2015-01-01

Associativity Forcing Commutativity in Left Nil Rings

Md Al Masum Bhuiyan
University of Texas at El Paso, mbhuiyan@miners.utep.edu

ASSOCIATIVITY FORCING COMMUTATIVITY IN LEFT NIL RINGS

MD AL MASUM BHUIYAN

Department of Mathematical Sciences

APPROVED:

____
Piotr J Wojciechowski, Chair, Ph.D.

____
Maria Christina Mariani, Ph.D.

____
Mahmud Shahriar Hossain, Ph.D.

____
Charles Amber, Ph.D.
Dean of the Graduate School

to my

MOTHER and FATHER

with love

ASSOCIATIVITY FORCING COMMUTATIVITY IN LEFT NIL RINGS

by

MD AL MASUM BHUIYAN

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Mathematical Sciences

THE UNIVERSITY OF TEXAS AT EL PASO

August 2015

Acknowledgements

With profound warmth, I would like to express my deep-felt gratitude to my advisor Dr. Pior J Wojciechowski of the Mathematical Science Department at The University of Texas at El Paso, who made me a part to UTEP family. His clarity of thoughts, insights and simplicity in exposition will remain inspirational to me. Beyond introducing me to beautiful ideas and teaching me techniques to make them precise, he has shown an exquisite sense for how to nurture me towards becoming an independent researcher, by daydreaming with me about what might be true, by giving me the space to slowly grope towards realizing those daydreams, and by offering the crucial insights when I need guidance. His generous offer to collaborate on the research of modern algebras gave me a jolt of confidence, and his ongoing support of me during that project has matured me considerably. I am going to miss our regular meeting conversations tremendously in the next few years.

I also wish to thank the other members of my committee, Dr. Maria Christina Mariani, chairman of the Mathematical Science Department and Dr. Shahriar Hossain of the Computer Science Department, both at The University of Texas at El Paso. Their suggestions, comments and additional guidance were invaluable to the completion of this work.

I want to thank the professors and staffs of Mathematical Science Department of The University of Texas at El Paso for all their hard work and dedication, providing me the means to complete my degree and prepare for a career as a mathematician.

Through all my endeavors, mathematical and otherwise, I am sustained by the love and trust of my family. In my mom, I found my intellectual role model and mentor. She taught me the joy of explaining an idea clearly, to ask for whats missing in my current point of view, and to have faith in my intuition but test its suggestions thoroughly. Beyond the intellectual, I admire and seek to emulate her devotion to the quirky joke and the thoughtful gesture. Through her deep engagement with my life, I am lucky never to have felt unloved or unimportant. Finally, I thank my father, who is my original teacher, interlocutor, and collaborator. My parent's unconditional support and love have allowed me to thrive and grow into an adult and mathematician.

NOTE: This thesis was submitted to my Supervising Committee on the June 18, 2015.

Abstract

Our study in this thesis is concentrated on the *AFC* groups (Associativity Forces Commutativity) and observing the ring structure there. We are looking for abelian groups yielding ring structures (K) where associativity forces commutativity. We call them K-AFC groups.

We consider that the only conditions put on a ring multiplication are the distributive laws over the additive group addition. After [1], We say that an abelian group $(G, +)$ is an AFC-group if (i) there exists a nonassociative and noncommutative ring $(G, +, \cdot)$ and (ii) all associative rings $(G, +, \cdot)$ are commutative.

We call a ring *left-nil* if its "left power sequence" contains only a finite number of nonzero terms, We first comment on the structure of one sided-nil rings and then study the *left-nil AFC* groups which are a type of a K-AFC group. We say that G is a *left-nil AFC* group if G satisfies the two conditions of *K-AFC groups*, where K is the class of left-nil rings.

The motivation of this study comes from the "Order Algebraic Structures"[4], where the issue of *K-AFC group* naturally arose and there is a considerable interest in this topic to investigate a relation between associativity and commutativity.

Table of Contents

Chapter 1

Introduction to Groups, Rings, Fields

Our thesis is based on two fundamental properties of modern algebra- *associativity* and *commutativity*. Before discussing the main content of this thesis, we explain the general context, background and some criteria of these properties which inspire us to start this research program. In this introductory chapter we mainly review some familiar properties of sets, groups, rings and the operations in these structures. We study the question- how these are related to each other.

1.1 Sets

Let S be a set. The cartesian product $S \times S$ is the set of ordered pairs of elements of S,

$$S \times S = \big\{(x, y) : x, y \in S\big\}$$

A binary operation $\phi : S \times S \to S$ is **commutative** if $\phi(x, y) = \phi(y, x),$ for all $x, y \in S$.
A binary operation $\phi : S \times S \to S$ is **associative** if $\phi\big(\phi(x, y), z\big) = \phi\big(x, \phi(y, z)\big),$ for all $x, y, z \in S$.

1.1.1 Example

let $S = \{1, 2, 3..................\}$ be the set of all positive integers.
Now,
$\phi(x, y) = x + y$ is a binary operation which is both commutative and associative.
$\phi(x, y) = (x + y)^2$ is a binary operation which is commutative, but not associative.
$\phi(x, y) = x^y$ is a binary operation which is neither commutative nor associative.

These operations are defined using the familiar operations of addition and multiplication on the positive integers.

1.2 Group

A group consists of a set G and a binary operation "$\cdot$" defined on G, for which the following conditions are satisfied :

1. **Associativity :** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, for all $a, b, c \in G$.
2. **Identity :** There exists an element $e \in G$ such that $a \cdot e = e \cdot a = a$, for all $a \in G$.
3. **Inverse :** Given $a \in G$, there exists $b \in G$ such that $a \cdot b = b \cdot a = e$.

1.2.1 Example

Let $\mathbb{Z} = \{... - 2, -1, 0, 1, 2, ...\}$ be the set of integers. Then, $(\mathbb{Z}, +, 0)$ is a group where "$+$" is the usual addition. However, $(\mathbb{Z}, \cdot, 1)$, where "$\cdot$" is the usual multiplication, is not group, because, multiplicative inverses can not always be found in the integers, violating property 3.

1.3 Abelian Group

An abelian group is a nonempty set A with a binary operation "$+$" defined on A such that the following condition holds:

(i) (**Associativity**) for all $a, b, c \in A$, we have $a + (b + c) = (a + b) + c$.

(ii) (**Commutativity**) for all $a, b, c \in A$, we have $a + b = b + a$.

(iii) (**Existence of an additive identity**) there exists an element $0 \in A$ such that $0+a = a$ for all $a \in A$.

(iv) (**Existence of an additive inverse**) for each $a \in A$ there exists an element $-a \in A$ such that $-a + a = 0$.

Groups are not satisfying this property are said to be nonabelian or noncommutative.

1.4 Subgroup

Let G be a group. A subset H of G is a *subgroup* of G if :

(a) H is closed under the group operation: If $a, b \in H$, then $a \cdot b \in H$.

(b) $e \in H$.

(c) If $a \in H$, then $a^{-1} \in H$.

1.4.1 Theorem

Any subgroup of an abelian group is abelian.

1.5 Direct Product of Groups

In the context of abelian groups, the direct product is sometimes referred to as the direct sum, and is denoted $G \oplus H$. Direct sums play an important role in the classification of abelian groups. According to the fundamental theorem of finite abelian groups, every finite abelian group can be expressed as the direct sum of cyclic groups.

1.5.1 Definition

i) Let G and H be two arbitary groups, the direct product $G \oplus H$ is defined as follows:

$$G \oplus H = \Big\{(g, h) : g \in G \text{ and } h \in H\Big\}$$

That is, the set of elements of $G \oplus H$ is the Cartesian product of the sets G and H.

ii) The binary operation on $G \oplus H$ is defined componentwise :

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2)$$

The resulting algebraic object satisfies the axioms for a group. Specifically :

• **Associativity:** The binary operation on $G \oplus H$ is indeed associative.

• **Identity :** The direct product has an identity element, namely (e_G, e_H), where e_G is the identity element of G and e_H is the identity element of H.

• **Inverses :** The inverse of an element (g, h) of $G \oplus H$ is the pair (g^{-1}, h^{-1}), where g^{-1} is the inverse of g in G, and h^{-1} is the inverse of h in H.

1.5.2 Example

Let $\mathbb{R}$ be the group of real numbers under addition. Then the direct product $\mathbb{R} \oplus \mathbb{R}$ is the group of all two-component vectors (x, y) under the operation of vector addition :

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

1.5.3 Elementary properties

• The order of a direct product $G \oplus H$ is the product of the orders of G and H :

$$|G \oplus H| = |G||H|.$$

• The order of each element (g, h) is the least common multiple of the orders of g and h

$$|(g, h)| = \text{lcm } (|g|, |h|).$$

In particular, if $|g|$ and $|h|$ are relatively prime, then the order of (g, h) is the product of the orders of g and h.

1.6 Cyclic group

A cyclic group is an abelian group that can be generated by a single element, in the sense that there is an elelment a such that all elements of the group are generated by a.

Equivalently, an element a of a group G generates G precisely if G is the only subgroup of itself that contains a [10].

1.6.1 Definition

A group G is called *cyclic* if $G =< a >$. Such an element a is called a *generator* of G.

1.6.2 Example

$\mathbb{Z}$ is a cyclic group under addition since $\mathbb{Z} =< 1 >=< -1 >$.

1.6.3 Theorem

Let m and n be positive integers.

1. If $gcd(m, n) = 1$ (i.e. m and n are relatively prime), then $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$, and $(1, 1)$ is a generator of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

2. If $gcd(m, n) \neq 1$, then $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is not cyclic. of every subgroup of $< a >$ is a divisor of n; and, for each positive divisor k of n, the group $< a >$ has exactly one subgroup of order k, namely $< a^{n/k} >$.

1.7 Homomorphism

If $(G, *)$ and $(H, \circ)$ are groups, then a function $f : G \to H$ is a homomorphism if

$$f(x * y) = f(x) \circ f(y), \text{ for all } x, y \in G.$$

1.7.1 Example

Consider R under addition and the group U of complex numbers z with $|z| = 1$ under multiplication.

Let $\phi : R \to U$ be the map $\phi(x) = e^{i2x}$. Since

$$\phi(x + y) = e^{i2(x+y)} = e^{i2x}e^{i2y} = \phi(x)\phi(y).$$

So, ϕ is a homomorphism.

1.8 Isomorphism

Let a function $f : G \to H$ be a homomorphism. If f is also a one-one correspondence, then f is called an isomorphism. Two groups G and H are called isomorphic, denoted by $G \cong H$, if there exists an isomorphism between them.

1.8.1 Example

The group of all real numbers with addition, $(\mathbb{R}, +)$, is isomorphic to the group of positive real numbers with multiplication $(\mathbb{R}+, \cdot) : (\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$

1.9 Fundamental Theorem of Finite Abelian Groups

Every finite abelian group (G) is isomorphic to a direct product of finite cyclic groups, that is,

$$G \cong \mathbb{Z}_{{p_1}^{r_1}} \oplus \mathbb{Z}_{{p_2}^{r_2}} \oplus ..... \oplus \mathbb{Z}_{{p_n}^{r_n}}$$

where the p_i's are primes ($r_i \geq 1$).

1.10 Finitely Generated Abelian Groups

First, let us examine a generalization of finite abelian groups. Suppose that G is a group and let $\{g_i\}$ be a set of elements in G, where i is in some index set I (not necessarily finite). The smallest subgroup of G containing all of the $\{g_i\}'s$ is the subgroup of G generated by the $\{g_i\}'s$. If this subgroup of G is all of G, then G is generated by the set $\{g_i : i \in I\}$. In this case the $\{g_i\}'s$ are said to be the set of generators of G. If there is a finite set of generators $\{g_i : i \in I\}$ of G, then G is *finitely generated*.

