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Challenges in Assessing College Students' Conception of Duality: The Case of Infinity

Grace Olutayo Babarinsa-Ochiedike
University of Texas at El Paso, ttazor@live.com

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GRACE OLUTAYO BABARINSA-OCHIEDIKE

Department of Teacher Education

APPROVED:

Mourat Tchoshanov, Ph.D., Chair

Olga Kosheleva, Ph.D.

Helmut Knaust, Ph.D.

Charles Ambler, Ph.D.
Dean of the Graduate School

Dedication

I dedicate this dissertation to the Ancient of Days; the author and the finisher of my faith. When my heart was overwhelmed, you led me beside the still waters and you restored my soul.

To my husband, Uzoma, with whom I am blessed to share my life. This journey begun 7 ½ years could not have been possible without your inspiration, understanding, and support. Thank you for bearing with me for eleven years as a wife, mother, instructor and a full time student.

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CHALLENGES IN ASSESSING COLLEGE STUDENTS' CONCEPTION OF DUALITY: THE CASE OF INFINITY

by

GRACE OLUTAYO BABARINSA-OCHIEDIKE, B.Sc., PG. Dip., M.S., M.Ed.

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Abstract

Interpreting students' views of infinity posits a challenge for researchers due to the dynamic nature of the conception. There is diversity and variation among students' process-object perceptions. The fluctuations between students' views however reveal an undeveloped duality conception. This study examined college students' conception of duality in understanding and representing infinity with the intent to design strategies that could guide researchers in categorizing students' views of infinity into different levels.

Data for the study were collected from N=238 college students enrolled in Calculus sequence courses (Pre-Calculus, Calculus I through Calculus III) at one of the southwestern universities in the U.S. using self-report questionnaires and semi-structured individual task-based interviews. Data was triangulated using multiple measures analyzed by three independent experts using self-designed coding sheets to assess students' externalization of the duality conception of infinity.

Results of this study reveal that college students' experiences in traditional Calculus sequence courses are not supportive of the development of duality conception. On the contrary, it strengthens the singularity perspective on fundamental ideas of mathematics such as infinity. The study also found that coding and assessing college students' conception of duality is a challenging and complex process due to the dynamic nature of the conception that is task-dependent and context-dependent.

Practical significance of the study is that it helps to recognize misconceptions and starts addressing them so students will have a more comprehensive view of fundamental mathematical ideas as they progress through the Calculus coursework sequence. The developed duality concept development framework called Action-Process-Object-Duality (APOD) adapted from the APOS theory could guide educators and researchers as they engage in assessing students' conception of duality. The results of this study could serve as a facilitating instrument to further analyze cognitive obstacles in college students' understanding of the infinity concept.

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Chapter 1: Introduction

The concept of infinity is one of the most important, and yet difficult links in the mathematics sequence for undergraduate science, technology, engineering and mathematics (STEM) students. Studies have confirmed that most students have extensive difficulty with the notions of infinity. The paradoxical nature of infinity makes it challenging for students to conceptualize. Research has indicated that students approach calculus courses with several misconceptions about infinity and that the first year of a calculus course is of no significant effect on students' conceptions of infinity (Monaghan, 1986; 2001). It is important that mathematics educators identify early any pre-conception or misconception the students may hold regarding infinity, to save them from building up rigid mental images in their cognition, that may later be difficult to give up, and to help the students develop a thorough perspective of mathematical principles and concepts.

1.1 Purpose of the Study

The main goal of this study was to investigate college students' duality conceptions of infinity through a series of structured activities. A number of studies on students' conceptions of infinity have been conducted at the elementary, secondary schools and college level, and all suggest the conflicting nature of the intuition of infinity in students. It was found that students' intuitive conceptions tend to reflect infinity as a process more than as an object (Falk, 2010; Fischbein, Tirosh, and Hess, 1979; Monaghan, 1986, 2001; Smith, Solomon, & Carey, 2005; Tirosh and Stavy, 1996). The challenge of studying preconceptions (initial ideas) of infinity is mostly based on the fact that our intuition of infinity is intrinsically contradictory. In some sense it is counterintuitive because our thinking is naturally adapted to finite objects and events (Fischbein, Tirosh, & Hess, 1979; Clegg, 2003; Maor, 1991). According to Selden (2002) "in order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate" (p. 10). A conceptual understanding of the process-object duality of infinity is essential for

having a well formed conception of infinity, which is critical for students to be successful in advanced mathematics courses. Infinity can be considered in different contexts: numerical, geometrical, descriptive, practical, theoretical etc. The context students use to represent infinity is dependent on the type of task. Monaghan (2001) regards 'context' as a problematic term in mathematics education, and 'tasks' "or activities that researchers give to students as presenting contexts for the discussion of infinite ideas" (p. 250). Hence, the type of tasks given to students is important. This study will focus on two different contexts related to infinity: theoretical and practical contexts, utilizing a scenario-based task and other types of tasks incorporated. Scholars have warned that care needs to be taken in interpreting students' representations of infinity (Monaghan, 2001, Fischbein et al., 1979) due to the danger of assumption that comes with determining students' process-object duality and the dynamic nature of the duality conception (Bingolbali & Monaghan, 2008; Falk, 2010). To address the complexity of the duality phenomenon, this study examines challenges in coding and assessing students' conception of duality as well as addresses diversity and variations among students' conception of duality (e.g., cases where the student's process view is dominant and the object view is recessive, cases where students' object view is dominant and process view is recessive, and the case where both process and object views are recessive (i.e. not strong or convincing)).

The purpose of this study is to examine college students' conception of duality in understanding infinity. More specifically, to determine whether or not the college students possess a dual process-object view of infinity.

1.2 Research Questions

The main research questions guiding this study are:

1. How is the duality conception externalized by college students at each course in the Calculus sequence?
2. To what extent does the type of a task impact the college students' external representation of infinity?

3. To what extent does the context of a task impact the college students' conception of duality?

1.3 Significance of the Study

Infinity is the conceptual foundation for mathematical topics such as the number line and infinite decimals. One of the five strands of mathematical proficiency outlined by the National Research Council, in Adding *It Up* (2001) is conceptual understanding. They define conceptual understanding as "comprehension of mathematical concepts, operations, and relations" (p. 5). The concept of infinity though complex remains crucial to mathematics studies. It is especially central to calculus because infinite processes form the basis for the concept of limit. Infinity also features in other important areas of mathematics (e.g. analysis and set theory) and also related to concepts such as sequences, functions, irrational numbers, probability and geometrical concepts (NCTM, 1989). The infinity concept in general has always been recognized as difficult and has historically been the origin of paradoxes and contradictions (Fischbein, 2001; Kleiner, 2001; Tall, 2001; Monaghan, 2001; Tsamir & Dreyfus, 2005) among "philosophers, mathematicians, mathematical historians, students, and mathematics education researchers" (Dubinsky, Weller, McDonald & Brown, 2005, pp. 336) and many others especially when it comes to the notion of actual infinity. Its major difficulty as explained by Kolar and Cadez (2012) is its abstract nature. Students struggle in trying to relate the concept of infinity to real-life situations as a way to understand it and therefore resolve to mental visualization. Dubinsky et al. (2005) state that "a first step in helping students overcome these difficulties is to understand their nature" (p. 264). Mathematics has a fascinating dual nature. "The ability to recognize the dual nature of mathematics concepts is crucial for the learning of mathematics" (Ng and Kwek, 2007, p. 2). Understanding of the process-object duality of infinity is therefore essential for having a well formed conception of infinity, which is critical for students to be successful in advanced mathematics courses like Calculus.

1.4 Theoretical Framework

This study is grounded in APOS theory (Dubinsky, Weller, McDonald, & Brown, 2005) to model the development of the duality conception. The term APOS is an acronym for Action, Process, Object, Schema, and these are the mental structures that an individual builds by the mental mechanism of interiorization and encapsulation. There are six categories of reflective abstractions in undergraduate mathematics education postulated by Dubinsky (1991) and Asiala et al. (1996) and they are interiorization, coordination, reversal, encapsulation, thematization, and generalization. Interiorization and encapsulation are the main mental mechanisms that describe the daily activities of everyone engaged in mathematical activities. However, Dubinsky et al. (2005a) suggest the possession of this mental mechanism by anyone does not necessarily mean anyone will use it when situation warrants it.

According to Dubinsky et al. (2005a), formation of mathematical concepts begins as one transforms an object to form another object. This transformation is referred to as action. Students are able to perform this explicitly based on specific instructions. When students repeatedly reflect on their action, they are able to interiorize their action into a mental process. A process is an action that has been interiorized. With regard to the perspective of Dubinsky et al. (2005a) when students repeatedly reflect on their action, they are able to interiorize their action into a mental process. Interiorizing infinity to a process relates to an understanding of potential infinity, whereby, infinity is imagined as performing an endless action, though without imagining the execution of each step. For example, when a student makes as many points as wanted on a line segment to represent infinite number. This study found out that there is possibility of students while interiorizing action into process that the emergent process may or may not be strong. Hence there is recessive process view at this stage which is referred to as the students' idiosyncratic process view (p) which when encapsulated, becomes the students' idiosyncratic object view ('o'). The moment students perceive the process as a totality and perform an action on the process, the process is then said to have been encapsulated "into a cognitive object" (Dubinsky et al., 2005a, p. 339). A process (e.g. counting natural numbers) can be transformed

into an object (e.g. set of natural numbers) by means of encapsulation. Encapsulating this endless process to a complete object relates to a conception of actual infinity (quantity that describes the cardinality or the size of a complete infinite set). For example, when a student assumes the infinite number of points on a line segment as a complete entity, such thinking is referred to as object conception. When a process has been transformed into an object, and students are able to see from dual perspective, the synthesized process and object becomes the person's infinity schema. This schema represents the "*process-object duality*" (Monaghan, 2001) the least studied of all constructs in APOS theory. This study constructs a duality concept development framework known as *Action-Process-Object-Duality* (APOD) (Figure 2.1) adapted from the APOS theory to diligently model the development of duality conception. This modified APOD Framework will be presented in detail in Chapter 2 and used throughout this study to interpret students' intuitions, and their attempts to conceive infinity as a process as well as an object.

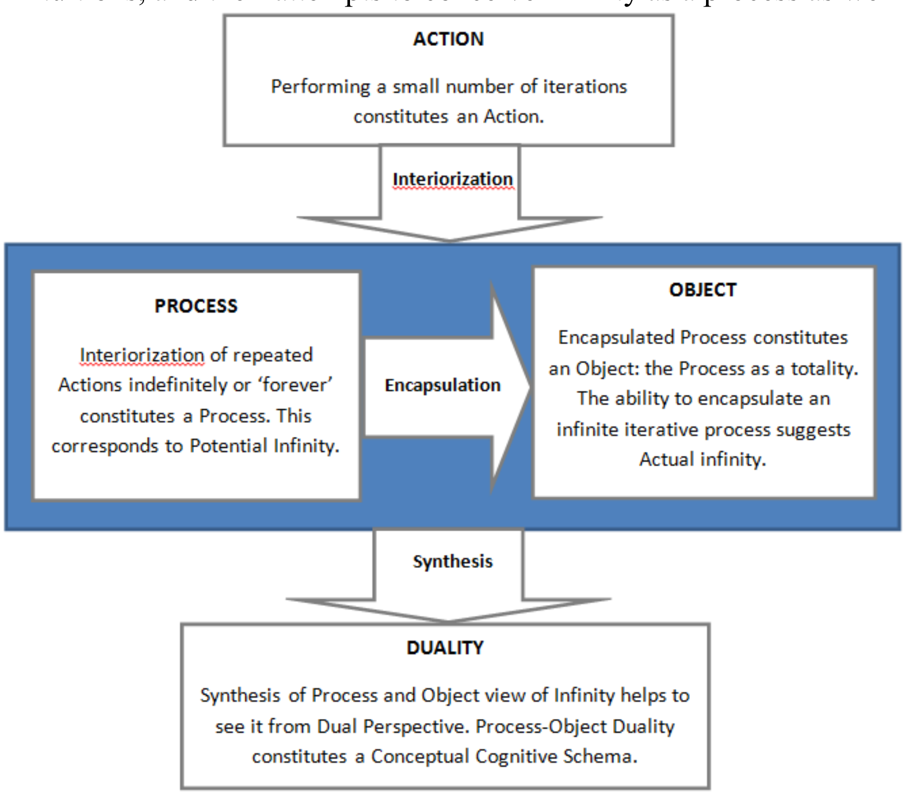


Figure 2.1: Overview - APOD Development Framework.

