M}(x)\}$.

The statement is proven, and so is the proposition.

3 From Set-Valued to General Type-2 Fuzzy Sets

Type-2 fuzzy sets: reminder. Instead of considering, for each x, a crisp set $M(x)$ of possible values of the degree of confidence $\mu(x)$, it makes sense to consider a more general case, when this set of possible values of the degree is fuzzy. Such situations are known as *type-2 fuzzy sets*; see, e.g., [4, 5].

In precise terms, for each x and for each real number $\mu \in [0, 1]$, instead of deciding whether this number is a possible value of the degree or not, we now have a *degree* $d(\mu, x)$ describing to what extent the number μ is a possible expert's degree of confidence that x satisfies the given property (e.g., "is small").

Centroid defuzzification: general type-2 case. We have a functional $c(\mu)$ defined for crisp function $\mu(x)$. In fuzzy techniques, a natural way to extend this functional to fuzzy-valued membership functions – i.e., to type-2 fuzzy sets – is to use Zadeh's extension principle.

It is known that this principle can be equivalently described in terms of α-cut: for any function $y = f(x_1, \ldots)$ and for fuzzy sets $X_1, \ldots$, the α-cut $Y(\alpha) \stackrel{\text{def}}{=} \{y : \mu_Y(y) \ge \alpha\}$ of the result $Y = f(X_1, \ldots)$ of the result of applying the function f to fuzzy sets $X_1, \ldots$ is equal to the range of the function on the alpha-cuts $X_i(\alpha) = \{x_i : \mu_i(x_i) \ge \alpha\}$ of the inputs X_i:

$$Y(\alpha) = f(X_1(\alpha), \ldots) = \{f(x_1, \ldots) : x_1 \in X_1(\alpha), \ldots\}.$$

In particular, for the centroid defuzzification, we start with a function $c(\mu)$ that depends on infinitely many real-valued inputs $\mu(x)$. For type-2 fuzzy sets, the inputs $\mu(x)$ are also fuzzy. Thus, the result of a centroid defuzzification is also a fuzzy set C. the α-cut $C(\alpha)$ of this fuzzy set C of defuzzification results is equal to the range of the values $c(\mu)$ under the condition that for all x, we have $\mu(x) \in M_x(\alpha) = \{\mu : d(\mu, x) \ge \alpha\}$.

Consequence of our main result: centroid defuzzification of a general type-2 fuzzy set can be reduced to the convex case. Our main result states that for each set-valued function $M(x)$, the range of the centroid defuzzification is equal to the range of its interval hull $\left[\underline{M}(x), \overline{M}(x)\right]$.

Thus, for each type-2 membership function $d(\mu, x)$, the range $C(\alpha)$ is equal to the range computed based on the interval hull $[\inf M_x(\alpha), \sup M_x(\alpha)]$ of the set $M_x(\alpha)$.

In other words, the result C of applying the centroid defuzzification $c(\mu)$ to the general type-2 fuzzy set is equal to the result of applying $c(\alpha)$ to an auxiliary fuzzy set in which each α-cut is a (convex) interval $[\inf M_x(\alpha), \sup M_x(\alpha)]$. Thus, *centroid defuzzification of a general type-2 fuzzy set can indeed be reduced to the convex case* – the case for which there exist efficient algorithms; see, e.g., [2].

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