So, an abelian group G is finitely generated if there are elements $x_1, \ldots, x_n \in G$ such that every element $x \in G$ can be written as

$$x = a_1x_1 + \cdots + a_nx_n, \text{ where } a_i \in \mathbb{Z}.$$

1.10.1 Example

1. All finite groups are finitely generated. For example, the group $\mathbb{Z}_p \oplus \mathbb{Z}_q$ (where p, q are primes) is generated by $\{(1,1)\}$.
2. The group $\mathbb{Z} \oplus \mathbb{Z}_n$ is an infinite group but is finitely generated by $\{(1,0),(0,1)\}$.
3. $\mathbb{Q}$ is not finitely generated.

1.11 Fundamental Theorem of Finitely Generated Abelian Group

Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form

$$\mathbb{Z}_{p_1{}^{r_1}} \oplus \mathbb{Z}_{p_2{}^{r_2}} \oplus \ldots.. \oplus \mathbb{Z}_{p_n{}^{r_n}} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots\ldots. \oplus \mathbb{Z}.$$

where the p_i are primes, not necessarily distinct, and the r_i are positive integers. The direct product is unique except for a possible rearrangement of the factors. The number of factors $\mathbb{Z}$ is called the Betti number of G.

1.12 Rings

Up to this point we have studied sets with a single binary operation satisfying certain axioms, but often we are interested in working with sets that have two binary operations. For example, one of the most natural algebraic structures to study is the set of integers with the operations of addition and multiplication. These operations are related to one another by the distributive property. If we consider a set with two such related binary operations satisfying certain axioms, we have an algebraic structure called a ring.

The ring structures on certain abelian groups was first defined by Ross Beaumont who considered rings on direct sums of cyclic groups [2]. Then Beaumont and Zuckerman described the rings on subgroups of the rationals [3].

1.12.1 Definition

A ring consists of a set R and two binary operations " $+$ " (addition) and " $\cdot$ " (multiplication) on R, for which the following conditions are satisfied :

1. **Additive associative :** $(a + b) + c = a + (b + c)$, for all $a, b, c \in R$.

2. **Additive commutative :** $a + b = b + a$, for all $a, b \in R$.

3. **Additive identity :** There exists an element $0 \in R$ such that for all $a \in R$, $0 + a = a + 0 = a$.

4. **Additive inverse :** for every $a \in R$, there exists $-a \in R$ such that $a + (-a) = (-a) + a = 0$.

5. **Left and right distributivity :** for all $a, b, c \in R$, it holds that $a \cdot (b + c) = a \cdot b + b \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

If we are working in an abelian environmment it is ususal to use additive notation : $x + y$. The reason for this is that while multiplication of various mathematical objects (functions, matrices, etc.) is noncommutative, addition invariably commutes. So, by using additive notation the commutativity seems perfectly natural. This is also true for the associativity.

Since in a ring we add and multiply elements by a distributive law, we want to figure out the connection between associativity and commutativity on multiplication. How does they imply or force each other on a particular ring structure? Are they always independent on all the ring structure?

In our thesis we have investigated such ring multiplications which admit a relation between these two properties.

Some restrictions involving the multiplication may be necessary at times. Suppose that a and b are two elements in a ring R. If a is non-zero and $a \cdot b = 0$, then b need not necessarily be 0. That is an interesting part in modern algebra. In such case a and b are called the zero-divisors.

In such stipulation, $a \cdot c = b \cdot c$ does not imply that $a = b$. However, we can impose additional conditions on the ring to ensure that this be true; namely make the ring into an integral domain.
Even stronger consideration lead to the definition of a field.

1.13 Field

A field consists of a set F and two binary operations "+" (addition) and "·"(multiplication), defined on R, for which the following conditions are satisfied :
1. $(F, +, \cdot)$ is a **ring**.
2. **Multiplicative commutative :** For any $a, b \in F$, $a \cdot b = b \cdot a$.
3. **Multiplicative associative :** For all $a, b, c \in F$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. **Multiplicative identity :** There exists $1 \in F$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in F$.
4. **Multiplicative inverse :** If $a \in F$ and $a \neq 0$, there exists $b \in F$ such that $a \cdot b = b \cdot a = 1$.

1.13.1 Example

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.

1.14 Integral Domain

An *integral domain* is a commutative ring R with identity $1_R \neq 0_R$ with no zero divisors, that is, $a \cdot b = 0_R$ implies that $a = 0_R$ or $b = 0_R$.

1.15 Associative Ring

If R is multiplicatively associative then R is called an *associative* ring.

For all $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Our rings not necessarily are *associative*.

1.16 Ring with identity

A ring with *identity* is a ring R that contains a multiplicative identity element 1_R such that $1_R \cdot a = a = a \cdot 1_R$, for all $a \in R$.

Our rings not necessarily have the *identity*.

1.16.1 Example

- $\mathbb{Z}, \mathbb{R}, \mathbb{Z}_n$ are rings with identity.
- The set of even integers is a ring without identity.

1.17 Ring with unity

Any element a in a ring R with identity which has an inverse u (i.e. $a \cdot u = 1_R = u \cdot a$) is called a *unit*.

1.18 Subring

Let R be a ring. A subring is a subset $S \subset R$ such that :

(a) S is closed under addition : If $a, b \in S$, then $a + b \in S$.

(b) The zero element of R is in S : $0 \in S$.

(c) S is closed under additive inverses : If $a \in S$, then $-a \in S$.

(d) S is closed under multiplication : If $a, b \in S$, then $ab \in S$.

1.19 Ideal

Let R be a ring. An ideal I of R is a subset $I \subset R$ such that :

(a) I is closed under addition : If $a, b \in I$, then $a + b \in I$.

(b) The zero element of R is in I : $0 \in I$.

(c) I is closed under additive inverses : If $a \in I$, then $-a \in I$.

(d) If $a \in R$ and $b \in I$, then $ab \in I$ and $ba \in I$.

In other words, I is closed under multiplication (on either side) by arbitrary ring elements.

$a = 0_R$ or $b = 0_R$.

Chapter 2

The Structure of a Nil Ring

The goal of this section is to look at several properties of the structure of nil rings. We will need then the development of the subject matter in the subsequent chapters.

2.1 Left Nil Rings

2.1.1 Left-power sequence

Let R be a nonassociative ring and $x \in R$. A *left-power-sequence* of x is defined as follows:

$$x_1 = x$$

$$x_2 = x \cdot x_1$$

....

$$x_{n+1} = x \cdot x_n$$

2.1.2 Left-nil ring

A ring R is called a *left-nil* if for every $x \in R$, its left-power sequence contains only a finite number of nonzero terms (which is equivalent to say that for some $n, x_n = 0$).

2.1.3 Example

In $\mathbb{Z} \oplus \mathbb{Z}$, we define the ring multiplication,

$$(a, b) \cdot (c, d) = (0, ac + bc)$$

where $a, b, c, d \in \mathbb{Z}$ and extend it linearly to make $\mathbb{Z} \oplus \mathbb{Z}$ an algebra over $\mathbb{Z}$.

Let $(a, b), (c, d)$ and (e, f) be any three elements of this group, where $a, b, c, d, e, f \in \mathbb{Z}$.

Now, we need to verify the distributive property of the ring $(\mathbb{Z} \oplus \mathbb{Z}, +, \cdot)$.

$(a, b) \cdot \big((c, d) + (e, f)\big) = (a, b) \cdot (c + e, d + f)$

$= (0, ac + ae + bc + be))$

$= (0, ac + bc) + (0, ae + be)$

$= (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$, which satisfies the left- distributivity on $(\mathbb{Z} \oplus \mathbb{Z}, +, \cdot)$.

Also, $\big((c, d) + (e, f)\big) \cdot (a, b) = (c + e, d + f) \cdot (a, b)$

$= (0, ca + ea + da + fa)$

$= (0, ca + da) + (0, ea + fa)$

$= (c, d) \cdot (a, b) + (e, f) \cdot (a, b)$, which satisfies the right- distributivity on $(\mathbb{Z} \oplus \mathbb{Z}, +, \cdot)$.

Now, $(a, b)^2 = (a, b) \cdot (a, b) = (0, a^2 + ba)$

and, $(a, b)^3 = (a, b) \cdot (a, b)^2$

$= (a, b) \cdot (0, a^2 + ba)$

$= (0, 0)$

Since a *left-cube* of any element of this ring is $(0, 0)$. $(\mathbb{Z} \oplus \mathbb{Z}, +, \cdot)$ is a *left-nil* ring.

2.2 Right Nil Rings

2.2.1 Right-power sequence

Let R be a nonassociative ring and $x \in R$. A *right-power-sequence* of x is defined analogously as follows :

$$x_1 = x$$

$$x_2 = x_1 \cdot x$$

$$x_3 = x_2 \cdot x$$

....

$$x_{n+1} = x_n \cdot x$$

2.2.2 Right-nil ring

A ring R is called a right-nil if for every $x \in R$, its right-power sequence contains only a finite number of nonzero terms (which is equivalent to say that for some $n, x_n = 0$).

2.2.3 Example

In $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, we define the ring multiplication,

$$(a, b) \cdot (c, d) = (0, 2bc)$$

for $a, c \in \mathbb{Z}_2$ and $b, d \in \mathbb{Z}_4$ where the products ab, ba, bc etc. are elements of $\mathbb{Z}_4$ with $\mathbb{Z}_2$ treated as a left and right module over $\mathbb{Z}_4$ in a natural way.
Also, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is a left-module over $\mathbb{Z}_4$:
for any $r \in \mathbb{Z}_4$, we define $r(a, b) = (ra, rb)$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_4$,
where, the product ra is defined as follows :

$r \cdot 0 = 0 \in \mathbb{Z}_2$

$$r \cdot 1 = \begin{cases} 0 & ; for, r = 0, 2 \\ 1 & ; for, r = 1, 3 \end{cases}.$$

Now, $(a, b)^2 = (a, b) \cdot (a, b) = (0, 2ba)$
$(a, b)^3 = (a, b)^2 \cdot (a, b)$
$= (0, 2ba) \cdot (a, b)$
$= (0, 4ba^2)$
$= (0, 0)$.
Since a right-cube of any element of this group is $(0, 0)$, $(\mathbb{Z}_2 \oplus \mathbb{Z}_4, +, \cdot)$ is a right-nil ring.

2.3 Relation between left-nil and right-nil ring

2.3.1 Remark

A *left-nil* ring may not be a *right-nil* ring.

For example, In $\mathbb{Z} \oplus \mathbb{Z}$, we define the ring multiplication,

$$(a, b) \cdot (c, d) = (0, ac + bc)$$

where $a, b, c, d \in \mathbb{Z}$ and extend it linearly to make $\mathbb{Z} \oplus \mathbb{Z}$ an algebra over $\mathbb{Z}$.

This is a *left-nil* ring [by Example 2.1.3]

Now $(1, 0)$ multiplied on the right,

$(1, 0)^2 = (1, 0) \cdot (1, 0) = (0, 1)$

$(1, 0)^3 = (1, 0)^2 \cdot (1, 0) = (0, 1) \cdot (1, 0) = (0, 1)$

$(1, 0)^4 = (1, 0)^3 \cdot (1, 0) = (0, 1) \cdot (1, 0) = (0, 1)$

$(1, 0)^5 = (1, 0)^4 \cdot (1, 0) = (0, 1) \cdot (1, 0) = (0, 1)$

....

so on.

Therefore, by the *right-power* sequence, $(1, 0)$ does not reach $(0, 0)$. Thus, the above ring is not a *right-nil* ring.

2.4 Nil Rings

Let R be an associative ring and $x \in R$. Then x is called a *nilpotent* element of R if there is a positive integer n such that $x^n = 0$.

A ring R is called a *nil ring* if every element of R is nilpotent.

2.4.1 Lemma

Every nil ring is both left- and right-nil.