1.5 Definition of Terms

The study finds the following definitions as more appropriate to describe the terms below.

Conceptual Understanding – One of the strands of mathematical proficiency defined by the National Research Council (2001, p. 5) as "comprehension of mathematical concepts, operations, and relations."

Concept image – Concept image refers to "a dynamic entity" (Bingolbali & Monaghan, 2008, p. 20) built or accumulated over the years as one grows through relative experiences. Tall and Vinner (1981) define concept image as the entire cognitive structure, which includes all mental pictures (pictorial, symbolic, and others), all mental attributes (conscious or unconscious) and associated processes in the individual's mind associated with a given concept (p. 152). According to Vinner (1991), concept image is "something non-verbal associated in our mind with the concept name" (p. 68).

Concept definition – According to Tall & Vinner (1981), concept definition is "the form of words used to specify that concept" (p. 152).

Duality – This is the quality or character of being twofold; dichotomy; the state or quality of being two or in two parts.

Schema – Clarke et al. (1997) describes schema as "a coherent collection of actions, processes, objects and other schemas that is invoked to deal with a new mathematical problem situation" (p. 346).

Infinity – is symbolized by ∞, and it is the quality of being infinite; an indefinitely great number or amount; "a concept of a value that is greater than any finite value" (McCombs, 2014, p. 13).

Infinite series – aka series is the sum of the terms of an infinite sequence. That is, limit of partial sum: $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$

Infinite set – is one which can be put in one-one correspondence with a proper subset.

Thus the natural numbers {1,2,3,..., n, ...} form an infinite set because they can be put into one-one correspondence with the even numbers {2,4,6,...,2n, ...} in which n corresponds to 2n. (Tall, 1992).

Processes – Fischbein (2001) describes processes as a dynamic form of infinity "which are, at every moment, finite, but continue endlessly" (p. 310).

Potential infinity – is the conception of the infinite as a process; the infinite presented over time (Dubinsky et al., 2005). Potential infinity is related to an ongoing process without an end (Kolar & Cadez, 2012). According to Fischbein (2001), *potential infinity* is a process that goes on forever.

Actual infinity – is the mental object obtained through encapsulation of that process (Dubinsky et al., 2005, p. 346).

Basic Metaphor of Infinity (BMI) – is a general cognitive mechanism that can occur by itself, as when one speaks of infinite as a "thing". (Lakoff & Nunez, 2000).

Process-view – conception of infinity as an endless, continuous operation.

Object-view – conception of infinity as a completed totality (Dubinsky et al., 2005).

Context-dependency – This is a term used to describe the form of representation of tasks and/or students' responses to the given problems or tasks (e.g. practical and theoretical context).

Task-dependency – This is the term used to describe the kind of task given by the researcher (e.g. scenario-based task, concept-definition task, concept-image task).

1.6 Chapter Overview

This study is organized into five chapters. Chapter 1 provides an introduction to the study through a discussion of the following topics: purpose of the study, research questions, significance of the study, theoretical framework that guided the design of the study and definition of terms. Chapter 2 provides in detail the APOS framework and the Cultural-Historical Epistemology of Duality in Calculus. It also features a brief discussion of other frameworks and an in-depth review of the literature related to the main areas addressed in this study which

includes: infinity concept, duality concept, and difficulties in understanding duality concept. Chapter 3 gives a detailed description of the methodology used in conducting this study, including the research design, selection of participants, instrument for data collection, data analysis, reliability and validity, and the ethical issues. Chapter 4 comprises the results from data collection with thorough description of the analysis of students' response on infinity survey and analysis of students' interview phase of the study. And finally, Chapter 5 provides a discussion of the results, summary, implications and limitations of the study, recommendations for future research, and conclusions.

Chapter 2: Review of Literature

The purpose of this study is to examine college students' conception of duality and determine whether or not they possess a dual process-object view of infinity. More importantly, we ask: (1) how is the duality conception externalized by college students, and (2) how can students' conception of duality be assessed? This chapter discusses the Cultural-Historical Epistemology of Duality in Calculus and gives an in-depth review of the literature related to the main areas addressed in this study which includes: infinity concept, duality concept, difficulties in understanding duality concept. With the purpose of gaining more insight into the various aspects of this study, this chapter provides in detail the theories that were used to interpret students' intuitions and their attempts to conceive infinity as a process as well as an object.

2.1 Cultural-Historical Epistemology of Duality in Calculus

The concept of mathematical infinity and the principle of duality can be traced to the Greek mathematicians. "Greek philosophers and mathematicians of the Golden Age, from Pythagoras to Zeno to Eudoxus and Archimedes, discovered much about the concept of infinity" (Aczel, 2000, p. 24). This infinity concept supports the development of calculus. Archimedes in the 3rd century used the method of exhaustion to determine the approximation to the area of a circle. This method basically depends on finite quantities of diminishing size. The approximation is done by adding a series of small parts of the figure – negligibly performing an infinite process. Archimedes' method of inscribing and circumscribing a circle by an infinite series of regular polygons with different number of sides was the foundation of integration which led to the approximation of the values of π. According to Maor (1991), "π is the limit of the values derived from these polygons as the number of sides tends to infinity" (p. 5).

In the fourth century, Zeno developed four paradoxes that posit puzzles for students to this day (Stadium, Achilles, Arrow and Dichotomy). Zeno's argument about motion and continuity is that under the assumption of infinite divisibility of space and time, motion is impossible (Aczel, 2000; Maor, 1991). He brought to bear several contradictions between the

discrete and the infinite. In Achilles' Paradox he described a race between a slow tortoise and Achilles, a fast runner of antiquity. He claimed that giving the tortoise a head start, Achilles would never even catch up with the tortoise. His rationale is that when Achilles arrives at the starting point of the tortoise, the tortoise will have already moved forward by several steps. When Achilles covers these several steps, the tortoise will have taken another several steps forward, and so on. This will result in an infinite number of steps indicating the process view of infinity. Zeno knows fully well that after a finite amount of time Achilles will arrive at the end point of the race. This is basically applying finite perceptive to infinite processes (process = discrete; object = continuity) which is rather absurd according to our present day logical perspective. This paradox lasted a space of twenty centuries to be resolved. This knowledge is what we conceive as the object view of infinity. Mathematically, this would be the summation $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...\frac{1}{2^n}+...=1$. This idea leads to contradiction of the concepts of "infinity" and "infinite partitions". The path which Achilles needs to take to catch up the tortoise can of course be partitioned infinitely many times; nevertheless, the length of that path is still finite, and Achilles can still cover that length within a finite period of time. This paradox is essentially the same with Zeno's Dichotomy paradox which claimed that for a person to get to a fixed point, he must first cover the midway mark, and then the midway mark of what remains, etc. These paradoxes reveal an important concept that infinitely many steps can sometimes lead to a finite end, which Aczel (2000) describes as *convergence*. However, these arguments and debates started human discussions about the concepts of limits and infinity, which paved the way for the development of calculus in the later centuries.

By the sixth century, the Greeks having been haunted by Zeno's paradoxes were the first to acknowledge the importance of infinity in mathematics. Aristotle introduced the two notions of potential and actual infinity, which marks the beginning of the dichotomy or the duality of infinity. In an attempt to define what is infinite in time and space, he defines infinite as an unending process. He gives examples of potential infinity to be integers or natural numbers

because of the process of counting them would require the whole of time, and thus the counting cannot be completed. He therefore claimed potential infinity "as being present over time" and actual infinity as being "present at a moment in time" (Dubinsky et al., 2005, p. 341). Aristotle could not comprehend the actuality of the notion of infinite quantity because he believed we as human were limited by time and so would be unable to think of an infinite process in its totality (Dubinsky et al., 2005). He therefore acknowledged the notion of potential infinity only and forbade that of actual infinity, example of which is the infinity of the number of points in a segment (Fischbein, 2001).

By the sixteenth century, the advance of dynamics and astronomy necessitated the discovery of methods for calculating areas, volumes and lengths of geometric (curved) figures. Galileo, Kepler and Cavalieri by the seventeenth century improved on Archimedes' method by using the method of indivisibility to discover several properties for different geometric figures. Cavalieri especially, used the method of indivisibility to find the area under a parabola and came close to discovery of the integral calculus (Maor, 1991). By this method, Kepler also "divided the area of the ellipse into very many "infinitesimal" triangles, then computed their areas and was able to see what limit of the total sum of areas would be as the number of triangles increase towards infinity" (Azcel, 2000, p. 53). He was known to have made a clever use of potential infinity by this statement. In 1612, he also used this infinitesimal method to find the volume of a wine bucket. In an attempt to count all the square numbers, Galileo established a one-to-one correspondence between all the integers and all the squares of integers and discovered what happens to be "the key property of infinite sets: An infinite set can be "equal" in number of elements to a smaller subset of itself – a set included as a smaller part of the original set" (p. 55). According to Aczel (2000), Galileo who only talked about the "discrete form of infinity" was the first to have made a big leap from the potential infinity of performing repeating sequence of actions to the actual infinity of completing the process, when he "explains the division of a circle into "infinitely many" infinitely small triangles. He argues that by bending a line segment into the shape of a circle one has "reduced to actuality that infinite number of parts into which, while

it was straight, were contained in it only potentiality." Thus the circle, he continues, is a polygon with an infinite number of sides." (p. 52).

With little improvement in the concepts of infinity, limits and integration in the seventeenth century was the "introduction of analytic geometry" which "together with the free use of the suggestive infinitesimal" led to the foundation of calculus by Newton and Leibniz (Boyer, 1959, p. 4). Newton and Leibniz discovered differential and integral calculus, which "revolved around the infinitely small, the infinitesimal" (Maor, 1991, p. 13). Boyer (1959) explains that the development of calculus was as a result of challenges the Greek mathematicians experienced in their effort to express their intuitive ideas about ratios or line proportionality which they imprecisely assumed to be continuous in terms of numbers and discrete. Newton's idea of calculus was based on geometry, while Leibniz was based on analysis. Later, Weierstrass not trusting intuition, decided to separate Calculus completely from geometry by establishing Calculus on the concept of numbers alone. Weierstrass resolves the question of the existence of a limit convergent by making the sequence itself the number or limit. This he accomplished by considering an unordered aggregate (Boyer, 1959). He was the first to formulate the static definition of a limit having greater clarity and precision that is used until today. Eliminating the infinitesimals and only using the real numbers, less than, and the operations of addition and subtraction (Hollingdale, 1989).

