Fuzzy-Inspired Hierarchical Version of the von~Neumann-Morgenstern Solutions as a Natural Way to Resolve Collaboration-Related Conflicts

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***Abstract*—In situations when several participants collaborate with each other, it is desirable to come up with a fair way to divide the resulting gain between the participants. Such a fair way was proposed by John von Neumann and Oscar Morgenstern, fathers of the modern game theory. However, in some situations, the von Neumann-Morgenstern solution does not exist. To cover such situations, we propose to use a fuzzy-inspired hierarchical version of the von Neumann-Morgenstern (NM) solution. We prove that, in contrast to the original NM solution, the hierarchical version always exists.**

I. Introduction to the Problem

Cooperative games: towards a formal description of collaborative situations. Situations when all participants collaborate with each other are known as *cooperative games*.

Let n denote the number of participants. For simplicity, the easier way to describe the participants is by simply numbering them, so we have participant number 1, participant number 2, etc.

The set $\{1, \ldots, n\}$ of all n participants is usually denoted by N.

The main question: what is a fair way to divide the gains of collaboration. When all the participants collaborate, as a single group N, they jointly gain some value $v(N)$. The question is: what is a fair way to divide this total amount $v(N)$ between n participants, i.e., how to allocate non-negative amounts $x_1, \ldots, x_n$ for which $\sum\limits_{i=1}^{n} x_i = v(N)$.

Such allocations are known as *imputations*.

Definition 1. *By an* imputation*, we mean a tuple* $(x_1, \ldots, x_n)$ *of non-negative numbers for which* $\sum\limits_{i=1}^{n} x_i = v(N)$.

What information we can use for this division. From the human prospective, fair means taking into account everyone's contribution:

- If a person did not bring anything to the table, i.e., when practically the same result could be obtained without him or her, then there is no need to reward this participant.
- On the other hand, if one of the participants brought in the lion's share of the gains – i.e., if this participant alone contributed most of the gain $v(N)$, then this person should take home most of this gain.

Same logic applies not only to individuals, but also to groups of individuals:

- if a group did not contribute anything, then it should not get much, and,
- vice versa, if a group contributed almost everything, it should take home almost everything.

In view of this, to make a fair division, we must take into account how much each individual and each group contributed.

From the mathematical viewpoint, groups of individuals are *subsets* $S \subseteq N$. Such subsets are called *coalitions*.

The contribution of each coalition can be described by describing, for each coalition S, the largest amount $v(S)$ that this coalition could earn if it collaborated only between themselves, without any help from others (and even with resistance by others).

These values $v(S)$ are the information that we need to use to describe a fair division of the overall gain.

It makes sense to only consider gains due to collaboration. Since we are considering collaborative situations, it makes sense to consider only gains due to collaboration, i.e., to subtract, from the gain $v(S)$, the amount $\sum\limits_{i \in S} v(\{i\})$ that individuals from the coalition S would have earned by themselves, without any collaboration.

In other words, instead of the original function $v(S)$, it makes sense to consider a new function $v'(S) \stackrel{\text{def}}{=} v(S) - \sum\limits_{i \in S} v(\{i\})$, for which $v'(\{i\}) = 0$ for all i.

Because of this, in the following text, without losing generality, we will consider functions $v(S)$ for which $v(\{i\}) = 0$ for all i.

Definition 2. *Let n be a positive integer. By a* cooperative game*, we mean a function that assigns, to each subset* $S \subseteq N \stackrel{\text{def}}{=} \{1, \ldots, n\}$*, a non-negative number* $v(S)$ *so that*

- $v(\{i\}) = 0$ *for all* i*, and*

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- *when $S \cap S' = \emptyset$, then $v(S \cup S') \ge v(S) + v(S')$.*

Comment. The requirement that $v(S \cup S') \ge v(S) + v(S')$ comes from the fact that we consider cooperative situations. So, if two disjoint coalitions S and S' collaborate, then jointly they should be able to gain no less than they would get on their own.

How do we enforce a fair solution? If a coalition S believes that in the proposed imputation y, its participants will not get a fair share, this coalition may be able to enforce a fairer solution. Let us describe this in precise terms.

If there is another imputation x which is:

- within the reach of S, i.e., for which $\sum\limits_{i \in S} x_i \le v(S)$, and
- in which all members of S gain more than in x, i.e., $x_i < y_i$ for all $i \in S$,

then S can force the group to switch from y to x.