2.4.2 Example

$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is a nil ring.

Proof : As an abelian group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to the group of all 3×3 upper triangular matrices with $0's$ on the diagonal of the form $\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$ where $a, b, c \in \mathbb{Z}$.

So, we endow $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ with the matrix multiplication. It becomes an associative ring.

Now, assume that $X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$

$$X^2 = X \cdot X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$X^3 = X \cdot X^2 = X^2 \cdot X$; [Matrix multiplication is associative]

$$= \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, the group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ admits a *left-nil* and *right-nil* structure.

Therefore, $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is both left- and right-nil ring. Moreover, it is a nilpotent ring in the sense that a product of every three elelments is 0.

2.5 Associative Nil Rings

2.5.1 Theorem

In an associative ring, if xy is nilpotent so is yx.

Proof : If $(xy)^n = 0$ then

$$(yx)^{n+1} = y(xy)^n x = y \cdot 0 \cdot x = 0 \cdot x = 0.$$

2.5.2 Remark

Every associative left or right nil is a nil ring. **Proof :** Let us consider, R to be an associative *left-nil* ring.

Let $x \in R$ and x_k be the k^{th} term of the left sequence, and let $x_k = 0$. But in an associative ring $x_k = x^k$, so $x^k = 0$, and R is nil.

2.5.3 Remark

A nil ring with identity is never nil or nilpotent, because the identity is not nilpotent.

2.6 Relation between nil ring and subring

Let R be a commutative ring and N be the set of all nilpotent elements of R. Then N is a subring of R.

Proof : Let R be a commutative and associative ring and let,

$$N = \{n \in R | n^k = 0 \text{ for some } k \in \mathbb{Z}^+\}.$$

N is non empty since $0 \in N$.

Also suppose that a and b are any two elements in N. In order to show that N is a subring of R, we must show that $ab \in N$ and $a - b \in N$.

Consider ab. We know that $a^p = 0$ and $b^q = 0$ for some positive integers p and q. Then, by using associativity and commutativity of R,

$$(ab)^{pq} = a^{pq}b^{pq} = (a^p)^q(b^q)^p = 0^p0^q = 0.$$

Therefore, $(ab)^{pq} = 0$, and $ab \in N$.

Now consider $a - b$. This can be written as $a + (-b)$. Since R is associative and commutative, we can use the binomial theorem on $(a + (-b))^{p+q}$ to form

$$(a + (-b))^{p+q} = \sum_{k=0}^{p+q} \binom{p+q}{k} a^{p+q-k}(-b)^k = \sum_{k=0}^{p+q} -\binom{p+q}{k} (a^{p+q-k}b^k).$$

Thus, whenever $k \leq q, p + q - k \geq p$. This gives us $a^{p+q-k} = 0$. Moreover, when $q \leq k \leq p + q$, the exponent of b is at least q. In this case, $b^k = 0$.
Hence, for each term in the binomial expansion , $a^{p+q-k} = 0$ or, $b^k = 0$ for every k.
Then every term reduces to 0, and so $(a - b)^{p+k} = 0$, and $a - b \in N$.
Therefore, the nilpotent elements of a ring R form a subring of R.

2.6.1 Remark

If R is noncommutative, then the set of all nilpotent elements of R may not be a subring of R.
For example, let $R = M_2(\mathbb{R})$, the 2×2 matrix ring over $\mathbb{R}$ and N is the set of all nilpotent elements of R.

Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ where $a, b \in N$.
Now, $a - b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is not nilpotent and hence not in N.
Therefore, N is not a subring of R.

2.7 Nilpotent Ideal

An Ideal I of R is called a nilpotent ideal if $I^n = 0$ for some positive integers n and I^n denotes the ideal of R consisting of all n-tuple products $i_1 \cdots i_n$ of elements in I.

2.7.1 Remark

Every zero ideal is a nilpotent ideal and every nilpotent ideal is necessarily a nil ideal but the converse need not be true in general.

2.7.2 Remark

If R is noncommutative, then the set of all nilpotent elements of R may not be a ideal of R. Since by Remark 2.3.7, may not even be a subring.

Chapter 3

AFC Groups

In this chapter, we study "*AFC groups*" which are abelian groups which admit a ring structure such that associativity is forcing commutativity. This chapter is based on [1]. The purpose of this chapter is mainly to observe several structures of *AFC* groups, which play a fundamental part in the classification of all $\mathbf{K}$-*AFC* groups. It will allow us to construct a class of $\mathbf{K}$-*AFC* groups which is called *left-nil AFC* groups.

3.1 Definition

Let $\mathbf{K}$ be the class of rings and G a group. We say that G is a $\mathbf{K}$-*AFC* group (associativity is forcing commutativity) if

(i) there exists a nonassociative and noncommutative ring $(G, +, \cdot) \in \mathbf{K}$ and
(ii) all associative rings $(G, +, \cdot) \in \mathbf{K}$ are commutative.

In case $\mathbf{R}$ is the class of all rings, the $\mathbf{R}$-*AFC* groups will be simply called *AFC* groups.

The following proposition is useful in any cases and follows immediately from the definition.

3.1.1 Proposition

If $\mathbf{K_1} \subseteq \mathbf{K_2}$ and for a group G the condition (i) fails in $\mathbf{K_2}$, then it also fails in $\mathbf{K_1}$. In particular, If $\mathbf{K_2} = \mathbf{R}$, then G is not a $\mathbf{K}$-*AFC* group for any class $\mathbf{K}$.

3.2 Associative and Noncommutative ring multiplication on some groups

3.2.1 Lemma

Let $G = A \oplus B$, and suppose that there is an associative and noncommutative multiplication "$\cdot$" on A. Then there is also an associative and noncommutative multiplication on G.

Proof : Let us define the product "$*$" on G by

$$(a_1, b_1) * (a_2, b_2) = (a_1 \cdot a_2, 0) \text{ where } a_1, a_2, b_1, b_2 \in A$$

This ring is noncommutative since $(a_1, b_1) * (a_2, b_2) = (a_1 \cdot a_2, 0)$
and $(a_2, b_2) * (a_1, b_1) = (a_2 \cdot a_1, 0)$, and there exist $a_1, a_2 \in A$ such that $a_1 \cdot a_2 \neq a_2 \cdot a_1$.
Let $(a, b), (c, d), (m, n)$ be any three elements in $G = A \oplus B$ where $a, c, m \in A$ and $b, d, n \in B$
Now, $(a, b) * \big((c, d) * (m, n)\big) = (a, b) \cdot (c \cdot m, 0)$
$= (a \cdot c \cdot m, 0)$.
On the other hand, $\big((a, b) * (c, d)\big) * (m, n) = (a \cdot c, 0) \cdot (m, n)$
$= (a \cdot c \cdot m, 0)$.
So, $(a, b) * \big((c, d) * (m, n)\big) = \big((a, b) * (c, d)\big) * (m, n)$.
Therefore, this is an associative and noncommutative ring on G.
This completes the proof.

3.2.2 Theorem

For every $n > 1$, $\mathbb{Z}_n \oplus \mathbb{Z}$ admits an associative and noncommutative multiplication.

Proof : Let us consider the operation,

$$(1, 0) \cdot (1, 0) = (0, 0);$$

$$(1, 0) \cdot (0, 1) = (0, 0);$$

$$(0, 1) \cdot (1, 0) = (1, 0);$$

$$(0, 1) \cdot (0, 1) = (0, 1);$$

and extend it naturally to the ring multiplication.

The ring is noncommutative, since

$$(1, 0) \cdot (0, 1) = (0, 0) \text{ and}$$

$$(0, 1) \cdot (1, 0) = (1, 0).$$

Let (a, b), (c, d) and (e, f) be any three elements on $\mathbb{Z}_n \oplus \mathbb{Z}$.

Now,

$(a, b) \cdot \big((c, d) \cdot (e, f)\big) = \big(a(1, 0) + b(0, 1)\big) \cdot \big((c(1, 0) + d(0, 1)) \cdot (e(1, 0) + f(0, 1))\big)$

$= \big(a(1, 0) + b(0, 1)\big) \cdot \big(ce(1, 0) \cdot (1, 0) + cf(1, 0) \cdot (0, 1) + de(0, 1) \cdot (1, 0) + df(0, 1) \cdot (0, 1)\big)$

$= \big(a(1, 0) + b(0, 1)\big) \cdot \big(ce(0, 0) + cf(0, 0) + de(1, 0) + df(0, 1)\big)$

$= \big(a(1, 0) + b(0, 1)\big) \cdot \big(de(1, 0) + df(0, 1)\big)$

$= ade(1, 0) \cdot (1, 0) + adf(1, 0) \cdot (0, 1) + bde(0, 1) \cdot (1, 0) + bdf(0, 1) \cdot (0, 1)$

$= ade(0, 0) + adf(0, 0) + bde(1, 0) + bdf(0, 1)$

$= (bde, bdf).$

On the other hand,

$\big((a, b) \cdot (c, d)\big) \cdot (e, f) = \big((a(1, 0) + b(0, 1)) \cdot (c(1, 0) + d(0, 1))\big) \cdot \big(e(1, 0) + f(0, 1)\big)$

$= \big(ac(1, 0) \cdot (1, 0) + ad(1, 0) \cdot (0, 1) + bc(0, 1) \cdot (1, 0) + bd(0, 1) \cdot (0, 1)\big) \cdot \big(e(1, 0) + f(0, 1)\big)$

$= \big(ac(0, 0) + ad(0, 0) + bc(1, 0) + bd(0, 1)\big) \cdot \big(e(1, 0) + f(0, 1)\big)$

$= \big(bc(1, 0) + bd(0, 1)\big) \cdot \big(e(1, 0) + f(0, 1)\big)$

$= bce(1, 0) \cdot (1, 0) + bcf(1, 0) \cdot (0, 1) + bde(0, 1) \cdot (1, 0) + bdf(0, 1) \cdot (0, 1)$

$= bce(0, 0) + bcf(0, 0) + bde(1, 0) + bdf(0, 1)$

$= (bde, bdf).$

Hence, $(a, b) \cdot \big((c, d) \cdot (e, f)\big) = \big((a, b) \cdot (c, d)\big) \cdot (e, f)$.

This is an associative and noncommutative ring multiplication.

3.3 Non-AFC groups

3.3.1 Theorem (Exterior direct sum of groups $A \oplus B$, $AB \neq \{0\}$)

If the group G is an additive group of a ring with an associative multiplication and A and B are ideals of G for which $AB \neq \{0\}$, then the exterior direct sum of groups $A \oplus B$ admits an associative noncommutative multiplication and thus it is not AFC.

Proof : Consider $M = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, a \in A, b \in B\}$. Then M and $A \oplus B$ are isomorphic as groups. If M has an associative and noncommutative multiplication ring, then so has G.
We know that matrix multiplication is associative, So M is an associative ring.
But since A and B are ideals in G and for some $a \in A, b \in B, ab \neq 0$, then M is noncommutative, because,

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix} \text{ and, } \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Therefore, } \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

which completes the proof.

3.3.2 Theorem ($A^+ \oplus M$, M is an A-module, AM is nontrivial)

If A is an associative ring and M is an A-module such that AM is nontrivial, then the group $A^+ \oplus M$ admits an associative, noncommutative multiplication and thus is not AFC.

Proof : We define the ring structure on $A^+ \oplus M$ by putting a coordinatewise addition, while the multiplication is given by

$$(a, b) \cdot (x, y) = (ax, ay) \text{ for } a, x \in A \text{ and } b, y \in M.$$

Let us consider $(c, d), (e, f)$ and (g, h) be any three elements on $A^+ \oplus M$,

Now, $(c, d) \cdot \big((e, f) \cdot (g, h)\big) = (c, d) \cdot (eg, eh)$

$= \big(c(eg), c(eh)\big)$.