While all these mathematicians mentioned were comfortable with the idea of a potential, unreachable infinity, where a quantity approached infinity, or where quantities approached zero, only Galileo attempted to discover an important property of actual infinity (Aczel, 2000). Cantor who is known as the father of set theory, by the nineteenth century finally solved the riddle of actual infinity. This he succeeded in doing while investigating the nature of *sets* (Aczel, 2000). As Galileo was able to show that there are as many squares of integers as the integers, Cantor was able to show as well that the rational numbers are countable, that is, there are as many rational numbers as there are integers. He used the concept of one-to-one correspondence to determine the equivalence of sets.

2.2 Infinity Concept

Aristotle differentiated between the two types of infinity – potential infinity and actual infinity. Potential infinity is "an ongoing activity that never ends" (Dubinsky et al., 2005a, p. 340). Lakoff and Núnez (2000) defines it as "ongoing processes or motions without end" (p. 158). Kolar and Cadez (2012) explain it as an infinite process so that we cannot determine its end or the last term of its sequence. E. g. the natural numbers 1,2,3,4… It is a thought process or practice of ever acquiring new numbers which cannot be actualized or realized. Potential infinity explains the process that creates infinite sets. Moore illustrated it as "that whose infinitude spread over time (existing 'in time')" (Kolar & Cadez, 2012, p. 390). According to Kattou, Thanasia, Kontoyianni, Christou, & George, (2009) potential infinity is "an everlasting activity that continues beyond time". They define actual infinity as "the not finite that is presented in a moment of time" (p. 1771). Aristotle defines actual infinity as "that whose infinitude exists at some point in time (existing 'all at once')" (Kolar & Cadez, 2012, p. 390). It is "a definite entity encompassing what was potential" (Dubinsky et al., 2005a, p. 340). According to Kolar and Cadez "actual infinity defines the state in infinity" (p. 390). Lakoff and Núnez (2000) by their *Basic Metaphor of Infinity* (BMI) define actual infinity as the ability to conceptualize metaphorically the "processes that go on indefinitely as having an end and an ultimate result" (p. 158). It is obtained as one speaks of "the infinite" as a thing.

Tall (2001) considered the potentially infinite collection, the counting sequence 1, 2, 3 … to describe natural numbers. This is obtained as one begins at 1 and continues to successively add one to the preceding number at every step. He described the ability to "think of the whole system in total, including all the numbers, all at one time" as actual infinity (p. 201), regarded as the totality of infinity (Tirosh, 1991). Similarly, Kattou et al., (2009) considered the potentially infinite collection, a line of infinite sets ($\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots$) to describe the set of natural numbers (Monaghan, 2001). Actual infinity is a completed infinite totality (Lakoff & Nunez, 2000). It is the ability to encapsulate the process into object of $N = \{1, 2, 3, \ldots\}$. This is obtained as one consider the set of all natural numbers without enumerating all the elements of the set or

"think of a collection in its entirety without reflecting on each of its elements, that is, without physically or mentally performing each step" (Kolar & Cadez, 2013, p. 390).

According to Lakoff and Núnez (2000) it is meaningless to think of infinity as a number because a number n equal to ∞ in the equation n = ∞ "means nothing". They argue that the "symbol ∞ means nothing at all except in the phrase "tends to infinity" and "approaches infinity" (p. 164) and that since there are three different cognitive uses of numbers, ∞ as a number is used in enumeration and comparison and not in calculation. For example, ∞ is assumed to be an endpoint in an enumeration 1, 2, 3, …, ∞, meaning "larger than any finite number and beyond all of them". Also, mathematicians use infinity as a number in enumeration, as in the sum of a sequence a_n from n = 1 to n = ∞:

$$\sum_{n=1}^{\infty} a_n$$

Lakoff and Núnez stated that they have never seen a case where people use ∞ as a number in calculations such as a senseless expression "17 times ∞, minus 473", but only in BMI, a special case to "indicate order of enumeration" to the integers (p. 165).

Many students consider the limit of a sequence to be the last term of the sequence (Mamona-Downs, 2001). Using Lakoff's 'basic metaphor of infinity', Ueno (2004) explained how people make a wrong use of infinity symbol to interpret an infinite sum as the limit of partial sums. They take infinite sum as a "result of adding an infinite number of terms', a sum 'up to the ∞^{th} term'" (p. 56) the ∞^{th} term being the limit. This is impossible as the ∞^{th} term' does not exist. That is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_\infty$$

Some researchers from part of a broader investigation identified some of the difficulties students experience in understanding the concept of infinity (Fischbein, 2001; Jirotková & Littler, 2004; Kolar & Cadez, 2010; Monaghan, 2001; Pehkonen & Hannula, 2006; Tall & Tirosh, 2001; Tsamir, 2001; Singer & Voica, 2007). Tall (1992) reported the outcome of

research experience (Wheeler and Martin, 1987, 1988) conducted at a large university with elementary pre-service teachers in enrolled in an upper-division course in mathematics methods. These teachers were asked to explain what the symbol ∞ and the final three dots in the expression "1, 5, 52, 125, 625, …" mean. The results showed that half of the pre-service teachers were not familiar with the symbolism and their responses to the meaning of the three dots predominantly evoked potential infinity. Their responses include: "unending process", "the numbers go on without stopping", or "no matter what number you say, there is always one greater simply by adding one to it".

Fischbein (2001) analyzed the effect of tacit models in reasoning with infinity. After considering the model of space properties for the interpretation and the measure of time (especially with reference to Zeno's paradoxes) he resolves that infinity is a source of difficulties and contradictions. He also considered the interpretation of infinity as corresponding with the inexhaustible and found that the tacit impact figural models on the logic of abstract geometric concepts when dealing with infinity leads to wrong or contradictory interpretations. Fischbein also attributes the difficulty in understanding the concept of infinity to the students' intuitive interpretations of infinity from the perspective of potential infinity. Students exhibit confusion on the notions of actual and potential infinity. An example of this intuitive cognition is the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$. Students consent to the addition of terms at every step of iteration and could not accept the series to be equal to 1. This is an indication of potential infinity filling their minds (Fischbein, 2001). By the concept of actual infinity the process can be encapsulated as an object to be number 1, meaning that students are able to think of the series as a totality or as a complete whole (Dubinsky, et al. 2005b; Kattou, et al., 2009; Monaghan, 2001).

Kattou, et al. (2009) examined elementary school teachers' perceptions of the notions of infinity as a process or an object. The researchers administered self-report questionnaire to 43 elementary school teachers in Cyprus. The tasks given to the study participants included finding the cardinality of two given sets, conceptualizing the inequalities 0.9999… = 1 and 0.3333… = 1/3, defining infinity and providing examples, and comparing infinite sets. They found that in the

first three tasks mentioned, a majority of the students comprehend infinity as an unlimited process. This agrees with the findings of other researchers (Monaghan, 2001). In the 'define infinity' tasks, the preservice teachers used phrases such as "it goes on forever", "it has no beginning, it has no end… always follows another number", the predominant notion of infinity evoked is *potential infinity*. 27.9% used object language such as "it is something countless", "it is an undefined set", or "it is a set with unlimited elements", indicating *actual infinity*. In the task of finding the comparison of infinite sets, using different representation, the majority (76.7%) of the teachers realized the cardinality of the given sets. A high percentage of geometric representation elicited a high percentage of one-to-one correspondence. It was found that geometric representations influenced this awareness. Tsamir (2001) in attempt to use research findings to develop an activity and assess its impact on students' comparison of infinite sets, tried the Infinite Sets Activity (ISA) with the secondary school students. When comparing infinite sets, the 'It's the Same Task' (IST) research-based activity used which consists of different representations (e.g. horizontal representation, vertical representation, numeric-explicit representation and geometric representation) applied the cognitive conflict approach as an instructional tool on the students' reported intuitive tendencies. Findings from these studies suggest that students were able to realize the inconsistencies in their own thinking, and that the use of two methods to compare infinite quantities leads to contradictory responses to the same mathematical task. IST activity was however found to be effective in triggering the use of one-to-one correspondence. These inconsistencies in students' responses to different representations of infinite sets is also observed in Tsamir and Tirosh (1999).

In the conceptualizing inequalities task, they reported that the preservice teachers were able to accept the validity of 0.333…… = 1/3 more easily than the equality 0.999…… =1 by use of the concept of limit. The intuitive conceptualization of the equalities 0.999…… = 1 and 0.333…… = 1/3 are one of the typical difficulties students encounter in constructing their understanding of real numbers. Studies show that students tend to disagree with the equality, thinking that the two numbers have an insignificant difference from one another. Students also

conceived the infinite sequence of the 9's in 0.999…… as a process, indicating potential infinity and by the concept of actual infinity, this process can be encapsulated as an object to be number 1. (Cornu, 1991; Monaghan, 2001; Fischben, 2001; Dubinsky et al., 2005) also students did not consider the limit as the value of infinity (Cornu, 1991) but rather as a boundary. Students intuitively consent 0.333.… to be the result of dividing 1 by 3, which only tends to 1/3 but never reaches it, something taken to be impracticable in the case of 0.999…… =1 (Edwards, 1997).

Jirotková and Littler (2004) investigated 44 Czech and 54 English students understanding of infinity in geometric context using a series of seven tasks to explore the mental processes students used when they are thinking about infinity. These students were aged 11-15 years. After describing each child's hypothetical statements in the task, their results claimed that 75 % of Czech and 59 % of English students considered the idea of two infinities choosing Adam, and 38% of students do not observe any contradiction between actual and potential infinity, indicating that the students' understanding of infinity is not clear. Actual infinity contradicts many of the students' intuitive ideas of infinity. They found out that most pupils are more comfortable discoursing about infinity in the context of numbers, which is also evident in Monaghan's (2001) research. The results also showed that from 12 years onward the students tested did not show stability of intuition of infinity when contexts were changed. This is contradictory to Fischbein, Tirosh & Hess's (1979) research. They suggest the use of different contexts to better help students in their understanding of infinity concept.

Singer & Voica (2007) using a variety of questions explore children's primary and secondary intuitions about infinity in a school context and other contexts, and found that young children have a structured representation about infinite sets. They suggest that building new knowledge from intuition in math class might be a way to reduce misunderstandings and misconceptions in fundamental areas of mathematics learning such as infinity. Pehkonen & Hannula (2006) in their study on the development of students' understanding of infinity examined surveys implemented with school students in grades 5, 7, 11 and elementary teacher students. Two tasks of the questionnaire aimed to examine the students' understanding of infinity

of natural numbers and their understanding of density of rational numbers. This study found out that most of the students did not have a proper view of infinity; not even those at the teacher education program. Only 20% of the students in the fifth grade had some understanding of the infinity of natural numbers, while just a few had any understanding of density of rational numbers. As the students get older, the situation improves. Students understand infinity of natural numbers earlier than density of rational numbers. Potential infinity is understood earlier than actual infinity. It was also observed that as students get older, the potential infinity becomes less frequent, and it seems as if the 11th grade students indicated an intermediate stage to the understanding of actual infinity in the context of density of rational numbers.

Primary teacher students' understanding of the different types of infinity was examined by Kolar & Cadez (2010) in their researching of the understanding of the concept of infinity. They found that respondents' understanding of infinity depends on the type of task given and the context of the task. Monaghan (2001) however warns about the methods mathematicians use in assessing young people's ideas of infinity. He argues that contexts and tasks given to students by researchers can pose a problem. Since students' views differ from the researchers, the context may not necessarily make sense to the students; therefore, care needs to be taken while interpreting the students' perceptions.