We say that an imputation x *dominates* the imputation y (and denote it by $x \succ y$) if there is a coalition that can enforce the switch (i.e., for which the above two conditions are satisfied).

The notion of dominance was introduced by John von Neumann and Oscar Morgenstern in their pioneering book [9] that started game theory as a mathematical analysis of conflict situations.

Definition 3. *We that an imputation x* dominates *an imputation y (and denote it by $x \succ y$) if the following two conditions are satisfied:*

- *$x_i > y_i$ for all $i \in S$ and*
- *$\sum\limits_{i \in S} x_i \le v(S)$.*

Ideal solution: the notion of a core. If an imputation y is dominated by another imputation x, this means that, in the opinion of some coalition S, the imputation y is not fair. It is therefore reasonable to consider only imputations which are not dominated by any other imputation.

The set of all non-dominated imputations is known as a *core*; see, e.g., [4], [9].

Definition 4. *A* core *is the set of all non-dominated imputations, i.e., of all imputations y for which $x \not\succ y$ for all x.*

Problem: not all games have cores. Some games have cores, but many do not.

As a result, no matter what imputation we select, it is always possible to switch to a different imputation – and this process can potentially continue forever, without reaching any equilibrium.

What should we do about it?

von Neumann's and Morgenstern's solution to this problem. To avoid the above, von Neumann and Morgenstern suggested that we adopt some *social norms* that would limit the set of possible imputations in such a way that no two imputations within this norm dominate each other.

The social norm has to be *enforceable* meaning that if someone proposes an imputation which is outside this norm, there should be a coalition that forces a switch to a solution within the norm. The resulting definition is known as a *von Neumann-Morgenstern solution* (or *NM-solution*, for short).

Definition 5. *A set C of imputations is called a* von Neumann-Morgenstern solution *if it satisfies the following two properties:*

- *if $x, y \in C$, then $x \not\succ y$;*
- *if $y \notin C$, then there exists an $x \in C$ for which $x \succ y$.*

How to use NM-solutions to make decisions. In this case, our decision making consists of two stages:

- first, all participants agree on an appropriate "social norm", i.e., on an appropriate set of imputations C within which they search for an imputation;
- then, once the set C is selected, the participants select an imputation x from this set.

The two conditions from Definition 5 guarantee that:

- if someone tries to violate an agreement and propose an imputation y outside the set C corresponding to the social norm, then we can force it back into C;
- second, that once an imputation x is selected, no coalition is interested in switching to a different socially acceptable imputation y.

Our goal is to compute the list of all possible social norms C (or at least compute one social norm C).

Comment: social norms do not have to be rational. Social norms may be motivated by some rational arguments, but they can also be rather arbitrary. For example, in the US, we drive on the right side of the road. There is nothing rational about selecting the right side, since in the UK and in several other countries, people drive on the left side, and it is OK – as long as the society selects some norm and everyone agrees to follow it.

First challenge: NM-solutions are difficult to compute. While the notion of NM solution sounds reasonable, it has several challenges. The first challenge is that this notion is difficult to compute.

While there exist algorithms for computing *approximate* NM-solutions [6], it is not even clear whether there exists a general algorithm that can compute exact NM-solutions [2], [8]. Moreover, in the discrete case, the corresponding problem is NP-hard [1], [7], which means that – unless P=NP, which few computer scientists believe to be true – no feasible algorithm can solve all particular cases of this problem.

This situation is not so bad: in most cooperation-related decision problems, there is no big rush, so we have time for computations.

Remaining problem. A more serious challenge is that some game do not have NM-solutions at all. What shall we do in such situations?

What we do in this paper. In this paper, we propose a natural fuzzy-inspired solution to this problem: namely, we propose a

hierarchical version of the NM solution, and we show that, in contrast to the original NM solution, the hierarchical solution always exists.

II. Hierarchical Version of von Neumann-Morgenstern Solution: Main Idea

Main idea. Let us consider a situation in which no NM-solution exists. In this case, we can still select an enforceable social norm, i.e., a set C of imputations such that every imputation y which is not in this set is dominated by some imputations x from this set.

In the case of an NM-solution C, once we select an imputation from the set C, the decision process is over: as long as we are restricting ourselves to socially acceptable imputations, no coalition can force us to change our mind.