On the other hand, $\big((c, d) \cdot (e, f)\big) \cdot (g, h) = (ce, cf) \cdot (g, h)$

$= \big((ce)g, (ce)h\big)$.

Since A is associative, $c(eg) = (ce)g$ and since M is the A-module, $c(eh) = (ce)h$,

$$(c, d) \cdot \big((e, f) \cdot (g, h)\big) = \big((c, d) \cdot (e, f)\big) \cdot (g, h).$$

So, $A^+ \oplus M$ is an associative ring.

Since $AM \neq \{0\}$, there are $a \in A$ and $m \in M$ such that $am \neq 0$.

But then $(a, 0) \cdot (0, m) = (0, am) \neq (0, 0)$ while $(0, m) \cdot (a, 0) = (0, 0)$.

So, the ring is noncommutative and $A^+ \oplus M$ is not *AFC*.

3.3.3 Corollary ($\mathbb{Z} \oplus G$)

For any group G, $\mathbb{Z} \oplus G$ is not *AFC*.

Proof : If G is trivial, then the direct sum is isomorphic to $\mathbb{Z}$. So it is not *AFC*. Otherwise G is a $\mathbb{Z}$-module such that $\mathbb{Z}G \neq \{0\}$, so the claim holds by Theorem 3.3.2.

3.3.4 Lemma ($\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ where $m, n \in \mathbb{N}$)

For any prime p and natural numbers m and n, the group $\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ admits an associative, noncommutative multiplication, and thus is not *AFC*.

Proof : Suppose that $m \geq n$ and let A denote the usual ring $\mathbb{Z}_{p^m}$ and let M be the group $\mathbb{Z}_{p^n}$. Then M becomes an A-module as an A-submodule of the $\mathbb{Z}_{p^m}$-module over $\mathbb{Z}_{p^m}$. Clearly $AM \neq \{0\}$, so the claim holds by Theorem 3.3.2.

We can now put together our results concerning the cyclic groups.

3.3.5 Theorem (Cyclic groups and subgroups of $\mathbb{Q}$)

Cyclic groups and all subgroups of $\mathbb{Q}$ are not *AFC*.

Proof : Let us consider, R be a cyclic ring, g be a generator of R^+(additive group of R) and $x, y \in R$. Then there exist $a, b \in \mathbb{Z}$ with $x = ag$ and $y = bg$. Since

$$xy = (ag)(bg) = (ab)g^2 = (ba)g^2 = (bg)(ag) = yx$$

it follows that R is clearly commutative. Therefore, part (i) of the Definition 3.1 is never satisfied.

For the subgroups of $\mathbb{Q}$, by [3] Corollary 3, all multiplications are commutative, so these groups are not *AFC*.

3.3.6 Theorem (Ascending chain of cyclic groups)

Let $\{A_\alpha\}$ be an ascending chain of cyclic groups. Then $A = \bigcup\limits_{\alpha} A_\alpha$ is not *AFC*.

Proof : We show that all multiplications on A are commutative. Suppose on the contrary that there is a noncommutative multiplication " $\cdot$ " on A, and let for some $a, b \in A$, $a \cdot b \neq b \cdot a$. Then for some i, $a, b \in A_i$.

Suppose that g generates A_i. Then for some m and n, $a = mg$ and $b = ng$.

So by distributivity,

$$a \cdot b = mg \cdot ng = mng^2 = ng \cdot mg = b \cdot a$$

which is contradiction. So A is not *AFC*.

3.3.7 Theorem (Direct product of cyclic groups)

If G is a direct product of cyclic groups, then G is not AFC.

Proof : If G is cyclic, then it is not AFC by Theorem 3.3.5. Otherwise G contains $\mathbb{Z}$ or $\mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ as a direct summand, and is not AFC by Corollary 3.3.3 or Lemma 3.3.4 and Lemma 3.2.1.

3.3.8 Corollary (Finitely generated group)

No finitely generated group is AFC.

Proof : By the Fundamental Theorem of Finitely Generated Abelian Groups, every such group is a direct sum of cyclic groups, so it is not AFC by Theorem 3.3.7.

3.3.9 Corollary (Finite group)

No finite group is AFC.

Proof : We know that every finite group is obviuosly finitely generated. So by Corollary 3.3.8, this completes the proof.

3.4 Non-AFC groups of a vector space over a field

3.4.1 Theorem

Let V be a nonzero vector space over a field then the group V is not AFC.

Proof : Let $\mathbb{K}$ be the base field of $\mathbb{F}$. Suppose first that $\dim_{\mathbb{K}} V > 1$ and take two linearly independent elements e and f of a basis. Let us define a multiplication on V using this basis by putting

$$e^2 = e, e \cdot f = f, f \cdot e = f^2 = 0,$$

and setting all the remaining products of the basis elements equal to 0.

The ring is noncommutative, since $e \cdot f = f$ and $f \cdot e = f^2$. Moreover,

Let x, y and z are any three elemtns in V such that $x = ae+bf$, $y = ce+df$ and $z = me+nf$ where $a, b, c, d, m, n \in \mathbb{K}$.

Now, $(x \cdot y) \cdot z = \big((ae + bf) \cdot (ce + df)\big) \cdot (me + nf)$

$= (ace^2 + ade \cdot f + bcf \cdot e + bdf^2) \cdot (me + nf)$

$= (ace + adf + 0 + 0) \cdot (me + nf)$

$= (ace + adf) \cdot (me + nf)$

$= acme^2 + acne \cdot f + admf \cdot e + adnf^2$

$= acme + acnf + 0 + 0$

$= acme + acnf$.

On the other hand, $x \cdot (y \cdot z) = (ae + bf) \cdot \big((ce + df) \cdot (me + nf)\big)$

$= (ae + bf) \cdot (cme^2 + cne \cdot f + dmf \cdot e + dnf^2)$

$= (ae + bf) \cdot (cme + cnf + 0 + 0)$

$= (ae + bf) \cdot (cme + cnf)$

$= acme^2 + acne \cdot f + bcmf \cdot e + bcnf^2$

$= acme + acnf + 0 + 0$

$= acme + acnf$.

Hence, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

This is an associative, noncommutative multiplication on V. Therefore the condition (ii) of the Definition 3.1 is not satisfied, so V is not *AFC*.

If, on the other hand, $\dim_{\mathbb{K}} V = 1$ then $V = \mathbb{K}$, so it is either $\mathbb{Z}_p$ (p-prime) or $\mathbb{Q}$. But then by Theorem 3.3.5, V is not *AFC*.

Chapter 4

Left-Nil AFC Groups

In this chapter we are going to study the "*left-nil AFC* groups". We consider "rings" in which the only conditions imposed on multiplication are the distributive laws over the addition. All groups considered here are abelian. By *order* of an elelment we always mean its order in the additive group. We assume neither associativity of multiplication nor existence of the unity. By a "nonassociative" ring we understand a ring which fails the associativity condition and similarity for a "noncommutative" ring.

4.1 Definition

Let $\mathbf{K}$ be the class of all *left-nil* rings and G a group. We say that G is a *left-nil AFC* group if

(i) there exists a nonassociative and noncommutative ring $(G, +, \cdot) \in \mathbf{K}$ and
(ii) all associative rings $(G, +, \cdot) \in \mathbf{K}$ are commutative.

4.2 Additive group of an integral domain with unity

4.2.1 Theorem

If R is the additive group of an integral domain with unity, then $R \oplus R$ is a *left-nil AFC* group.

Proof : Define a ring multiplication on $R \oplus R$:

$$(a, b) \cdot (c, d) = (0, ac + bc) \text{ where } a, b, c, d \in R$$

and extend it linearly to make $R \oplus R$ an algebra over R by [11].

Let $(a, b), (c, d)$ and (e, f) be any three elements of this group, where $a, b, c, d, e, f \in R$.

Now, we need to verify the distributive property of the ring $(R \oplus R, +, \cdot)$.

$(a, b) \cdot \big((c, d) + (e, f)\big) = (a, b) \cdot (c + e, d + f)$

$= (0, ac + ae + bc + be))$

$= (0, ac + bc) + (0, ae + be)$

$= (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$, which satisfies the left- distributivity on $(R \oplus R, +, \cdot)$.

Also, $\big((c, d) + (e, f)\big) \cdot (a, b) = (c + e, d + f) \cdot (a, b)$

$= (0, ca + ea + da + fa)$

$= (0, ca + da) + (0, ea + fa)$

$= (c, d) \cdot (a, b) + (e, f) \cdot (a, b)$, which satisfies the right- distributivity on $(R \oplus R, +, \cdot)$.

Now, $(a, b)^2 = (a, b) \cdot (a, b) = (0, a^2 + ba)$

and, $(a, b)^3 = (a, b) \cdot (a, b)^2$

$= (a, b) \cdot (0, a^2 + ba)$

$= (0, 0)$

Since a *left-cube* of any element of this ring is $(0, 0)$. $(R \oplus R, +, \cdot)$ is a *left-nil* ring.

To prove that the group $R \oplus R$ is an AFC group, we need to verify that $R \oplus R$ satisfies the two conditions of the Definition 4.1.

The ring is noncommutative since $(1, 0) \cdot (0, 1) = (0, 0)$ and $(0, 1) \cdot (1, 0) = (0, 1)$.

Let, $u = (0, 1)$, $v = (1, 1)$, $w = (1, 0)$ to be any three elements of $(R \oplus R, +, \cdot)$.

Now, $u \cdot (v \cdot w) = (0, 1) \cdot \big((1, 1) \cdot (1, 0)\big)$

$= (0, 1) \cdot (0, 2)$

$= (0, 0)$.

On the other hand, $(u \cdot v) \cdot w = \big((0, 1) \cdot (1, 1)\big) \cdot (1, 0)$

$= (0, 1) \cdot (1, 0)$

$= (0, 1).$

Hence, $u \cdot (v \cdot w) \neq (u \cdot v) \cdot w$

This is a nonassociative and noncommutative ring. So, part (i) of the Definition 4.1 is satisfied.

For the second condition of the *left-nil AFC* group, we need to prove that every associative multiplication is commutative.

So suppose that $R \oplus R$ is an associative *nil-ring* and put $x = (1, 0)$ and $y = (0, 1)$.

Let, $x^2 = cx + dy$ for some $c, d \in R$.

Using associativity,

$x \cdot (x \cdot x) = (x \cdot x) \cdot x$

$\Rightarrow x \cdot x^2 = x^2 \cdot x$

$\Rightarrow x \cdot (cx + dy) = (cx + dy) \cdot x$

$\Rightarrow cx^2 + dx \cdot y = cx^2 + dy \cdot x$; [Using distributivity]

$\Rightarrow dx \cdot y = dy \cdot x$

$\Rightarrow d(x \cdot y - y \cdot x) = 0.$

There are two cases based on d :

Case 1 : $d \neq 0$,

Since R is an integral domain, then, $x \cdot y - y \cdot x = 0$

$\Rightarrow x \cdot y = y \cdot x$, so the ring is commutative.

Case 2 : $d = 0$,

then $x^2 = cx$. We show that $x^2 = y^2 = (0, 0)$.

If $x^2 \neq (0, 0)$ then for some $k > 2, x^k = (0, 0)$. Suppose this $k \in \mathbb{N}$ is minimal.

We have, $x^{k-2} \cdot x^2 = x^{k-2} \cdot cx$

$\Rightarrow x^k = cx^{k-1}$; [Using associativity]

$\Rightarrow 0 = cx^{k-1}$

$\Rightarrow c = 0$; [Since $x^{k-1} \neq 0$ by the choice of k and the fact that R is an integral domain]

Hence, $x^2 = 0$. Similarly $y^2 = 0$.

Now let, $x \cdot y = ax + by$, for some $a, b \in R$.