As can be observed from previously reviewed studies (Kattou et al., 2009; Lakoff and Núnez, 2000; Tall, 1992), the language the research participants use to express their views also posit another source of difficulties in coming to understand students' infinity concept. Language such as: "it goes on forever", "it has no beginning, it has no end… always follows another number", "the numbers go on without stopping", "keeps going and going", and "it's finite". Monaghan (2001) suggests that it is common practice among children when discussing about infinity for their language to repeatedly reflect infinity as a process. Infinity is perceived as the act of going on and on (potential infinity) and not as a realized thing (actual infinity). She said in addition that caution must be observed when interpreting students' responses. That a student uses the statement "It's going towards infinity" does not necessarily mean the object conception is

assumed. There exists a thin line between process and object views that may not be well defined in the minds of the students. She stated emphatically "Although children may say "towards infinity", this does not rule out 'infinity as a process' coloring their thoughts." (p. 245).

2.3 Duality Concept

Duality conception is the ability to conceive abstract notions as a process as well as an object. Many studies have acknowledged process-object duality as a model of mathematics concept development (Dubinsky, 1991; Dubinsky, Weller, McDonald, & Brown, 2005b; Gray and Tall, 1994; Selden, 2002; Sfard, 1991; Tall, 1991) and have also taken the interpretation of abstraction level as reflection of this process–object duality (Hazzan and Zazkis, 2003). The notion of duality is one of the central concepts in geometry and analysis, and in both cases defined using very concrete structures (Artstein-Avidan and Milman, 2009). Gray and Tall (1994) call the symbolism that intrinsically represents the amalgam of process and concept ambiguity a '*procept*'. The procept theory by Tall (1991) define concept as an object by reason of encapsulation and think of mathematical entities in terms of process and objects (i.e., process-object). Several concepts in mathematics, such as function, can be viewed both as processes (having computational or procedural aspects) and as objects (entities that can be acted upon, possibly by other processes) (Selden, 2002). She posits that "in order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate" (Selden, 2002, p.10). A single mathematical notation can be used to describe both process and object conceptions. For instance, 2+3x indicates the process of adding 2 to the product of 3 and x, as well as the product of that process, the expression "2+3x", which is the object.

Sfard (1991), having investigated the role of algorithms in mathematical thinking and analyzed different mathematical definitions and representations through ontological and psychological perspective, claims that the ability to conceive a function or number both as a process and as an object (i.e., process-object) is crucial for deep understanding of mathematics.

She states that this dual nature of mathematical construct can be observed verbally as well as through diverse symbolic representations. Sfard proposed "*operational-structural duality*" and describes duality as "inseparable, though dramatically different, facets of the same thing" (p. 9). She conjectures two ways of developing a mathematical concept – structurally as an object, and operationally as a process. The structural conception she described to be an actual infinity: static, timeless, instantaneous and integrative. It also means the idea can be recognized "at a glance" and it can be manipulated "as a whole without going into details" (p. 4). An example of this is the infinity of the number of points in a segment. The operational conception on the other hand she describes as a potential infinity: dynamic, sequential, and detailed. It infers the process of performing algorithms and actions. This duality emphasizes the wholeness or unity of mathematical knowledge which distinguishes it from other dichotomies such as conceptual/procedural and instrumental/relational knowledge, which decomposes "mathematical knowledge into two separate components (e.g., concepts vs. procedures)" (p. 8). According to Sfard (1991), theories based on process-object duality though differentiates between a process conception and an object conception of mathematical notions, affirm that when learning a mathematical concept, the process conception precedes the object conception and that the process conception is less abstract being on a lower reduced level of abstraction than the object conception.

Hazzan & Zazkis (2005) assert that the means by which students reduce abstraction is neither exhaustive nor mutually exclusive. For example, students were given this problem in an attempt to test their abilities to perform the conversion of square units: A length of 3 cm on a scale model corresponds to a length of 10 m in a park. A lake in the park has an area of 3600 m^2. What is the area of the lake in the model? One of the students in her solution assigned the dimensions 90 × 40 to the lake, converted each length separately and then calculated the area of lake in the model, while some of her classmates measured the lake to be a 36 × 100 rectangle or a 60 × 60 square. Majority of the students obtained correct answers by randomly assigning units and restricting the lake shape to either a square or a rectangle, however, it was difficult for

anyone to explain why the final answer to the area was not influenced by and specific choice of shape and measurements. The description of the area as a specific multiplication of two sizes can be interpreted as students' conception of area as a process, rather than as an object that assigns a measure to a shape (Hazzan & Zazkis, 2005).

2.4 Difficulties in Understanding Duality Concept

The dual nature of mathematical constructs can be observed through various kinds of students' representations (Sfard, 1991; Gray & Tall, 1994). To develop conceptual understanding of the concept of infinity, it is important for teachers to connect potential and actual infinity with concrete real life examples (Singer & Voica, 2003). The majority of recent studies have applied the APOS theory to interpret individual students' perception about infinite processes (e.g., Brown, McDonald, & Weller, 2010; Dubinsky, Weller, Stenger, & Vidakovic, 2008, Mamolo & Zazkis, 2008, Weller, Arnon, & Dubinsky, 2009, 2011).

Studies have found that coding and assessing college students' conception of duality is a challenging and complex process due to the dynamic nature of the conception that is (1) task-dependent and (2) context-dependent (Kolar & Cadez, 2012, Monaghan, 2001).

2.4.1 Context-Dependency and Task-Dependency

Monaghan (2001) regards 'context' as a problematic term in mathematics education, and 'tasks' "or activities that researchers give to students as presenting contexts for the discussion of infinite ideas" (p. 250). It is very challenging assessing students' perception of infinity due to its dynamic nature that is context-dependent and tasks-dependent. Inconsistency exists between tasks and the context in which infinity is presented. Kolar and Cadez (2012) claim that "the type of infinity task can influence the rate of success in the recognition of infinite sets" (p. 402). The students' view and the kind of response students give to a task are determined by the context in which a task is presented (Jirotkova & Littler, 2004, Monaghan, 2001). Different tasks correspond to different types of infinity (e.g. infinitely large, infinitely close, infinitely many), and infinitely large and infinitely many tasks are much easier to explain, whereas, respondents'

intuitions have a way of getting in their recognizing the type of infinite set, thus posing an obstacle.

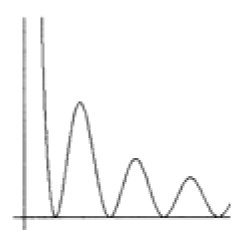
Singer (2000) suggests that introducing students to the concept of infinity requires the use of many examples from different contexts. Infinity could be considered in different contexts: numerical, geometric, descriptive, symbolic, etc. Monaghan elaborates on the following contexts for duality: numeric vs. geometric, counting vs. measuring, static and dynamic. She says, in considerations of this apparent duality, the types of tasks given to students are important.

Numeric vs. Geometric:

Monaghan describes a *numeric context* as “a situation that evokes general arithmetic principles” and a *geometric context* as “a situation that evokes spatial consideration” (p. 250). According to her, there exist an indistinct borderline between these two contexts. Relating this to one of Cornu (1997)’s epistemological obstacles in the history of limit (The failure to link geometry with numbers). The table below is a typical example from Monaghan’s (1986) study found in Monaghan (2001, p. 251). Responses suggested that students’ perception of the existence of a limit of a convergent function, presented graphically is stronger than their perception of the existence of a limit of a convergent numeric sequence.

Table 2.1: Example from Monaghan (1986).

Does the sequence 1, 0, 0.1, 0, 0.01, ... have a limit?		Does this curve have 0 as a limit?	
Yes	79	Yes	133
unsure	20	unsure	11
No	91	No	46



Counting vs. Measuring:

Monaghan describes *counting context* as a situation evoked by discrete sets, where problems are solved by counting or one-to-one correspondence. *Measuring context* is a situation evoked by continuous measures, where “problems are solved by comparing continuous quantities

or sets in one, two or three dimensions” (1986, p. 312). She related this to the cardinality problem of Tall (1980), who in an attempt to develop a mathematically rigorous concept of infinity from a measuring paradigm, as opposed to a Cantorian counting paradigm, asserts that a line twice the length of another may be seen as having twice as many points. He later realizes that interpreting students’ responses to problem of cardinality can lead to a misrepresentation of the students’ conception. Consider the examples from Monaghan’s (1986) study found in Monaghan (2001, p. 253). Percentage response in [ ]

[Measuring context]

Consider all the decimal numbers between 0 and 1 and between 0 and 10. Are there:

i.	more between 0 and 1?	[1]
ii.	more between 0 and 10?	[79]
iii.	same number in both?	[48]
iv.	can't compare?	[60]

[Counting context]

Consider the two sequences of numbers 1, 2, 3, 4 … and 2, 4, 6, 8… Are there:

i.	more in the first row?	[31]
ii.	more in the second row?	[4]
iii.	same in both?	[86]
iv.	can't compare?	[66]

There is greater number of ‘more’ responses in the measuring context as compared to the counting, which resonates with Tall (1980)’s assertion. Also, the measuring context suggests that whatever holds for the finite case also holds for the infinite case as an evaluatory scheme’. This again explains that measuring context can lead to an ordering of infinities. Hence, Monaghan warms that ‘care must be taken not to over-interpret children's thoughts” (Monaghan, 2001, p. 252).

Static vs. Dynamic:

The context of a response is termed *dynamic context* if it evokes indefinite process (motion) in some sense, otherwise, it is termed *static context,* if it evokes no sense of 'becoming', that is, no motion involved. Monaghan asserts that the difference between these contexts is dependent on how the individual interprets the question and not the question itself. To explain this, she considered the question: What is $\frac{1}{1-0.\overline{9}}$? According to her, possible responses or interpretations by students from a *static context* would be: "*1-0.$\overline{9}$ is infinitely small, so the reciprocal is infinitely large"* or *" $\frac{1}{10}$ is undefined".* And a possible response from a *dynamic context* would be: "$\frac{1}{0.1} = 10$, $\frac{1}{0.01} = 100$, *...., so the answer becomes infinitely large".* Thus $\frac{1}{1-0.\overline{9}}$ is seen as "*defined"* (1986, p. 316). "Behind dynamic interpretations of infinite phenomena is the idea of infinity as a process. Dynamic contexts are, however, less general than infinity as a process as an evaluatory scheme" (2001, p. 253). To illustrate this she considers the cardinality question discussed above about 1, 2, 3, 4 ... and 2, 4, 6, 8 ... Her results suggest that the dominant response was 'same in both'. However, the result from interviews revealed to Monaghan that it was due to 'infinity as a process' even though the interpretation of question was not stated in a particularly dynamic way.

Findings from literature reviewed indicate that students often use their intuition when responding to problems related to infinity. They use their experience of comparing finite sets to compare infinite sets, and when they use more than one context for the same problem, they run into a contradiction (Tall, 1991; Vinner, 1991). This contradiction is what Brousseau (1997) described as an epistemological obstacle: A kind of knowledge that is just limited to one specific context and not generalizable. Jirotkova & Littler (2004) from their experience express that students are more comfortable at responding to infinity problems in the contexts of numbers. Physical representation of geometrical problems on infinity posits a problem. However, students rely on physical reality when thinking about geometrical problems. When students attempt to relate geometrical problems to concrete or physical situations, there is a transformation: "the abstract mathematical description which represents an infinite set becomes finite" (Kolar & Cadez, 2012, p. 402).