In contrast, in the no-NM-solutions case, even after we restrict ourselves to imputations from the set C, switching is still possible.

What should we do?

A natural idea is to the further restrict imputations. In other words, instead of a previous two-stage approach, now we have a multi-stage approach:

- first, we select the sets $C_1 \supset C_2 \supset C_3 \supset \ldots \supset C_k$;
- then, we first force an imputation to be in the set C_1;
- after that, we force the imputation to be in the narrower set C_2,
- then we force the imputation to be in the still narrower set C_3, etc.,
- until we reach an imputation from the final set C_k in which no two imputations dominate each other.

Here, we cannot enforce C_k in one step, but we can enforce C_k in several steps:

- first, we enforce C_1;
- then, we enforce $C_2 \subset C_1$;
- after that, we enforce $C_3 \subset C_2$; etc.
- finally, after enforcing C_{k-1}, we enforce $C_k \subset C_{k-1}$.

From the idea to the precise definition. To produce the precise definition, let us consider a general set of imputations I with a binary relation $\succ$, i.e., in mathematical terms, a directed graph.

Theoretically, the set of possible imputations is infinite, so we can have an infinite graph, but in practice, only finitely many outcomes are possible – for one reason that we need to describe the solution, the length of a realistic description is bounded by some big number, and there are only finitely many descriptions of given length. Thus, from practical viewpoint, it is sufficient to assume that I is a finite graph.

The fact that we can enforce going from C_i to C_{i+1} means that for every element $x \in C_i - C_{i+1}$, there exists an element $y \in C_{i+1}$ that dominates it, i.e., for which $y \succ x$. In the final set, we should have no dominations.

Thus, we arrive at the following definition.

Definition 6. *Let $(I, \succ)$ be a directed graph. By a* hierarchical von Neumann-Morgenstern solution*, we mean a finite nested sequence of sets $C_0 = I \supset C_1 \supset C_2 \supset \ldots \supset C_k$ with the following two properties:*

- *if $x, y \in C_k$, then $x \not\succ y$, and*
- *for every $i \geq 0$, if $y \in C_i - C_{i+1}$, then there exists an $x \in C_{i+1}$ for which $x \succ y$.*

Comment. One can easily see that in the particular case $k = 1$, we get exactly the original definition of the NM-solution.

Discussion. The above definition may be reasonable from the mathematical viewpoint, but does it make sense? Actually, this is exactly how we make decisions. Let us consider an example.

The first thing the society does is enforces restrictions to legal solutions only. This is our set C_1. However, law cannot predict everything, there are always situations when a certain decision may be legally OK, but ethically wrong.

For example, there are laws restricting how much noise we can inflict on our neighbors. However, even if we are within these restrictions, it is not always ethical to always produce a loud noise – e.g., when a neighbor is sick and needs some sleep.

As a result, within the set C_1 of legal actions, there is a subset C_2 of actions which are not only legal but also ethical.

But this is only a first approximation. In reality, there are several levels of ethical behavior – ranging from not harming your neighbor all the way to actively helping the neighbors. In our descriptions, these levels of more and more ethical behavior correspond to sets C_2, C_3, etc.

Depending on which of the sets C_i the action belongs to, its degree of morality increases:

- the lowest degree corresponds to legal actions from the set C_1;
- the next degree corresponds to actions from the smaller set C_2;
- an even higher level of morality corresponds to actions from the yet smaller set C_3, etc.,
- until we finally reach the set C_k of perfectly moral actions.

Relation to fuzzy. The appearance of degrees is in line with the general ideas of fuzzy logic (see, e.g., [3], [5], [10]), where everything is a matter of degree:

- some people are clearly young, some people are clearly not young, but many people are young to a certain degree;
- some actions are clearly moral, some are clearly immoral, but many actions are moral to a certain degree.

The result about NM-solutions explains the need for fuzziness. We cannot say anything about the need for degree of youth – this is just a fact of life that this is how we make judgments.

However, for morality, the need for degrees, as we have shown, can be mathematically justified: it follows from the mathematical result that some games do not have NM-solutions.

Remaining question: do hierarchical solutions always exist? It looks like hierarchical von Neumann-Morgenstern

solutions are reasonable, the question is: do they always exist? This is what we will prove in the next sections.