$\Rightarrow x \cdot (x \cdot y) = x \cdot (ax + by)$

$\Rightarrow x^2 \cdot y = ax^2 + bx \cdot y$; [Using associativity and distributivity]

$\Rightarrow 0 = 0 + bx \cdot y$;

$\Rightarrow bx \cdot y = 0$.

There are two cases based on b :

Case (i) : $b \neq 0$, then $x \cdot y = 0$ which is a contradiction.

Case (ii) : $b = 0$.

We show that a must be 0. Suppose, on the contrary that $a \neq 0$. Then $x \cdot y = ax$

And since $x^2 = y^2 = (0, 0)$,

$(x \cdot y) \cdot y = (ax) \cdot y$

$= a(x \cdot y)$

$= a(ax)$

$= a^2x \neq (0, 0)$.

Also, $x \cdot (y \cdot y) = x \cdot (y^2)$

$= x \cdot (0, 0)$

$= (0, 0)$.

So, $(x \cdot y) \cdot y \neq x \cdot (y \cdot y)$. This contradicts associativity of $R \oplus R$, so $a = 0$.

Hence, $x \cdot y = 0$. Similarly $y \cdot x = 0$. So the ring is commutative.

Therefore, in all cases the ring is commutative and thus associativity forced commutativity in $R \oplus R$ in the class *left-nil* rings.

This completes the proof.

4.2.2 Corollary

In particular, $\mathbb{Q} \oplus \mathbb{Q}$, $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{Z}_p \oplus \mathbb{Z}_p$ are *left-nil-AFC* groups.

4.2.3 Corollary

$\mathbb{Z} \oplus \mathbb{Z}_p$ is a *left-nil-AFC* group.

Proof : Define the multiplication on $\mathbb{Z} \oplus \mathbb{Z}_p$ by

$$(m, a) \cdot (n, b) = (0, \overline{mn} + \overline{n}a)$$

where $m, n \in \mathbb{Z}_p$, $a, b \in \mathbb{Z}$ and $\overline{n}$ denotes the element from $\mathbb{Z}_p$ which is the homomorphic image of n with

$$n \longmapsto \overline{n} \text{ in } \mathbb{Z}_p \cong \mathbb{Z}/_{<p>}$$

It is the remainder from division of n by p.

Let $(a, b), (c, d)$ and (e, f) be any three elements of this group, where $a, c, e \in \mathbb{Z}$ and $b, d, f \in \mathbb{Z}_p$.

Now, we need to verify the distributive property of the ring $(\mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

$(a, b) \cdot \big((c, d) + (e, f)\big) = (a, b) \cdot (c + e, d + f)$

$= (0, \overline{ac + ae} + \overline{c + e}b))$

$= (0, \overline{ac} + \overline{ae} + \overline{c}b + \overline{e}b)$

$= (0, \overline{ac} + \overline{c}b) + (0, \overline{ae} + \overline{e}b)$

$= (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$, which satisfies the left- distributivity on $(\mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

Also, $\big((c, d) + (e, f)\big) \cdot (a, b) = (c + e, d + f) \cdot (a, b)$

$= (0, \overline{ca + ea} + \overline{a}(d + f))$

$= (0, \overline{ca} + \overline{ea} + \overline{a}d + \overline{a}f)$

$= (0, \overline{ca} + \overline{a}d) + (0, \overline{ea} + \overline{a}f)$

$= (c, d) \cdot (a, b) + (e, f) \cdot (a, b)$, which satisfies the right- distributivity on $(\mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

Now, $(a, b)^2 = (a, b) \cdot (a, b) = (0, \overline{a^2} + \overline{a}b)$

and, $(a, b)^3 = (a, b) \cdot (a, b)^2$

$= (a, b) \cdot (0, \overline{a^2} + \overline{a}b) = (0, 0)$

Since a *left-cube* of any element of this ring is $(0,0)$. $(\mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$ is a *left-nil* ring.

To prove that the group $\mathbb{Z} \oplus \mathbb{Z}_p$ is an AFC group, we need to verify that $\mathbb{Z} \oplus \mathbb{Z}_p$ satisfies the two conditions of the Definition 4.1.
We obtain a noncommutative ring since $(1,0) \cdot (0,1) = (0,0)$ and $(0,1) \cdot (1,0) = (0,1)$.

Let, $u = (0,1)$, $v = (1,1)$, $w = (1,0)$ to be any three elements of $(\mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$
Now, $u \cdot (v \cdot w) = (0,1) \cdot \big((1,1) \cdot (1,0)\big)$
$= (0,1) \cdot (0, \overline{2})$
$= (0,0)$.
On the other hand, $(u \cdot v) \cdot w = \big((0,1) \cdot (1,1)\big) \cdot (1,0)$
$= (0,1) \cdot (1,0)$
$= (0,1)$.
Hence, $u \cdot (v \cdot w) \neq (u \cdot v) \cdot w$.
This is a nonassociative and noncommutative ring. So, part (i) of the Definition 4.1 is satisfied.
For the second condition of the *left-nil* AFC group, we need to prove that every associative multiplication is commutative.
So suppose that $\mathbb{Z} \oplus \mathbb{Z}_p$ is an associative and *nil-ring* and put $x = (1,0)$ and $y = (0,1)$.
Let, $x^2 = cx + dy$ for some $c \in \mathbb{Z}$ and d is an integer from 0 to $p-1$.
Using associativity,
$x \cdot (x \cdot x) = (x \cdot x) \cdot x$
$\Rightarrow x \cdot x^2 = x^2 \cdot x$
$\Rightarrow x \cdot (cx + dy) = (cx + dy) \cdot x$
$\Rightarrow cx^2 + dx \cdot y = cx^2 + dy \cdot x$; [Using distributivity]
$\Rightarrow dx \cdot y = dy \cdot x$
$\Rightarrow d(x \cdot y - y \cdot x) = 0$.
There are two cases based on d :

Case 1 : $d \neq 0$,

Since $\mathbb{Z}$ and $\mathbb{Z}_p$ are integral domains, then, $x \cdot y - y \cdot x = 0$

$\Rightarrow x \cdot y = y \cdot x$, so the ring is commutative.

Case 2 : $d = 0$,

then $x^2 = cx$. We show that $x^2 = y^2 = (0, 0)$.

If $x^2 \neq (0, 0)$ then for some $k > 2, x^k = (0, 0)$. Suppose this $k \in \mathbb{N}$ is minimal.

We have, $x^{k-2} \cdot x^2 = x^{k-2} \cdot cx$

$\Rightarrow x^k = cx^{k-1}$; [Using associativity]

$\Rightarrow 0 = cx^{k-1}$

$\Rightarrow c = 0$; [Since $x^{k-1} \neq 0$ by the choice of k and the fact that $\mathbb{Z}$ and $\mathbb{Z}_p$ are integral domains]

Hence, $x^2 = 0$. Similarly $y^2 = 0$.

Now let, $x \cdot y = ax + by$, for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_p$.

$\Rightarrow x \cdot (x \cdot y) = x \cdot (ax + by)$

$\Rightarrow x^2 \cdot y = ax^2 + bx \cdot y$; [Using associativity and distributivity]

$\Rightarrow 0 = 0 + bx \cdot y$

$\Rightarrow bx \cdot y = 0$.

There are two cases based on b :

Case (i) : $b \neq 0$, then $x \cdot y = 0$ which is a contradiction.

Case (ii) : $b = 0$.

We show that a must be 0. Suppose, on the contrary that $a \neq 0$. Then $x \cdot y = ax$.

And since $x^2 = y^2 = (0, 0)$,

$(x \cdot y) \cdot y = (ax) \cdot y$

$= a(x \cdot y)$

$= a(ax)$

$= a^2x \neq (0, 0)$.

Also, $x \cdot (y \cdot y) = x \cdot (y^2)$

$= x \cdot (0, 0)$

$= (0, 0)$.

So, $(x \cdot y) \cdot y \neq x \cdot (y \cdot y)$. This contradicts associativity of $\mathbb{Z} \oplus \mathbb{Z}_p$, so $a = 0$.

Hence, $x \cdot y = 0$. Similarly $y \cdot x = 0$. So the ring is commutative.

Therefore, in all cases the ring is commutative and thus associativity forced commutativity in $\mathbb{Z} \oplus \mathbb{Z}_p$ in the class *left-nil* rings.

This completes the proof.

4.3 Cyclic groups and subgroups of $\mathbb{Q}$

4.3.1 Theorem

Cyclic groups and subgroups of $\mathbb{Q}$ are not *left-nil AFC* group.

Proof : By Proposition 3.1.1, we know that if $\mathbf{K_1} \subset \mathbf{K_2}$ and for a group G the condition (i) of AFC group fails in $\mathbf{K_2}$, then it fails in $\mathbf{K_1}$.

Let us consider $\mathbf{K_2} = AFC$ group and $\mathbf{K_1} =$ *left-nil AFC* group. So, $\mathbf{K_1} \subset \mathbf{K_2}$.

Now, by Theorem 3.3.5, we know that cyclic groups and subgroups of $\mathbb{Q}$ are not AFC group. Because, all multiplications are commutative for these group.

Therefore, part(i) of the definition is never satisfied.

Since condition (i) fails in $\mathbf{K_2}$ for cyclic groups and subgroups of $\mathbb{Q}$, then it fails in $\mathbf{K_1}$.

Therefore, cyclic groups and subgroups of $\mathbb{Q}$ are not *left-nil AFC* group.

4.3.2 Corollary

Let A_α be an ascending chain of cyclic groups. Then $A = \cup A_\alpha$ is not *left-nil AFC* group.

Proof : By Proposition 3.1.1, we know that if $\mathbf{K_1} \subset \mathbf{K_2}$ and for a group G the condition (i) of AFC group fails in $\mathbf{K_2}$, then it fails in $\mathbf{K_1}$.

Let us consider $\mathbf{K_2} = AFC$ group and $\mathbf{K_1} =$ *left-nil* AFC group. So, $\mathbf{K_1} \subset \mathbf{K_2}$.

Now, by Theorem 3.3.6, we know that $\cup A_\alpha$ are not AFC group where A_α be an ascending chain of cyclic groups. Because, all multiplications are commutative for this group.

Therefore, part(i) of the definition is never satisfied.

Since condition (i) fails in $\mathbf{K_2}$ for $\cup A_\alpha$, then it fails in $\mathbf{K_1}$.

Therefore, A is not *left-nil* AFC group.

4.4 Vector Space over a field

4.4.1 Theorem

Let V be a vector space with dimension ≥ 3 over a field then the group V is not a *left-nil* AFC group.

Proof : Let $\mathbb{K}$ be the base field of $\mathbb{F}$.

Suppose first that $\dim_{\mathbb{K}} V \geq 3$ and take three linearly independent elements e, f and g of a basis on V.

Now, we can extend $\{e, f, g\}$ to a basis $\mathbb{B}$ of V over $\mathbb{K}$. Let us define a multiplication on V using the basis by putting,

$$e^2 = e \cdot f = f \cdot e = f^2 = f \cdot g = g \cdot f = g \cdot e = g^2 = 0,$$

$$e \cdot g = f$$

and setting all the remaining products of the basis elements equal to 0.

The ring is noncommutative, since $g \cdot e = 0$ and $e \cdot g = f$. Moreover,

let $x = \alpha e + \beta f + \gamma g$ be any element in V where $\alpha, \beta, \gamma \in V$.