2.5 Theoretical Perspectives

The APOD duality concept development framework adapted from APOS theory (Dubinsky, et al., 2005a, 2005b) appeared to be the most appropriate for this study to model the development of the duality conception and to interpret individuals' perception about duality because central to it are the process and object conceptions. This framework is based on constructivist philosophy of learning and rooted in Piaget's reflective abstraction (Dubinsky, 1991). Reflective abstraction is "the construction of mental objects and of mental actions on these objects" and is developed around the notion of *schema*. "A schema is a more or less coherent collection of objects and processes" (Dubinsky, 1991, p. 102). The main tenet for Piaget's constructivist theory of knowing is that knowledge develops as individuals reflect on their actions on objects (Piaget, 1970), and as they construct more organized structure in an attempt to make sense of the things they experience (Sfard, 1994).

APOS Theory is mostly used in the analysis of mathematical ideas at the college level, especially calculus concepts (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas & Vidakovic, 1996; Dubinsky et al., 2005a, 2005b; Martinez-Planell, Gonzalez, DiCristina & Acevedo, 2012; McDonald, Mathews & Strobel, 2000). Some studies have shown that it is also a valuable tool in studying students' understanding of more basic mathematical concepts (Dubinsky & MacDonald, 2001). Specific components in my research questions, data collection and data analysis are also informed by other theories and frameworks such as *concept image* and *concept definition* theory (Tall & Vinner, 1981), Reducing Abstraction (Hazzan, 1999), Reification theory (Sfard, 1991), and Procept theory (Gray & Tall, 1994). These learning theories are developed to focus research on learning mathematics by undergraduate students, using the lens of constructivism to interpret human cognition. The underlying assumption of these theories is that learning necessitates construction of knowledge. Other theories that were helpful in interpreting the collected data for the study will be discussed next.

Hazzan (1999) considers the capability to abstract as an important skill for implementing meaningful mathematics. He described reducing abstraction as when students use familiar procedures to make sense of unfamiliar problems and making abstract concepts more concrete. I has also been considered as a vital tool to analyze the ways in which students conceive abstract mathematical concepts. While reducing the level of abstraction, students tend to ignore the meaning of the stated circumstances in the problem and cling to the familiar mathematics entities they are able to produce by reason of experience. Hazzan suggests also that while reducing abstraction "students apply mental strategy that makes the unfamiliar mathematical language more familiar for them" (p. 80). "Abstractness of mathematical concepts can be reduced by connecting them to real life situations" (Wijeratne, 2013, p. 684).

2.5.1 Concept Image and Concept Definition

Another conceptual framework this study will use to assess college students preconception of infinity are *concept image* and *concept definition*. The terms concept definition and concept image distinguish between a formal mathematical definition and a person's ideas about a particular mathematical concept, such as function (Tall & Vinner, 1981). Tall and Vinner define concept image as the entire cognitive structure, which includes all mental pictures (graphical, pictorial, symbolic, verbal and others) all mental attributes (conscious or unconscious) and associated processes in the individual's mind that is associated with a given concept (p. 152). It is assumed that concept image is not formed once, hence not static. Bingolbali & Monaghan, (2008) refer to it as "a dynamic entity" (p. 20) built or accumulated over the years as one grows and through relative experiences.

Concept definition, according to Tall & Vinner is "the form of words used to specify that concept", that is "a form of words that the student uses for his own explanation of his (evoked) concept image" (p. 152). They said further that concept definition can be learnt meaningfully or memorized, and it may or may not be coherent with the formal definition accepted by the mathematical community at large. Also the ability to memorize a concept definition does not

necessarily mean you understand the concept. Understanding is guaranteed through the concept image formed about a thing.

Vinner (1983) posits that some concepts have both concept definition and concept image while many others do not. Take for example a ‘house’ or an ‘orange’. No definition is presented to teach such concepts, they are learned by means of ‘ostensive definition’, yet we have perfect concept images for them. The other case is the word ‘forest’ learnt by means of definition, say: ‘many trees together are a forest’. As we visualize the phrase ‘many trees together’ we form a concept image. So he claimed that: in order to acquire concepts one needs a concept image and not a concept definition; and that “concept definitions (where the concept was introduced by means of a definition) will remain inactive or even will be forgotten” (p. 293) as illustrated in Figure 2.1. The different ways in which a cognitive system might function is represented by the arrows in the figures.

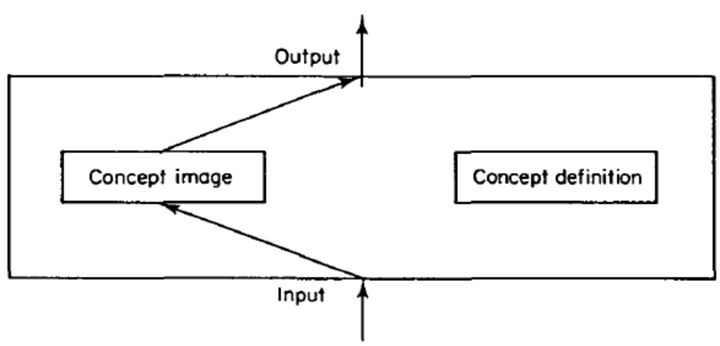


Figure 2.1: Interplay between concept image and concept definition.

According to Vinner (1991) acquiring a concept simply means forming a concept image for it. Hence, he described concept image as “something non-verbal associated in our mind with the concept name” (p. 68). Concept image of a concept can be likened to multiple representation of the concept, gathered experiences and impressions, all of which can be translated into verbal forms. Multiple representations can be pictorial/visual, abstract/analytical, descriptive, tables, etc. He also likened a person’s concept image to his or her visual representation of the concept,

mental pictures and gathered experiences and impressions, all of which can be translated into verbal forms. Vinner stated further that concept image is specific with individuals. Since what goes on in individual's mind differs, so is the way we see and perceive things.

At every juncture and experience with different tasks, the brain activates certain concept images, based on stimulus. This instantaneous response is what (Tall & Vinner, 1981, p. 152) refer to as "*evoked concept image*", which means there is possibility of conflicting images at other times. The concept image may contain the *formal image*, i.e., that part of the concept image that is formally deduced from axioms (Tall, 2001, p. 204).

Tall and Vinner also described an occurrence where a part of the concept image or concept definition conflicts with another part of the concept image or concept definition as a "*potential conflict factor*" (p. 153). For instance, it is very common to see students form a concept image of "$(S_n) \to s$" to suggest "S_n gets close to s as n gets large, but does not actually reach s until infinity". Actual cognitive conflict includes the perception that 0.9999… is less than 1, stating that the process of getting closer to 1 goes on forever without ever being completed. If such factor is not identified and an intervention designed to correct such conflicts, to help students connect to the right concept image, it will hinder the learning of a formal theory.

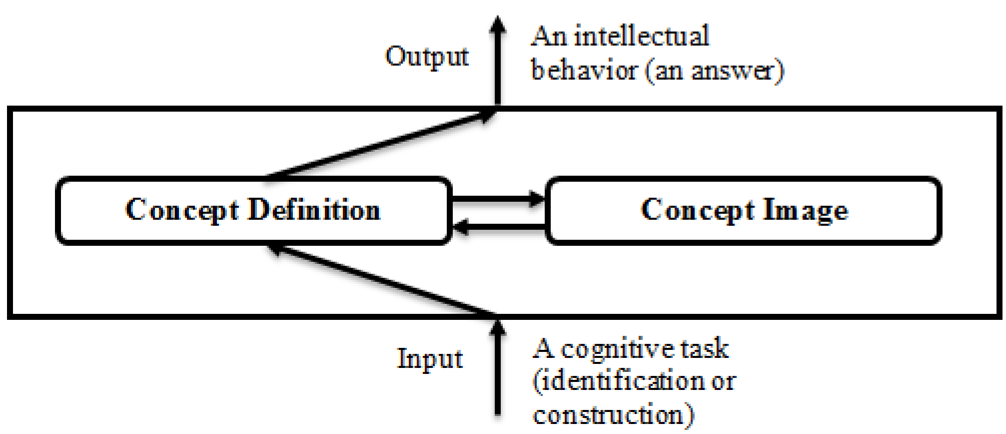


Figure 2.2: Interplay between concept image and concept definition (rigorous task).

The notions of concept definition and concept image as a lens to analyze students' conceptions is as a result of the conflict that exist when students are able to state the formal definition of a concept but they are not able to apply this definition to the properties of the concept. Tall & Vinner rightly asked “what mental picture can one have of a function f whose derivatives f’ exists yet f’ may not be continuous?” (1981, p. 169). Or how could students possibly visualize a function which is continuous everywhere but differentiable nowhere? Rasslan and Vinner (1998) studied the concept definitions and concept images of the increasing/decreasing function concept of 180 Israeli Arab high school students and the results of their study indicated that while 68 percent of the students could state the definition, 28 percent of them applied the definition erratically, and only 36 percent of the students correctly applied the definition. Vinner (1991) claimed that students often depend on their concept image, and do not consult their concept definition in problem solving process (Figure 2.3). Ideally students are not supposed to formulate a solution before consulting the concept definition as represented by Figure 2.4.

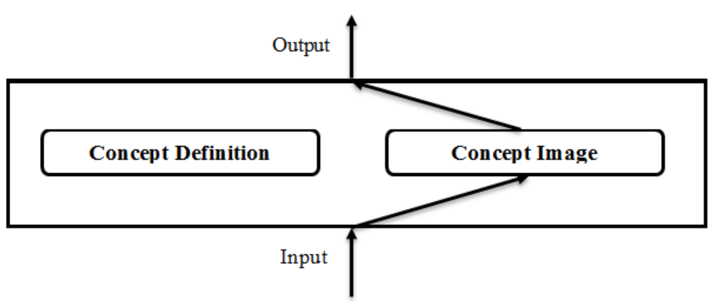


Figure 2.3: Intuitive response.

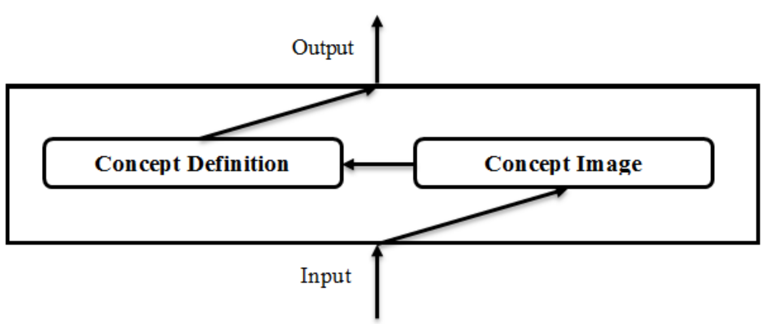


Figure 2.4: Deduction following intuitive thought.