III. First Result: Hierarchical von Neumann-Morgenstern Solutions Always Exist

Proposition 1. *Every directed finite graph $(I, \succ)$ has a hierarchical von Neumann-Morgenstern solution.*

Proof. We have the set $C_0 = I$. Let us inductively construct the desired sequence $C_0 \supset C_1 \supset \ldots \supset C_k$.

let us assume that we already have constructed the sequence $C_0 \supset C_1 \supset \ldots \supset C_i$ for which, for every $j \leq i-1$ and for every $y \in C_j - C_{j+1}$, there exists an $x \in C_{j+1}$ for which

$$x \succ y.$$

If no two elements $x, y \in C_i$ dominate each other, i.e., if $x \not\succ y$ for all $x, y \in C_i$, then the sequence

$$C_0 \supset C_1 \supset \ldots \supset C_i$$

is a hierarchical NM-solution.

If there are elements $x, y \in C_i$ for which $x \succ y$, then we can take $C_{i+1} = C_i - \{y\}$. Indeed, in this case, the only element y from $C_i - C_{i+1}$ is dominated by some element from C_i: namely, by the element x.

At each step, we decrease the size of the set C_i. Since we started with a finite graph $C_0 = I$, this process cannot go on indefinitely, so it will stop and we will get the desired hierarchical von Neumann-Morgenstern solution.

The proposition is proven.

IV. Towards a Better Definition

Discussion. The above proof shows the deficiency of the above definition, since it allow an unreasonably huge number of layers – as many as there are elements in the original set of imputations I.

It is therefore desirable to decrease the number of such layers. One possibility is to require that for each set C_i, the next set C_{i+1} should not be just a subset of C_i, it should also be as small as possible. In other words, in addition to the requirement that every element from the difference $C_i - C_{i+1}$ be dominated by some element from C_{i+1}, we should also require that we cannot have a smaller set with such property.

Thus, we arrive at the following definition.

Definition 7. *Let $(I, \succ)$ be a directed graph, and let $C \subset I$ be its subset.*

- *We say that the set $C' \subset C$* is a possible next level for *C if for every $y \in C - C'$, there exists an $x \in C'$ for which $x \succ y$.*
- *We say that the set $C' \subset C$* is a minimal next level for *C if it is a possible next level for C and no proper subset of C' is a next level for C.*

Comment. One can easily check that if C' is a minimal next level for C if and only for every element $x \in C'$, the set $C' - \{x\}$ is *not* a possible next level for C, i.e., that for every $x \in C'$, one of the two possible properties hold:

- either x is not dominated by any other element from C',
- or there exists an element $y \in C - C'$ which is dominated by x, but not by any other element from the set c'.

Proposition 2. *If a subset C of a finite graph $(I, \succ)$ contains two elements x and y for which $x \succ y$, then there exists a set $C' \subset C$ which is a minimal next level for C.*

Proof. We start with the set $C' = C - \{y\}$ which is a possible next level for C. If this set is minimal, we are done.

If this set is not minimal, this means that there exists a subset $C'' \subset C'$ ($C'' \neq C'$) which is also a possible next level for C. If this set C'' is minimal, we are done.

If the set C'' is not minimal, this means that there exists an even smaller possible next level set C''', etc.

Since we started with a finite set, and decrease the size by at least 1 on each iteration, eventually, we will find a minimal next level set.

The proposition is proven.

Definition 8. *Let $(I, \succ)$ be a directed graph. By a* strong hierarchical von Neumann-Morgenstern solution*, we mean a finite nested sequence of sets $C_0 = I \supset C_1 \supset C_2 \supset \ldots \supset C_k$ with the following two properties:*

- *if $x, y \in C_k$, then $x \not\succ y$, and*
- *for every $i \geq 0$, the set C_{i+1} is a minimal next level for C_i.*

Proposition 3. *Every directed finite graph $(I, \succ)$ has a strong hierarchical von Neumann-Morgenstern solution.*

Proof. We start with the set $C_0 = I$.

Once we have found the sets $C_1 \supset C_1 \supset \ldots \supset C_i$, if in C_i, there are no connected elements $x \succ y$, then we are done. If in the set C_i, there are connected elements, then we can use Proposition 2 to find a minimal next level set for C_i. This set is what we take as C_{i+1}.

At each step, we decrease the size, so this procedure will eventually stop and we will thus get the desired strong hierarchical von Neumann-Morgenstern solution.

The propositions is proven.

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