Now, $x^2 = x \cdot x = (\alpha e + \beta f + \gamma g) \cdot (\alpha e + \beta f + \gamma g)$

$= (\alpha^2 e^2 + \alpha\beta e \cdot f + \alpha\gamma e \cdot g + \beta\alpha f \cdot e + \beta^2 f^2 + \beta\gamma f \cdot g + \gamma\alpha g \cdot e + \gamma\beta g \cdot f + \gamma^2 g^2)$

$= (0 + 0 + \alpha\gamma f + 0 + 0 + 0 + 0 + 0 + 0)$

$= \alpha\gamma f$

$x^3 = x \cdot x^2 = (\alpha e + \beta f + \gamma g) \cdot x^2$

$= (\alpha e + \beta f + \gamma g) \cdot \alpha\gamma f$

$= \alpha^2\gamma e \cdot f + \beta\alpha\gamma f^2 + \alpha\gamma^2 g \cdot f$

$= 0 + 0 + 0$

$= 0.$

Since a *left-cube* of any element of this ring is 0. So, V is a *left-nil ring*.

Let u, v and w be any three elements in V such that $u = ae + bf + cg$, $v = le + mf + ng$ and $w = pe + qf + rg$ where, $a, b, c, l, m, n, p, q, r \in V$.

Now, $(u \cdot v) \cdot w = \big((ae + bf + cg) \cdot (le + mf + ng)\big) \cdot (pe + qf + rg)$

$= (ale^2 + ame \cdot f + ane \cdot g + blf \cdot e + bmf^2 + bnf \cdot g + clg \cdot e + cmg \cdot f + cng^2) \cdot (pe + qf + rg)$

$= (0 + 0 + anf + 0 + 0 + 0 + 0 + 0 + 0) \cdot (pe + qf + rg)$

$= anf \cdot (pe + qf + rg)$

$= anpf \cdot e + anqf^2 + anrf \cdot g$

$= 0 + 0 + 0$

$= 0.$

On the other hand, $u \cdot (v \cdot w) = (ae + bf + cg) \cdot \big((le + mf + ng) \cdot (pe + qf + rg)\big)$

$= (ae + bf + cg) \cdot (lpe^2 + lqe \cdot f + lre \cdot g + mpf \cdot e + mqf^2 + mrf \cdot g + npg \cdot e + nqg \cdot f + nrg^2)$

$= (ae + bf + cg) \cdot (0 + 0 + lrf + 0 + 0 + 0 + 0 + 0 + 0)$

$= (ae + bf + cg) \cdot lrf$

$= alre \cdot f + blrf^2 + clrg \cdot f$

$= 0 + 0 + 0$

$= 0.$

Hence, $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.

This is an associative and noncommutative multiplication on V. Therefore, the condition (ii) of the Definition 4.1 is not satisfied, so V is not a *left-nil AFC* group.

4.4.2 Theorem

Let V be a vector space with dimension 2 over a field of characteristic 0, then the group V is a *left-nil AFC* group.

Proof : Let $\mathbb{K}$ be the base field of $\mathbb{F}$.

Suppose first that $\dim_{\mathbb{K}} V = 2$ and take two linearly independent elements e and f of a basis on V. Let us define a ring multiplication on V using the basis by putting,

$$e^2 = e \cdot f = f^2 = 0$$

$$f \cdot e = f$$

and setting all the remaining products of the basis elements equal to 0.

The ring is noncommutative, since $e \cdot f = 0$ and $f \cdot e = f$. Moreover,

let $x = \alpha e + \beta f$ be any element in V where $\alpha, \beta \in V$.

$x^2 = x \cdot x = (\alpha e + \beta f) \cdot (\alpha e + \beta f)$

$= (\alpha^2 e^2 + \alpha\beta e \cdot f + \beta\alpha f \cdot e + \beta^2 f^2)$

$= (0 + 0 + \beta\alpha f + 0 + 0 + 0 + 0 + 0 + 0)$

$= \beta\alpha f$

$x^3 = x \cdot x^2 = (\alpha e + \beta f) \cdot x^2$

$= (\alpha e + \beta f) \cdot \beta\alpha f = \alpha^2\beta e \cdot f + \beta^2\alpha f^2$

$= 0 + 0$

$= 0.$

Since a *left-cube* of any element of this ring is 0. So, V is a *left-nil ring*.

To prove that V is a *left nil AFC* group, we need to verify that V satisfies the two conditions of the Definition 4.1.

Let, $a = f, b = e + f, c = e$ on V

$a \cdot (b \cdot c) = f \cdot \big((e + f) \cdot e\big)$

$= f \cdot \big((e+f) \cdot e\big)$

$= f \cdot (e \cdot e + f \cdot e)$

$= f \cdot (0 + f)$

$= f \cdot f$

$= 0.$

On the other hand, $(a \cdot b) \cdot c = \big(f \cdot (e+f)\big) \cdot e$

$= (f \cdot e + f \cdot f) \cdot e$

$= (f + 0) \cdot e$

$= f \cdot e$

$= f.$

Hence, $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c.$

This is a nonassociative and noncommutative ring. So, part(i) of the Definition 4.1 is satisfied.

For the second condition of the *left-nil AFC* group, we need to prove that every associative multiplication is commutative.

Suppose that V admits an *associative nil-ring* structure and let x, y be the basis of V over $\mathbb{K}$.

Let, $x^2 = cx + dy$ for some $c, d \in \mathbb{K}$.

Using associativity,

$x \cdot (x \cdot x) = (x \cdot x) \cdot x$

$\Rightarrow x \cdot x^2 = x^2 \cdot x$

$\Rightarrow x \cdot (cx + dy) = (cx + dy) \cdot x$

$\Rightarrow cx^2 + dx \cdot y = cx^2 + dy \cdot x$; [Using distributivity]

$\Rightarrow dx \cdot y = dy \cdot x$

$\Rightarrow d(x \cdot y - y \cdot x) = 0.$

There are two cases based on d :

Case 1 : $d \neq 0$,

Since V is a vector space over a field of characteristic zero, then, $x \cdot y - y \cdot x = 0$

$\Rightarrow x \cdot y = y \cdot x$. So the ring is commutative.

Case 2 : $d = 0$,

then $x^2 = cx$. We show that $x^2 = y^2 = (0, 0)$.

If $x^2 \neq (0, 0)$ then for some $k > 2, x^k = (0, 0)$. Suppose this $k \in \mathbb{N}$ is minimal.

We have, $x^{k-2} \cdot x^2 = x^{k-2} \cdot cx$

$\Rightarrow x^k = cx^{k-1}$ [Using associativity]

$\Rightarrow 0 = cx^{k-1}$

$\Rightarrow c = 0$; [Since $x^{k-1} \neq 0$ by the choice of k and the fact that V is a vector space over a field of characteristic zero]

Hence, $x^2 = 0$. Similarly $y^2 = 0$.

Now let, $x \cdot y = ax + by$, for some $a, b \in \mathbb{K}$.

$\Rightarrow x \cdot (x \cdot y) = x \cdot (ax + by)$

$\Rightarrow x^2 \cdot y = ax^2 + bx \cdot y$; [Using associativity and distributivity]

$\Rightarrow 0 = 0 + bx \cdot y$;

$\Rightarrow bx \cdot y = 0$.

There are two cases based on b :

Case (i) : $b \neq 0$, then $x \cdot y = 0$ which is a contradiction.

Case (ii) : $b = 0$.

We show that a must be 0. Suppose, on the contrary that $a \neq 0$. Then $x \cdot y = ax$.

And since $x^2 = y^2 = (0, 0)$,

$(x \cdot y) \cdot y = (ax) \cdot y$

$= a(x \cdot y)$

$= a(ax)$

$= a^2 x \neq (0, 0)$.

Also, $x \cdot (y \cdot y) = x \cdot (y^2)$

$= x \cdot (0, 0)$

$= (0, 0)$.

So, $(x \cdot y) \cdot y \neq x \cdot (y \cdot y)$. This contradicts associativity of V, so $a = 0$.

Hence, $x \cdot y = 0$. Similarly $y \cdot x = 0$. So the ring is commutative.

Therefore, in all cases the ring is commutative and thus associativity forced commutativity on V over $\mathbb{K}$ in the class *left-nil-algebras*.

This completes the proof.

4.5 $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$ (where α or $\beta > 1$)

4.5.1 Theorem

$\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$ (where α or $\beta > 1$), is not a *left-nil-AFC* group.

Proof : (without loss of generality, let us consider $\beta > 1$), We want to show that there exists a ring on the group $G = \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$ with an associative and noncommutative multiplication. Then, part (ii) of the definition of *AFC* group would not be satisfied.

Let, $R = \{0, p^{\beta-\alpha}, 2p^{\beta-\alpha}, ..., p^{\beta+1-\alpha}, ..., (p^\alpha - 1)p^{\beta\alpha}\}$.

Then $R \subseteq \mathbb{Z}_{p^\beta}$ is a subgroup and $R \cong \mathbb{Z}_{p^\alpha}$. We show the theorem for $R \oplus \mathbb{Z}_{p^\beta}$.

Define the multiplication on $R \oplus \mathbb{Z}_{p^\beta}$ by

$$(r, a) \cdot (s, b) = (0, asp^{\beta-1})$$

where $r, s \in R$ and $a, b \in \mathbb{Z}_{p^\beta}$.

Let $(a, b), (c, d)$ and (e, f) be any three elements of this group, where $a, c, e \in R$ and $b, d, f \in \mathbb{Z}_{p^\beta}$.

Now, we need to verify the distributive property of the ring $(R \oplus \mathbb{Z}_{p^\beta}, +, \cdot)$.

$(a, b) \cdot \big((c, d) + (e, f)\big) = (a, b) \cdot (c + e, d + f)$

$= (0, b(c + e)p^{\beta-1})$

$= (0, bcp^{\beta-1} + bep^{\beta-1})$

$= (0, bcp^{\beta-1}) + (0, bep^{\beta-1})$

$= (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$, which satisfies the left- distributivity on $(R \oplus \mathbb{Z}_{p^\beta}, +\cdot)$.

Also, $\big((c, d) + (e, f)\big) \cdot (a, b) = (c + e, d + f) \cdot (a, b)$

$= (0, (d + f)ap^{\beta-1})$

$= (0, dap^{\beta-1} + fap^{\beta-1})$

$= (0, dap^{\beta-1}) + (0, fap^{\beta-1})$

$= (c, d) \cdot (a, b) + (e, f) \cdot (a, b)$, which satisfies the right- distributivity on $(R \oplus \mathbb{Z}_{p^\beta}, +, \cdot)$.

Now, $(a, b)^2 = (a, b) \cdot (a, b) = (0, bap^{\beta-1})$

and, $(a, b)^3 = (a, b) \cdot (a, b)^2$

$= (a, b) \cdot (0, bap^{\beta-1})$

$= (0, 0)$

Since a *left-cube* of any element of this group is $(0, 0)$, $(R \oplus \mathbb{Z}_{p^\beta}, +, \cdot)$ is a *left-nil* ring.

We need to verify the associative property of this ring $(R \oplus \mathbb{Z}_{p^\beta}, +, \cdot)$.

Now, $(a, b) \cdot \big((c, d) \cdot (e, f)\big)$

$= (a, b) \cdot (0, dep^{\beta-1})$

$= (0, 0)$.

On the other hand, $\big((a, b) \cdot (c, d)\big) \cdot (e, f)$

$= (0, bcp^{\beta-1}) \cdot (e, f)$

$= (0, bcep^{\beta-1}p^{\beta-1})$

$= (0, bcep^{2\beta-2})$

$= (0, 0)$; [In $\mathbb{Z}_{p^\beta}$, $bcep^{2\beta-2} = 0$ since $p^\beta | p^{2\beta-2}$ if $\beta > 1$]

So, $(a, b) \cdot \big((c, d) \cdot (e, f)\big) = \big((a, b) \cdot (c, d)\big) \cdot (e, f)$.

Therefore, $R \oplus \mathbb{Z}_{p^\beta}$ is an associative and noncommutative ring and thus $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$ is not a *left-nil AFC* group.

4.5.2 Corollary

$\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus H$ is not a *left-nil-AFC* group where α or $\beta > 1$ and H is any finite group.