2.6 Linguistic Perspectives

Lakoff and Nunez (2000) define the notion of actual infinity (e.g. infinite sets, infinite intersection, limits of infinite series, least upper bounds and points at infinity) by engaging a linguistic perspective to conceptualize that infinite iterative processes – processes that iterate without end, can have an end and an ultimate result. Their Basic Metaphor of Infinity (BMI) is based on cognitive mechanisms such as image schemas, conceptual metaphor, and "what linguistics calls the *aspectual system*" (p.155) - imperfective aspect (in ongoing actions) and perfective aspect (in complete actions). "BMI is the metaphor which changes potential infinity into actual infinity" (Ueno, 2004, p.56). The ongoing actions or process is an everyday life experience that is likened to *potential infinity*. The complete actions by which an unending process is conceptualized as a realized '*thing'* is considered *actual infinity*. Lakoff and Nunez argue that actual infinity is a concept and cannot be experienced in real life.

Tall, Thomas, Davis, Gray and Simpson (2000) elaborate on the object conception by use of language markers. They suggest that the way a person talks or write about a concept will help to elucidate whether the person has constructed a mental object as regarding the concept in question. They believe that objects by nature "are described by their properties, their relationships with other objects, and the ways in which they can be used" (p. 8). Language plays a significant role in interview analysis of infinity conception. Because of the ambiguities inherent in everyday or natural language, the language marker a person use to represent or describe a thing, its properties and relation to other things will help to determine if the person truly regard the thing as an object or not. Tall et al. (2000) give the following example: a person saying that 5 is a prime number, a third prime, and a second odd prime conceives of "5" as a mental object. "It is the use of language in a way that intimates properties, relationships, and usage of a concept which indicates that the individual is, in fact, conceiving "5" as an object" (p. 8).

Chapter 3: Chapter Methodology

In this chapter, a detailed description of the methodology used in conducting this study is presented, including the research design, selection of participants, instruments for data collection, data analysis, reliability and validity, and the ethical issues.

3.1 Research Design

The purpose of this study is to examine college students' conception of duality and determine whether or not the college students possess a dual process-object view of infinity. More specifically we ask: (1) how is the duality conception externalized and expressed by college students, and (2) how could students' conception of duality be assessed? The main research questions guiding this study are:

1. How is the duality conception externalized and expressed by college students at each course in the Calculus sequence?
2. To what extent does the type of a task impact the college students' external representation of infinity?
3. To what extent does the context of a task impact the college students' conception of duality?

To address these research questions Sequential Mixed Methods Nested Design (Onwuegbuzie & Leech, 2006) was utilized in addition to the theoretical framework to examine college students' preconceptions of infinity and determine whether or not they possess a dual process-object view of infinity from a triangulated perspective. Mixed method according to Johnson et al. (2007) is "the type of research in which a researcher or team of researchers combines elements of qualitative and quantitative research approaches (e.g., use of qualitative and quantitative viewpoints, data collection, analysis, inference techniques) for the broad purposes of breadth and depth of understanding and corroboration. (p. 123). The purpose for the sequential mixed method design is to use the qualitative data to explain the initial quantitative results. It also permits the opportunity of varieties of theoretical perspectives. In the first phase of

this design, the researcher will collect and analyze quantitative data, followed by the collection of qualitative data to help explain, or elaborate on the quantitative results obtained in the first phase. The second, qualitative phase builds on the first, quantitative phase. The researcher identifies specific quantitative results that call for additional explanation and uses these result to guide in the development of the qualitative strand - selecting participants for interviewing and developing interview protocols. Next, the researcher collects and analyzes the qualitative data to explore the insights it brings to the quantitative results "and what overall is learned in response to the study's purpose" (Creswell and Plano Clark, 2011, p. 83). The rationale for this approach is that the quantitative data and their subsequent analysis provide a broad understanding of the research problem. That is, analyzing the qualitative data will help the researcher explain the statistical results obtained from the quantitative data by exploring participants' views in more depth and drawing more robust conclusions (Creswell and Plano Clark, 2011; Teddlie & Tashakkori, 2009).

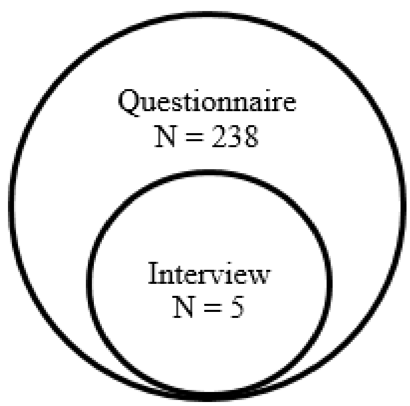


Figure 3.1: Sequential Mixed Method Nested design of the study.

Creswell and Plano Clark (2011) determined some strengths and challenges of this mixed method design as follows:

Strengths

- The design appeals to quantitative researchers because it often begins with strong quantitative orientation.

- The implementation is straightforward because the two methods (quantitative and qualitative) are conducted in separate phases, and the design allows collecting one type of data at a time.
- It does not require a research team but a single researcher can conduct the design.
- The final report is straightforward to write, with the quantitative section preceding the qualitative section.
- The design lends itself to emergent approaches because the quantitative first phase determines the qualitative second phase design.

Challenges

- Lengthy amount of time is required for implementing the two phases.
- Researcher will have to tentatively frame the qualitative phase of the study, which is not fully determined until phase one is realized, in other to secure institutional review board (IRB) approval.
- Researcher will have to determine which quantitative results need further explanation.
- Researcher will have to decide the criteria for selecting participants for the second qualitative phase.

3.2 Participants

Since the concept of infinity is central to calculus because infinite processes form the basis for the concept of limit and are also related to concepts such as sequences and functions, which are traditionally taught in Precalculus and the Calculus sequence, and also to allow for a variety of different conceptions of infinity from different points in students' learning of Calculus, I chose to select 238 college students enrolled in the Calculus sequence courses (Pre-Calculus, Calculus I through Calculus III) at one of the southwestern universities in the U.S. I used cluster random sampling (Onwuegbuzie & Collins, 2007) to select participants that are willing to participate in the study. Cluster random sampling involves selecting intact groups representing clusters of individuals instead of choosing individuals one at a time. Five instructors teaching the

Calculus sequence courses were contacted to request if they would allow part of their class time to administer the infinity survey to members of their class who were willing to partake in the study. Only two of these instructors gave their consent in letting their students partake of the survey. The questionnaire was administered during class time of the Pre-Calculus, Calculus I, II, and III sections of the two instructors who were willing to let their students participate. After been introduced to the students, I briefly explained the purpose of my research and thanked the students for their willingness to participate in the survey. 238 students who volunteered completed the Infinity Questionnaire. However, the duration of time to administer the survey varied for each class, since it was administered towards the end of class section in all cases.

The demographic information of study participants is shown in Table 3.1.

Table 3.1: Demographic Information of college students

Courses	Male	%	Female	%	Other	%	Total
Precalculus	37	29	31	29	2	67	70
Calculus I	60	47	62	58	0	0	122
Calculus II	19	15	7	7	1	33	27
Calculus III	13	10	6	6	0	0	19
Total	129		106		3		238

The table showed that Calculus I students represent about half (51%) of the participants. Initially seventy four students were sampled from Calculus I but because there was just one participant who volunteered to be interviewed at that period, the researcher contacted professors from the Calculus sequence courses (Pre-Calculus, Calculus I through Calculus III) the following year but only two Calculus I professors were willing to give up part of their class time to be used to administer the survey to their students. The questionnaire was then administered to a new set of Calculus I students who were willing to participate in the survey and also to be interviewed. Forty eight participants from the two Calculus I classes of these two professors completed the survey. In all by gender, 54% of the participants were male and 44% were female. Three of the participant did not disclose their gender on the survey.

3.3 Data Collection

Data for the study made use of two instruments to gather both quantitative and qualitative data that were collected in two distinct phases: (a) a self-reporting questionnaire given to college calculus students during the sampling stage and (b) semi-structured individual task-based interviews. NVivo 10 software was used for the open-coding of the self-reporting questionnaire responses to capture students' conception of infinity as a process, object or process-object and to categorize them into various nodes. Data collected from the questionnaire was used to categorize college students' response by their level of perception of duality and to select participants for the interview. The interviews were audio taped using a Sony audio recorder in order to capture all the information shared by the interviewees and later transcribed literally by the researcher using Express scribe software. There are two important features that make transcribing an audio file using Express scribe much easier and to check against recordings for accuracy: The playback speed can be adjusted if the speaker talks too fast to keep up with typing, and one can also play the digital audio file and listen to it in Express scribe while typing on Microsoft word document without having to switch back and forth between the two programs (Express scribe and Microsoft word). Data collection for the study was triangulated using multiple measures analyzed by three independent experts using a self-designed coding scheme to assess students' externalization of their conception of infinity. Students' views were classified into four levels to determine their conception of duality.

3.3.1 Infinity Questionnaire

Of the 238 college students that completed the Infinity questionnaire, 70 were enrolled in Precalculus class, 122 enrolled in Calculus I class, 27 enrolled in Calculus II class and 19 enrolled in Calculus III class. The participants were given a self-reporting questionnaire designed in form of a survey. The survey instrument are consisted of open-ended question items with the intent of obtaining written responses about college students' conception of duality.

The first part of the questionnaire collects students' demographic information such as students' gender, ethnicity, and total math GPA. Many participants refused to disclose some of this information. The second part consisting of four tasks aimed to investigate the college students' conception of duality. Pape and Tchoshanov (2001) emphasize the use of explanations, justifications and "representations in the service of supporting arguments" (p. 126). The first task 'define infinity' task was modeled after tasks used by many researchers such as Kattou et al. (2010), Singer and Voica (2003, 2008), Bingolbali & Monaghan (2008), and Wawro, Sweeney, & Rabin (2011). The second task was a scenario-based task, created by researchers as a form of Zeno's paradox, similar to Piaget's tasks discussed in Monaghan (2001), to access the impact of contexts on students' understanding of duality concept. The third task of the questionnaire, created by the researchers was a 'draw infinity' task with students explaining their drawings. The first and third task in the questionnaire were used to engage students in externalization of their concept images and concept definitions of infinity because they relate to an individual's cognitive structure associated with the concept, and has the potential to reveal the particular conception of infinity and the associated misconceptions of infinity that the college students may hold. The self-reported Task 4 was a multiple-choice of infinity task, modeled after a task used by Kolar and Cadez (2012). Respondents were required to identify their conception of infinity among the options given. The questionnaire required that students completed the four tasks individually and to justify their responses. The questionnaire was collected after 20 minutes. The questionnaire tasks are presented on a 2-page handout (See Appendix B).

3.3.2 Interview

The main source of the qualitative data necessary for understanding the phenomenon in question is interviewing (Merriam, 2009). A semi-structured interview protocol (Fraenkel & Wallen, 2006; Goldin, 2000) was employed since it does not require "the exact wording nor the order of questions" to be predetermined (Merriam, 2009, p. 114). It also allowed freedom in selecting follow up questions tailored to individual interviewee's responses during the interview.

After the questionnaires were collected and analyzed by three independent experts using self-designed coding scheme to assess students' externalization of their conception of infinity, twenty three students (N=23) were selected and invited to participate in the semi-structured individual task-based interview (see Appendix C for interview protocol) based on the fluidity of their view, coursework, the four categorical levels used to determine students' positioning toward duality of infinity concept, students' response to the multiple-choice Task 4 and its disconnection from the first three tasks, or a statement or drawing that needed more explanation for better clarification. Five (N=5) of the selected students from the Calculus I class agreed to participate in the interview and each participant represented a category level of duality conception of infinity. Pseudonyms are used in this paper when referring to the interviewee.