Proof : By Theorem 4.5.1, we know that $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$ is not a *left-nil AFC* group. Because, there is an associative and noncommutative ring multiplication on $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$.

Then, by Lemma 3.2.1, we know that if $G = A \oplus B$ and there is an associative and noncommutative multiplication " $\cdot$ " on A, then there is also an associative and noncommutative multiplication on G.
Since $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}$ has an associative and noncommutative multiplication, then so does $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus H$.
Therefore, $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus H$ is not a *left-nil-AFC* group.

4.6 $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{q^\beta}$ (where $\beta \geq 1, p \neq q$)

4.6.1 Theorem

$\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{q^\beta}$ (where $\beta \geq 1, p \neq q$) is not a *left-nil-AFC* group.

Proof : We need to show that there exists a ring on the group $G = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{q^\beta}$ with an associative and noncommutative multiplication . Then, part (ii) of the definition of *AFC* group would not be satisfied.
Now, the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{q^\beta}$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_{pq^\beta}$ (where $\beta \geq 1$). By the isomorphism, if $\mathbb{Z}_p \oplus \mathbb{Z}_{pq^\beta}$ has an associative and noncommutative multiplication, then so does $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{q^\beta}$.
Let $R = \{0, q^\beta, 2q^\beta, ...(p-1)q^\beta\}$.
Then $R \subseteq \mathbb{Z}_{pq^\beta}$ is a subgroup and $R \cong \mathbb{Z}_p$. We show the theorem for $R \oplus \mathbb{Z}_{pq^\beta}$.

Define the multiplication on $R \oplus \mathbb{Z}_{pq^\beta}$ by

$$(r, a) \cdot (s, b) = (0, aspq^{\beta-1})$$

Let $(a, b), (c, d)$ and (e, f) be any three elements of this group, where $a, c, e \in R$ and $b, d, f \in \mathbb{Z}_{pq^\beta}$.

Now, we need to verify the distributive property of the ring $(R \oplus \mathbb{Z}_{pq^\beta}, +, \cdot)$.

$(a, b) \cdot \big((c, d) + (e, f)\big) = (a, b) \cdot (c + e, d + f)$

$= (0, b(c + e)pq^{\beta-1})$

$= (0, bcpq^{\beta-1} + bepq^{\beta-1})$

$= (0, bcpq^{\beta-1}) + (0, bepq^{\beta-1})$

$= (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$, which satisfies the left- distributivity on $(R \oplus \mathbb{Z}_{pq^\beta}, +, \cdot)$.

Also, $\big((c, d) + (e, f)\big) \cdot (a, b) = (c + e, d + f) \cdot (a, b)$

$= (0, (d + f)apq^{\beta-1})$

$= (0, dapq^{\beta-1} + fapq^{\beta-1})$

$= (0, dapq^{\beta-1}) + (0, fapq^{\beta-1})$

$= (c, d) \cdot (a, b) + (e, f) \cdot (a, b)$, which satisfies the right- distributivity on $(R \oplus \mathbb{Z}_{pq^\beta}, +, \cdot)$.

Now, $(a, b)^2 = (a, b) \cdot (a, b)$

$= (0, bapq^{\beta-1})$

and, $(a, b)^3 = (a, b) \cdot (a, b)^2$

$= (a, b) \cdot (0, bapq^{\beta-1})$

$= (0, 0)$

Since a *left-cube* of any element of this group is $(0, 0)$, $(R \oplus \mathbb{Z}_{pq^\beta}, +, \cdot)$ is a *left-nil* ring.

We need to verify the associative property of this ring $(R \oplus \mathbb{Z}_{pq^\beta}, +, \cdot)$.

Now, $(a, b) \cdot \big((c, d) \cdot (e, f)\big)$

$= (a, b) \cdot (0, depq^{\beta-1})$

$= (0, 0)$.

On the other hand, $\big((a,b)\cdot(c,d)\big)\cdot(e,f)$

$= (0, bcpq^{\beta-1})\cdot(e,f)$

$= (0, bcepq^{\beta-1}pq^{\beta-1})$

$= (0, bcep^2q^{2\beta-2})$

$= (0,0)$; [In $\mathbb{Z}_{p^\beta}$, $bcep^2q^{2\beta-2} = 0$ since $p^\beta | p^2q^{2\beta-2}$ if $\beta > 1$]

So, $(a,b)\cdot\big((c,d)\cdot(e,f)\big) = \big((a,b)\cdot(c,d)\big)\cdot(e,f)$.

Therefore, $R \oplus \mathbb{Z}_{pq^\beta}$ is an associative and noncommutative ring and thus $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{q^\beta}$ is not a *left-nil-AFC* group.

4.7 Classification of finitely generated AFC groups

For the further steps in the classification of *left-nil* finitely generated groups we need the following theorem :

4.7.1 Theorem (Additive group of the form $A \oplus A \oplus A$)

If A is an additive group of an associative ring, then the group of the form $A \oplus A \oplus A$ is not a *left-nil AFC* group.

Proof : Suppose that $G = A \oplus A \oplus A$ and $K =$ the group of 3×3 upper triangular matrices over the ring A, with $0'$s on the diagonal.

$$= \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \text{ where } a, b, c \in A.$$

Then, G and K are isomorphic as groups. So, K has an associative and noncommutative multiplication ring structure iff G has.

Let,

$$X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \text{ where } a, b, c \in A,$$

$$Y = \begin{bmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} \text{ where } d, e, f \in A \text{ and}$$

$$Z = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \text{ where } x, y, z \in A.$$

Then,

$$X^2 = X \cdot X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X^3 = X \cdot X^2 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, the group $G = A \oplus A \oplus A$ admits a *left-nil* structure.

We know that the matrix multiplication is associative, $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$.

Now, let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ on G.

Then

$$A \cdot B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } B \cdot A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $A \cdot B \neq B \cdot A$.

G admits a *left-nil* associative and noncommutative structure.
This completes the proof.

4.7.2 Theorem (Additive group of the form $A \oplus A \oplus \mathbb{Z}$)

If A is an additive group of an associative ring, then the group of the form $A \oplus A \oplus \mathbb{Z}$ is not a *left-nil AFC* group.

Proof : Suppose that $G = A \oplus A \oplus \mathbb{Z}$.
We define the ring multiplication on G as follows:

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (0, a_1c_2, 0)$$

where $a_1, a_2, b_1, b_2 \in A$ and $c_1, c_2 \in \mathbb{Z}$ and $a_1c_2 \in \mathbb{Z}$
Now, let (x, y, z) be any element in $A \oplus A \oplus \mathbb{Z}$.
$(x, y, z)^2 = (x, y, z) \cdot (x, y, z) = (0, xz, 0)$
$(x, y, z)^3 = (x, y, z) \cdot (x, y, z)^2 = (x, y, z) \cdot (0, xz, 0)$
$= (0, 0, 0)$.
Since a *left-cube* of any element of this group is $(0, 0, 0)$, $(A \oplus A \oplus \mathbb{Z}, +, \cdot)$ is a *left-nil* ring.
Let us consider, $a = (x, y, z), b = (e, f, g), c = (m, n, p)$ to be any three elements in G where

$x, y, e, f, m, n \in A$ and $z, g, p \in \mathbb{Z}$.

$a \cdot (b \cdot c) = (x, y, z) \cdot \big((e, f, g) \cdot (m, n, p)\big)$

$= (x, y, z) \cdot (0, ep, 0)$

$= (0, 0, 0)$.

On the other hand, $(a \cdot b) \cdot c = \big((x, y, z) \cdot (e, f, g)\big) \cdot (m, n, p)$

$= (0, xg, 0) \cdot (m, n, p)$

$= (0, 0, 0)$.

So, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Again, let $u = (1, 1, 1)$ and $v = (1, 1, 0)$ be two elements in G.

Now, $u \cdot v = (1, 1, 1) \cdot (1, 1, 0) = (0, 0, 0)$ and $v \cdot u = (1, 1, 0) \cdot (1, 1, 1) = (0, 1, 0)$.

So, $u \cdot v \neq v \cdot u$.

Therefore, G has an associative and noncommutative ring multiplication.

This completes the proof.

4.7.3 Corollary

$\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z} \oplus H$ is not a *left-nil AFC* group. (H is any finite group).

Proof : By Theorem 4.7.2, we know that $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}$ admits an associative and noncmmutative multiplication.

Then, by Lemma 3.2.1, we know that if $G = A \oplus B$ and there is an associative and noncommutative multiplication " $\cdot$ " on A, then there is also an associative and noncommutative multiplication on G.

Since $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}$ has an associative and noncommutative multiplication, then so has $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z} \oplus H$

Therefore, $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z} \oplus H$ is not a *left-nil AFC* group.

4.7.4 Theorem (Additive group of the form $\mathbb{Z} \oplus A \oplus B$)

If A and B are two additive groups of associative rings, then the group of the form $\mathbb{Z} \oplus A \oplus B$ is not a *left-nil AFC* group.

Proof : Suppose that $G = \mathbb{Z} \oplus A \oplus B$. We define the ring multiplication on G as follows :

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (0, a_1 c_2, 0)$$

where $a_1, a_2 \in \mathbb{Z}$, $b_1, b_2 \in A$, $c_1, c_2 \in B$ and $a_1 c_2 \in \mathbb{Z}$.

Let (x, y, z) be any element in $\mathbb{Z} \oplus A \oplus B$.

$(x, y, z)^2 = (x, y, z) \cdot (x, y, z) = (0, xz, 0)$

$(x, y, z)^3 = (x, y, z) \cdot (x, y, z)^2 = (x, y, z) \cdot (0, xz, 0)$

$= (0, 0, 0)$.

Since a *left-cube* of any element of this group is $(0, 0, 0)$, $(\mathbb{Z} \oplus A \oplus B, +, \cdot)$ is a *left-nil* ring.

Let us consider $a = (x, y, z)$, $b = (e, f, g)$, $c = (m, n, p)$ to be any three elements in G. We need to verify the associative property of this ring $(\mathbb{Z} \oplus A \oplus B, +, \cdot)$.

$a \cdot (b \cdot c) = (x, y, z) \cdot \big((e, f, g) \cdot (m, n, p)\big) = (x, y, z) \cdot (0, ep, 0)$

$= (0, 0, 0)$.

On the other hand, $(a \cdot b) \cdot c = \big((x, y, z) \cdot (e, f, g)\big) \cdot (m, n, p)$

$= (0, xg, 0) \cdot (m, n, p)$

$= (0, 0, 0)$.

So, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Thus the ring is associative.

Suppose that $u = (1, 1, 1)$ and $v = (1, 1, 0)$ be two elements in G.

$u \cdot v = (1, 1, 1) \cdot (1, 1, 0) = (0, 0, 0)$ and $v \cdot u = (1, 1, 0) \cdot (1, 1, 1) = (0, 1, 0)$.

So, $u \cdot v \neq v \cdot u$.

Since G has an associative and noncommutative ring multiplication, $\mathbb{Z} \oplus A \oplus B$ is not a *left-nil AFC* group.

4.7.5 Theorem ($\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$)

$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ is not a *left-nil-AFC* group.

Proof : We want to show that there exists a ring on the group $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ with an associative and noncommutative multiplication. Then, part (ii) of the definition of the AFC group would not be satisfied.
By [5] (Example-2. P-283), the first two coordinates of any product with z on $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ must be zero. So we define the following multiplication on $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ by

$$(a, b, c) \cdot (d, e, f) = (0, 0, \overline{ae})$$

where $a, b, d, e \in \mathbb{Z}$ and $c, f \in \mathbb{Z}_p$. Here, $\overline{ae}$ denotes the element from $\mathbb{Z}_p$ which is the homomorphic image of ae with

$$ae \longmapsto \overline{ae} \text{ in } \mathbb{Z}_p \cong \mathbb{Z}/_{<p>}$$

Let $(a, b, c), (d, e, f)$ and (m, n, r) be any three elements of this group, where $a, b, d, e, m, n \in \mathbb{Z}$ and $c, f, r \in \mathbb{Z}_p$.
Now, we need to verify the distributive property of the ring $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.
$(a, b, c) \cdot \big((d, e, f) + (m, n, r)\big) = (a, b, c) \cdot (d + m, e + n, f + r)$
$= (0, 0, \overline{ae + an}))$
$= (0, 0, \overline{ae}) + (0, 0, \overline{an})$
$= (a, b, c) \cdot (d, e, f) + (a, b, c) \cdot (m, n, r)$, which satisfies the left- distributivity on $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

Also, $\big((d, e, f) + (m, n, r)\big) \cdot (a, b, c) = (d + m, e + n, f + r) \cdot (a, b, c)$
$= (0, 0, \overline{db + mb})$

$= (0, 0, \overline{db}) + (0, 0, \overline{mb})$

$= (d, e, f) \cdot (a, b, c) + (m, n, r) \cdot (a, b, c)$, which satisfies the right- distributivity on $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

Now, $(a, b, c)^2 = (a, b, c) \cdot (a, b, c) = (0, 0, \overline{ab})$

and, $(a, b, c)^3 = (a, b, c) \cdot (a, b, c)^2$

$= (a, b, c) \cdot (0, 0, \overline{ab})$

$= (0, 0, 0)$

Since a *left-cube* of any element of this ring is $(0, 0, 0)$. $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$ is a *left-nil* ring.

We need to verify the associative property of this ring $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

Now, $(a, b, c) \cdot \big((d, e, f) \cdot (m, n, r)\big)$

$= (a, b, c) \cdot (0, 0, \overline{dn}) = (0, 0, 0)$.

On the other hand, $\big((a, b, c) \cdot (d, e, f)\big) \cdot (m, n, r)$

$= (0, 0, \overline{ae}) \cdot (m, n, r)) = (0, 0, 0)$.

So, $(a, b, c) \cdot \big((d, e, f) \cdot (m, n, r)\big) = \big((a, b, c) \cdot (d, e, f)\big) \cdot (m, n, r)$. Thus the ring is associative.

Suppose that $u = (1, 0, 0)$ and $v = (0, 1, 0)$ to be any two elements in $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p, +, \cdot)$.

$u \cdot v = (1, 0, 0) \cdot (0, 1, 0) = (0, 0, 1)$ and $v \cdot u = (0, 1, 0) \cdot (1, 0, 0) = (0, 0, 0)$

So, $u \cdot v \neq v \cdot u$.

Therefore, $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_p$ has an associative and noncommutative ring multiplication.

This completes the proof.

4.8 Finite left-nil AFC groups

We can now classify all finite *left-nil* groups by the following theorem :

4.8.1 Theorem

A finite group G is *left-nil AFC* group iff $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, for some p.

Proof :

Let G be a finite *left-nil AFC* group. By the Fundamental Theorem of Finite Abelian Groups, G is isomorphic to the direct sum of a finite number of $\mathbb{Z}_{p^r}$, with various prime numbers p and natural r's.

There are two cases based on the repetition of p :

i) p never repeats,

ii) p repeats.

Case i : If $p = p_1$ never repeats, then the finite group $\mathbb{Z}_{p_1^{r_1}} \oplus \mathbb{Z}_{p_2^{r_2}} \oplus \mathbb{Z}_{p_3^{r_3}} \oplus \ldots\ldots\ldots \oplus \mathbb{Z}_{p_n^{r_n}}$ is cyclic. By Theorem 4.3.1, we know that cyclic groups are not *left-nil AFC* group.
So, G is not a *left-nil AFC* group.

Case ii : If p repeats, then there are three cases based on the number of summands as follows:

a) G is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$;

b) G is isomorphic to $\mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus H$, where $H \neq \{0\}$ and at least one $\alpha, \beta > 1$;

c) G is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus H$, for some finite group $H \neq \{0\}$.

These three cases can be described as follows :

case a : If $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, then by Theorem 4.2.1, G is a *left-nil AFC* group.

case b : If $G \cong \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus H$ where $H \neq \{0\}$ and at least one $\alpha, \beta \geq 1$, then by Corollary 4.5.2, G is not a *left-nil AFC* group.

case c : If $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus H$, for some finite group $H \neq \{0\}$, then there are three cases based on H.

case c_1 : If H contains a summand $\mathbb{Z}_p$, then by Theorem 4.7.1, G is not a *left-nil AFC* group.

case c_2 **:** If H contains a summand $\mathbb{Z}_{p^\alpha}$ and $\alpha > 1$, then by Corollary 4.5.2, G is not a *left-nil AFC* group.

case c_3 **:** If H contains a summand $\mathbb{Z}_{q^\alpha}$, then by Theorem 4.6.1, G is not a *left-nil AFC* group.

Therefore, the only *left-nil AFC* group is $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

This completes the proof.

4.9 Finitely generated left-nil AFC groups

4.9.1 Theorem

A finitely generated abelian group G is *left-nil AFC* group iff $G \cong \mathbb{Z} \oplus \mathbb{Z}$ or, $\mathbb{Z} \oplus \mathbb{Z}_p$ or, $\mathbb{Z}_p \oplus \mathbb{Z}_p$, for some p.

Proof : Suppose that G is a finitely generated *left-nil AFC* group. By the Fundamental Theorem of Finitely Generated Abelian Groups, G is isomorphic to the direct sum of a finite number of cyclic groups.

So, there are two cases based on the number of summands as follows :

i) G is finite,

ii) G is infinite.

Case i : If G is finite, then by Theorem 4.8.1, G is *left-nil AFC* group iff $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, for some p.

Case ii : If G is infinite, then we have the following cases :

a) G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$;

b) G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus H$, for some group $H \neq \{0\}$;

c) G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_p$;

d) G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{p^\alpha}$, where $\alpha > 1$;

e) G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{p^\alpha} \oplus H$, where $\alpha \geq 1$ and some group $H \neq \{0\}$.

These five cases can be described as follows :

case a : If $G \cong \mathbb{Z} \oplus \mathbb{Z}$, then by Theorem 4.2.1, G is a *left-nil AFC* group.

case b : If $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus H$ where $H \neq \{0\}$, then there are two cases based on H.

case b_1 : If H contains a summand $\mathbb{Z}$, then by Theorem 4.7.1, G is not a *left-nil AFC* group.

case b_2 : If H contains a summand $\mathbb{Z}_p$, then by Theorem 4.7.5, G is not a *left-nil AFC* group.

case c : If $G \cong \mathbb{Z} \oplus \mathbb{Z}_p$, then by Corollary 4.2.3, G is a *left-nil AFC* group.

case d : If $G \cong \mathbb{Z} \oplus \mathbb{Z}_{p^\alpha}$ where $\alpha > 1$, then by Theorem 4.5.1, G is not a *left-nil AFC* group.

case e : If $G \cong \mathbb{Z} \oplus \mathbb{Z}_{p^\alpha} \oplus H$ where $H \neq \{0\}$ and $\alpha \geq 1$, then there are two cases based on H.

case e_1 : If H contains a summand $\mathbb{Z}_{p^\alpha}, \alpha \geq 1$, then by Theorem 4.7.2, G is not a *left-nil AFC* group.

case e_2 : If H contains a summand $\mathbb{Z}_{q^\beta}, \beta \geq 1$, then by Theorem 4.7.4, G is not a *left-nil AFC* group.

Therefore, the only *left-nil AFC* group is $\mathbb{Z} \oplus \mathbb{Z}$ or, $\mathbb{Z} \oplus \mathbb{Z}_p$ or, $\mathbb{Z}_p \oplus \mathbb{Z}_p$, for some p.
This completes the proof.

4.10 Conclusion

We proved that $R \oplus R$ is a *left-nil AFC* group, where R is the additive group of an integral domain with unity. Similarly a vector space with dimension 2 over a field at characteristic zero is a *left-nil AFC* group, while the property fails in higher dimension.

We completely classified finitely-generated *left-nil AFC* groups which cover finite abelian and infinite abelian groups. Namely, a finite group G is a *left-nil AFC* group iff $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, for some p and a finitely generated abelian group G is *left-nil AFC* group iff $G \cong \mathbb{Z} \oplus \mathbb{Z}$ or, $\mathbb{Z} \oplus \mathbb{Z}_{p^\alpha}$, $\alpha \geq 1$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$, for some p.

Now, in above groups it is clear that associativity forcing commutativity in the left-nil ring structure.

Chapter 5

Concluding Remarks

5.1 Significance of the Result

In various areas of algebra it is of interest, whether associativity forces commutativity. As part of our work we investigated a concrete correlation between these properties and defined a special class of ring structure. We provided a comprehensive and complete classification of *left-nil AFC* groups. The results add knowledge in the interplay between group and ring structures.

5.2 Future works

Encouraged by a possibility to classify *finitely-generated left-nil AFC* groups, we wish to study the topic further by relaxing the *finitely-generated* assumption. Although the subject of *AFC* groups is very vast, the stipulation that the resulting rings should be *left-nil* provides a significant restriction and brings hope of further classifications and interesting examples. Focusing on major classes of abelian groups such as torsion groups, torsion-free groups of a given rank[5],[6],[7] or torsion free periodic rings[8],[9] will provide background for the future work.

References

[1] Piotr J Wojciechowski, *"Associativity forces commutativity"*, Department of Mathematical Sciences, UTEP, Texas, 2015.

[2] Beaumont R. A., *"Rings with additive group which is the direct sum of cyclic groups"*, Duke Math. J. 15 (1948), 367 - 369.

[3] R.A.Beaumont and H.S.Zuckerman, *"A characterization of the subgroups of the additive ratio- nals"*, Pcific J. Math. 1 (1951) 169-177.

[4] G. Birkhoff and R. S. Pierce, *"Lattice-ordered rings"*, An. Acad. Brasil. Cienc. 28, (1956)

[5] L. Fuchs, *"Infinite Abelian Groups"*, Acad. Press. 1970, 1973.

[6] Beaumont R. A., Pierce R. S., *"Torsion free groups of rank two"*, Mem. Am. Math. Soc. 38, 41 p. (1961).

[7] Beaumont R. A., Pierce R. S., *"Torsion-free rings"*, Ill. J. Math. 5, 61 - 98 (1961).

[8] Beaumont R. A., *"A survey of torsion-free rings, Topics in Abelian Groups"*, (Proc. Sympos., New Mexico State Univ., 1962) pp. 41 50 Scott, Foresman and Co., Chicago, Ill. 1963.

[9] Abu-Khuzam H. 1987, *"A commutativity theorem for periodic rings"*, Math.Japon. 32 (1987), 1 - 3.

[10] Joseph A. Gallian, Contemporary Abstract Algebra/Cole 2006.

[11] K. E. Eldridge, *"Orders for finite noncommutative rings with unity"*, Amer. Math. Monthly 75 (1968), 512-514.

Curriculum Vitae

Md Al Masum Bhuiyan was born on January 19^{th}, 1988. The second son of Md Abdul Mannan Bhuiyan and Parul Akter, obtained a Master of Science (in Applied Mathematics) and Bachelor of Science (in Mathematics) from The University of Dhaka, one of the top ranked universities for science research in Bangladesh. He entered the University of Texas at El Paso in the fall of 2013. While pursuing his master's degree in Mathematical Sciences he worked as a Teaching Assistant, and as a instructor of ENSO program in the summer of 2014.

Using this research, he participated in the Dynamica Expo Conference 2014 and recieved the Runner-Up award in basic science category.

He is a member of Bangladesh Mathematician Club and American Mathematical Society.

Residential address: 1127, Los-Angeles Dr., Apt-03
El Paso, Texas 79902