Two weeks after completing the survey, the five participants who consented were interviewed individually for an interval of 15 - 20 minutes in a private classroom and were provided with writing materials. The researcher informed participants that the interview was designed to follow up on their thinking process and not to judge their responses. The emphasis of the interview was to determine the conception level of the participants' (dual-idiosyncratic) responses to the tasks. Talk aloud protocols (Monaghan, 2001) were implemented as participants were required to explain everything they were doing and thinking while working on the new tasks. As the participants were asked to reflect on their responses to the questionnaire items, slight intermissions occurred during the interviews to get clarification on the wording of the questions or explanation from participants as they justified their responses to the tasks given.

The interviews with students were conducted for two important reasons: One was to gain additional insight into the students' view of infinity as they recalled their ways of thinking about the written responses to the questionnaire tasks and worked through related tasks in the researcher's presence; check for consistency in their language used to describe infinity as students clarified ambiguous responses to their personal concept definition of infinity, and since most of the participants provided relatively short and simple responses to the open-ended questions. A second reason for conducting the interviews with students was to better probe

students' response to the multiple-choice Task 4 and its disconnection from the first three tasks. In this way students were able to explicitly talk about their conception of infinity as a process, object or process-object, and the researcher was able to gain further insight into their understanding of infinity and categorize their views as either a process, object or process-object. Because grounded theory is verificational in nature (Corbin & Strauss, 1994), this study engaged its use to guide the qualitative interview coding.

The semi-structured interview protocol consisted of two tasks related to the Cookie monster task but presented in different context since it has been established that introducing students to the concept of infinity requires the use of multiple examples from different contexts. During the interviews, some of the participants were first asked to complete the two interview tasks while others were interviewed beginning with their questionnaire responses that required clarification, to get further interpretations on their thinking and externalization of their conception of infinity. In such a situation, students were asked to reflect on their responses to the questionnaire tasks, asked if that view still held and for them to elaborate more on their explanations. Talk aloud protocols (Monaghan, 2001) were implemented as participants worked through the questionnaire tasks and other related semi-structured interview tasks designed to further elicit their views and to what extent students construe coherent argument consistently. During the interview, the participants had opportunity to clarify their conception. Information about students' mathematics background, which included math courses they had taken in high school and what their first encounter with the concept of infinity was also collected. The previous math courses included: (1) Algebra I, (2) Algebra II, (3) Geometry, (4) Precalculus and (5) Calculus I. The task-based interviews were audio taped and transcribed for analysis using Express scribe software.

3.4 Data Analysis

After the infinity questionnaire was administered, the qualitative data obtained which addressed the students' concept images of infinity were coded by three independent expert raters

using self-designed coding scheme to assess students' externalization of their conception of infinity. Coding which involved attaching one or more keywords to a text segment was concept driven (Kvale & Brinkmann, 2009) because they were codes developed in advance by the researchers. The first round of coding (Iteration One) involved each of the three expert raters independently coding each item of the survey to categorize students' view of infinity as: PROCESS view (P), OBJECT view (O), or PROCESS-OBJECT view (PO). Then, the three expert raters shared some of the surveys that they were unsure or not confident in their interpretations about. From that discussion, the expert raters realized that there existed fluctuations in the students' views of infinity and that there were a variety of process-object views being reported. Among a variety of process-object views, either the process or the object view was *dominating*. This posited challenges for interpreting students' perceptions of infinity as either a process or an object, and especially in determining the students' process-object duality conception. They agreed that focusing on the dominant views was insufficient to determine individual student's duality conception.

To overcome these challenges, rather than just focusing on *dominance*, the expert raters decided to further code and organize the PROCESS-OBJECT view into two major views – the dominant views and the recessive views. Another round of coding (Iteration Two) incorporated this concept and among the process-object views there were three recessive sub-categories: PROCESS-object view (Po - case where the process view is dominant and the object view recessive), OBJECT-process view (Op – case where the object view is dominant and the process view recessive), and process-object view (po – case where both views are recessive). The evolved sub-categories were further used to develop four levels of duality conception (Babarinsa-Ochiedike, Tchoshanov & McDermott, 2013). A modified duality concept development framework known as *Action-Process-Object-Duality* (APOD) adapted from the APOS theory presented in Illustration 3.1 was designed and used as well as Tall & Vinner's (1981) "concept definition and concept image" throughout this study to interpret students' intuitions, and their attempts to conceive infinity as a process as well as an object.

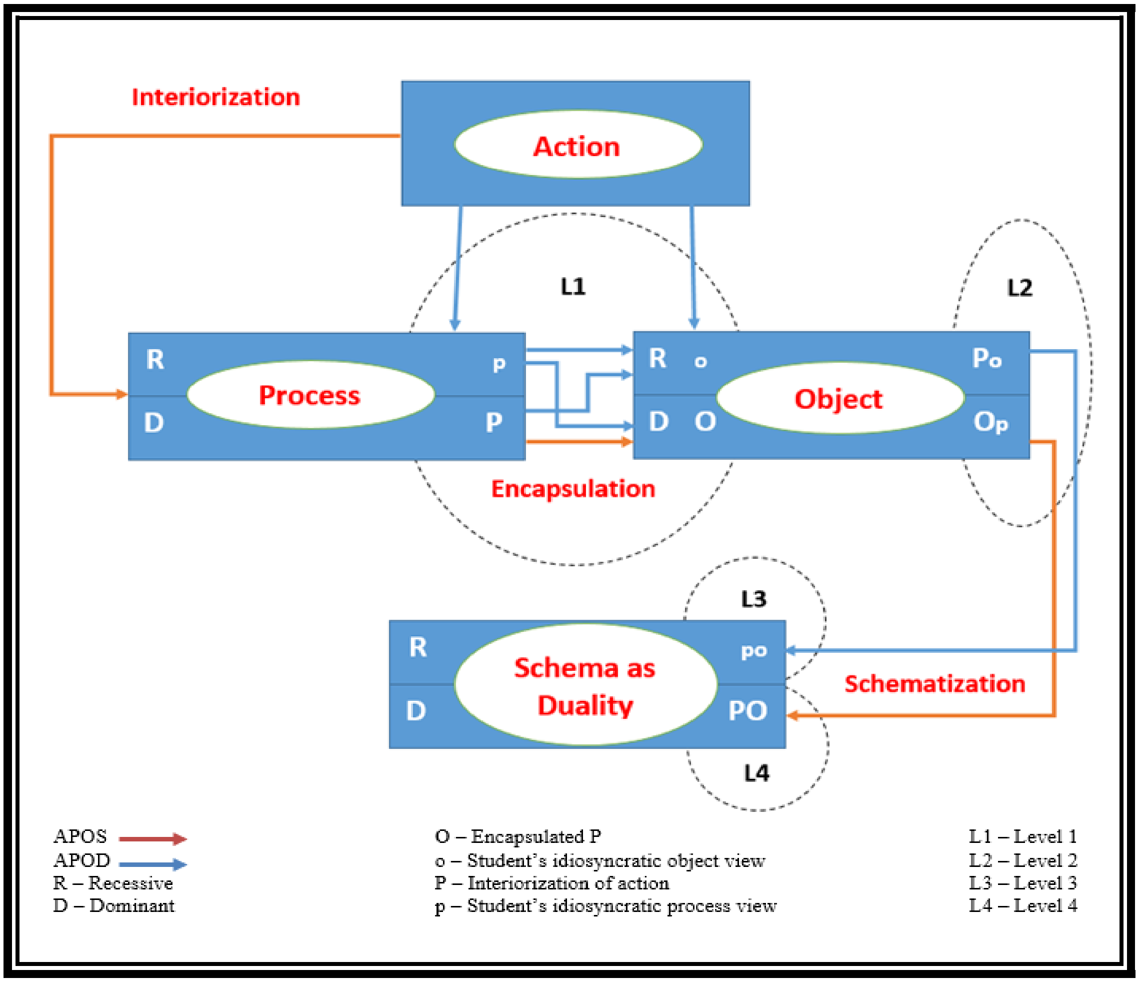


Figure 3.2: APOD Framework development.

3.4.1 Levels of Duality Conception

Level 1: Isolated singular view

This level represents the isolated singular view that is either dominant of recessive. In this case, only one view is displayed by the student, which could either be the process view (P) or the object view (O). The strength of the conviction/view could either be dominant ('P' or 'O') or recessive ('p' or 'o'), which is insignificant in determining the Level 1.

Level 2: Semi-isolated dominant view

In this case, students tend to display both the process and object view, depending on the task or context. Either the strength of the students' object view is dominant and process view is

recessive ('Op') or the strength of the students' process view is dominant and the object view recessive ('Po').

Level 3: Dual-idiosyncratic view

This level represents the case where both process and object views are recessive (i.e. not strong or convincing). We believe this case to be an indication of equality of students' process and object views ('p' and 'o' denoted as 'po').

Level 4: Duality view

This case also indicated equality of views ('P' and 'O' denoted as 'PO') except that both the process and object views are dominant (i.e. strong and convincing).

3.4.2 Quantifying the Data

We believe that having one view dominant over the other is an indication of strength in the dominant view, which makes us to classify both Level 1 and Level 2 as students' singularity conception. In contrast, Level 3 and Level 4 are classified as students' duality conception.

The third round of coding (Iteration Three) applied this concept (Levels of duality conception) to develop the final coding scheme presented in Table 3.2. Since conceptions cannot be quantified, the textual qualitative data representing the students' conceptions of infinity was quantified. It was understood that depending on the task of the survey instrument, a different view was being expressed by the participants. An overall rating of the level of infinity conception was then determined by the frequency of the process and object language coded in each participant's responses across the four tasks. This was a more adequate explanation of the view of infinity elicited by the individual participant. Students' level of infinity conception was scored 0 to 4. Items with no response and items not determinable were identified and scored 0. Data obtained from the analysis was used to construct the descriptive statistics (frequency) using Excel software. The quantitative results obtained were then used to determine how the duality conception was externalized by college students at each course in the Calculus sequence, select

participants for the interview phase and to triangulate the qualitative data obtained from interviews explained in Chapter 4. The result to this analysis is presented in Chapter 4.

Table 3.2: Coding scheme for duality conception of infinity

Conception	**Levels**	**Views**	**Code**	**Score**
Not determinable	None	Blank	ND	0
Singularity conception	Level 1	Isolated-singular view	‘P’ or ‘O’	1
	Level 2	Semi-isolated dominant view	‘Op’ or ‘Po’	2
Duality conception	Level 3	Dual-idiosyncratic view	po (p and o)	3
	Level 4	Duality view	PO (P and O)	4

3.4.3 Interview Coding and Analysis

Coding is “the most widely used categorizing strategy in qualitative data analysis” (Maxwell, 2012, p. 111) and has been identified as the preliminary process toward data interpretation and analysis (Saldaña, 2009). According to Kvale and Brinkmann “Meaning and language are intertwined” and this mode of analysis view “knowledge as preexisting elements that can be collected” from what is already in the texts (p. 196). To accurately replicate the intent and meaning represented by college students’ in their responses the mode of interview analysis employed was the meaning coding which is supported by linguistic analysis (Kvale & Brinkmann, 2009).

Open coding implicates “word by word, line by line analysis questioning the data in order to identify concepts and categories” which can be further broken apart (Grbich, 2007, p. 74). It is the contention of this study that implementation of open coding of text meaning enabled the expert raters to capture the fullness of students’ understanding and to categorize their responses into the views of infinity that they hold and to make meaning of students’ responses. It also enabled them to quantify the frequency of specific themes as they are addressed in the text, taking into consideration the common strands of the description of infinity expressed by the

students in their responses and arguments presented during the interview, and categorizing them appropriately as either process, object or process-object.

Linguistic analysis of students' responses by way of checking for the use of grammar, personal and impersonal pronouns, nouns and use of metaphor enabled them to better understand their object of encapsulation. Following the model of Moru (2009) to analyze the students' encapsulation of process into object, the common properties of students' descriptions of infinity was considered and classified into categories and the language markers for the object conceptions held by the students was identified.

To successfully make meaning of the interview data, segments in the data containing meaningful phrases for process and object views were identified and extracted from each transcript and color coded by hand by the individual experts. Coding by hand here instead of NVivo allowed the researchers to be closer to the data in order to examine all language markers that are potential tool for categorizing the obtained information into useful patterns, which enabled the researchers to interpret and make meaning of what was "seen and read" (Merriam, 2009, p. 176) . A color-coding scheme for displaying information by using different colored highlights enabled the researcher to view how students' process-object conception was externalized. Color-coding helped to keep things organized and simple. The APOD theoretical framework discussed in the next section which was used for analyzing the questionnaire data was also used for analyzing students' responses to the task-based interview, to identify different levels of students' understanding of infinity concept. An overall rating was performed using the coding scheme and the results compared to the questionnaire results to find out if the interview data supported or contradicted the conception presented in the questionnaire. Cross-expert analysis was implemented on the areas of concern (inconsistencies). The researcher interviewed the other two experts on their interpretation of meaning to specific language markers to arrive complete understanding of conception. The emerged themes identified are discussed in Chapter 5.

3.4.4 Analysis using APOD Theoretical Framework

The model below describes the four mental constructions from the APOD framework that the students might make in developing their understanding of the duality concept. This model is proposed as the "genetic decomposition" of the sum of an infinite geometric series.

At an action level, a student can transforms an object to form another object

$$\frac{1}{2}$$
$$\frac{1}{2}+\frac{1}{4}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$$

At the process level, a student interiorized the action into a mental process.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots<1$$

At the object level, a student sees this partial sum as a totality and then encapsulates it into an object.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

Finally at the duality level the actions, processes, and objects are organized and linked into a coherent duality framework. This framework includes the synthesis of process and object view of infinity to evaluate the sum of an infinite geometric series to become:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\frac{1}{2^n}<1 \quad \text{and} \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\frac{1}{2^n}=1$$

With regards to Item 2 of the Infinity Questionnaire, the students responding "No" to the Cookie Monster problem (Appendix B) have a predominantly 'process' view of infinity. The action of eating half of cookie remaining can be imagined to continue indefinitely. This type of thinking by students signifies a process conception (potential infinity). Students explaining their responses from the 'ordinal' perception of infinity, the partial sums of the series will always be less than 1:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^n}+\cdots=\frac{2^n-1}{2^n}<1.$$

Students responding "Yes" have the 'object' view of infinity, the completed infinite process of eating half of cookie remaining; and that is acknowledging that the last crumb of

cookie is been swallowed. This type of thinking by students could be based on a ‘cardinal’ perception of the above series as an object (a sum of an infinite geometric series) which is:

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

Synthesizing these two views will entail the dual process-object view of infinity concept in the context of the Cookie Monster problem.

3.4.4 Inter-Rater Reliability

Two other expert raters and I coded and scored the 238 students’ infinity questionnaires. One of the raters is a faculty member in the mathematics education department and an expert in the study of Calculus conception and in the early development of Advanced Mathematics Concepts. The third expert rater, a doctoral student was chosen because of his experience working on preservice teachers’ infinity conception. After the questionnaire responses were analyzed by the individual experts using the coding scheme, a reliability check was performed to determine if our coding was identical to the overall conception (Iteration Four). The Delphi method and Fleiss’s Kappa measure of inter-rater reliability were utilized during the process of iteration. “The Delphi method is well suited as a research instrument when there is incomplete knowledge about a problem or phenomenon” (Skulmoski, Hartman & Krahn, 2007, p. 1). Among the notable characteristics inherent with using Delphi method as explained in Hsu and Sandford (2007) are the ability to provide anonymity to respondents, a controlled feedback process, and the suitability of a variety of statistical analysis methods to interpret data. The number of Delphi iterations can vary from three to five. Fleiss’s Kappa is an extension of Cohen’s Kappa to assess reliability of agreements between multiple raters, without applying weighting. Fleiss’s Kappa measures the overall agreements between all the raters. Usually, a Kappa of <0.2 is considered poor agreement, 0.21-0.4 fair, 0.41-0.6 moderate, 0.61-0.8 strong, and more than 0.8 near complete agreement. The expert raters discussed and reviewed their ratings in accordance with one another’s perspectives or view and justification and made

appropriate changes to resolve the disagreements identifying the levels of conception (Iteration Five). A consensus was reached. The percentage of agreement was 72% and we concluded that the inter-rater reliability is satisfactory, because the obtained Kappa of .7165 is more than the generally applied criterion of .70.

3.5 Reliability and Validity

The intent of reliability is "minimize the errors and biases in the study' (Yin, 2003, p. 37). Internal validity according to Cohen, Manion, and Morrison (2007) is defined as "the explanation of a particular event, issue or set of data which a piece of evidence provides can actually be sustained by the data" (p. 135). Lincoln and Guba (1985) suggested credibility instead of internal validity in qualitative research. To ensure reliability and credibility, this study used both questionnaire and interview to address the research questions for triangulation. Two independent expert raters developed the coding scheme together with the researcher to analyze the questionnaire and they discussed evolved discrepancies until agreement was achieved. The results of students' responses to the two instruments (questionnaire and interview) were compared. In order to meet the credibility the researches need to identify and describe the subject and to take into account subject's point of view, hence multiple authenticities. Experts' raters also engaged in peer review at each step of data analysis.

External validity refers to the degree to which the results of the study can be generalized to other cases, settings, to the wider population. In this study, subjects were limited to 238 college students enrolled in the Calculus sequence courses (Pre-Calculus, Calculus I through Calculus III) at one of the southwestern universities in the U.S. So, it is not the intention of the researcher of this study to generalize the result of this study to whole population. Nevertheless, the researcher provides rich descriptions about the context and participants and their responses in instruments so that other researchers intending to transfer the results of this research have a rich base of information. Moreover, in this research, the modified APOD framework developed from

APOS theory was used to interpret the student responses in the interview with the aim of analytic generalization.

3.6 Ethical Issues

In qualitative studies, ethical issues are of great importance since to a large extent, the validity and reliability of a study are dependent on the ethical stance of the researcher (Merriam, 2009). The participation in the study was voluntary and in order to avoid coercion or undue influence participants were asked to complete and sign a consent form (Appendix A). Participants were also informed that they may be contacted later to participate in the interview phase of the study. There are no known risks associated with this research. Participants' rights and welfare were protected. In order to avoid conflict of interest, the researcher is not an instructor of record of sections where the survey was administered. Each participant was assigned a generic number code to maintain anonymity. Pseudonyms were used in order to protect the privacy of the participants. No photos or videotaping were used in this study.

The questionnaire data and the interview data, respectively will be presented in Chapter 4, detailing ways in which college students externalized their conception of infinity and the impacts of tasks and contexts on students' representation of views

Chapter 4: Results and Findings

In this chapter, the results of the mixed method study will be discussed in order to answer the research questions. Detail analysis of the results of the questionnaire tasks is presented and the discussions of students' responses during the interview are presented using the theoretical framework of APOD. For each questionnaire task the purpose of the task and samples of students' responses to the task is presented. Following the analysis of questionnaire tasks are sections addressing individual research questions that guided the study.

4.1 Analysis of the Questionnaire Tasks

A self-reporting questionnaire was administered during class time of the college students enrolled in Calculus sequence courses (Pre-Calculus, Calculus I through Calculus III). Four tasks aimed to investigate the college students' conception of duality. The questionnaire requires that students complete the four tasks and justify their responses. Two of these tasks (Task 1 and 3) were used to prompt how college students use mathematical constructs like definitions, graphics and symbols to represent their reasoning.

4.1.1 Analysis of Responses to Questionnaire Task Q1

The first task of the questionnaire to be discussed is the 'define infinity' task with the following statement:

Task Q1: "When you think of infinity what comes to your mind?"

The students' explanation of what comes to their mind when they think of infinity is their personal concept definition of infinity, and is a product of their personal experiences with the concept of infinity and partly or fully the explanation of their concept image as well (Vinner, 1991). Define infinity task generated a large variety of responses by the students. Results of responses to Questionnaire Task Q1 and levels of students' conception are displayed in Tables 4.1 and 4.2.

The majority (85%) of the responses were categorized as singularity conception at Levels 1 and 2 while one tenth were categorized as duality conception at Levels 3 and 4. The remaining

5% of students' responses could not be determined and were categorized as ND, either the participant did not respond to the question or there was not sufficient information for the view to be determined. Examples of such responses are "*to infinity and beyond*", "*a Buzz light year*", and *"∞", the symbol*. More than half (81 out of 139) participants in Level 1 used process language to describe infinity while the remaining 58 participants (42%) used the object language. Although 37% of the participants used both the language of process and object to define infinity, about three quarter of them used a semi-isolated dominant view (Level 2), indicating that the strength of the students' process view is dominant or the object view is recessive and vice versa. Having one view dominant over the other is an indication of strength in the dominant view.

Table 4.1: Results of responses to Questionnaire Task Q1.

Conception	Views	Pre Cal N	Pre Cal %	Cal 1 N	Cal 1 %	Cal 2 N	Cal 2 %	Cal 3 N	Cal 3 %	Total N	Total %
Not determinable	ND	3	**4**	7	**6**	2	**7**	0	**0**	12	**5**
Singularity conception	P	15	**21**	46	**38**	10	**37**	10	**53**	81	**34**
	O	21	**30**	31	**25**	3	**11**	3	**16**	58	**24**
	Po	16	**23**	7	**6**	2	**7**	2	**11**	27	**11**
	Op	11	**16**	19	**16**	2	**7**	4	**21**	36	**15**
Duality conception	po	3	**4**	12	**10**	7	**26**	0	**0**	22	**9**
	PO	1	**1**	0	**0**	1	**4**	0	**0**	2	**1**

Table 4.2: Levels of conception to Questionnaire Task Q1.

Conception	Levels	Pre Cal N	Pre Cal %	Cal 1 N	Cal 1 %	Cal 2 N	Cal 2 %	Cal 3 N	Cal 3 %	Total N	Total %
Not determinable	ND	3	**4**	7	**6**	2	**7**	0	**0**	12	**5**
Singularity conception	Level 1	36	**51**	77	**63**	13	**48**	13	**69**	139	**58**
	Level 2	28	**40**	26	**22**	4	**14**	6	**32**	64	**27**
Duality conception	Level 3	2	**3**	12	**10**	7	**26**	0	**0**	21	**9**
	Level 4	1	**1**	0	**0**	1	**4**	0	**0**	2	**1**
	Total	70		122		27		19		238	

The word cloud in Figure 4.1 displays 40 of the most frequent process and object terms the college participants used to define infinity. Participants used terms such as: *something, numbers, ending, forever, amount, symbol, and limit etc*. to describe infinity. The ability to conceive of infinity as a "*number*", an "*amount*" or "*something*" are considered to be indications of object conception and the ability to conceive of infinity as "*continues*", "*endless*" or "*forever*" are considered to be indications of process conception. Examples of students' responses are illustrated in the next section.

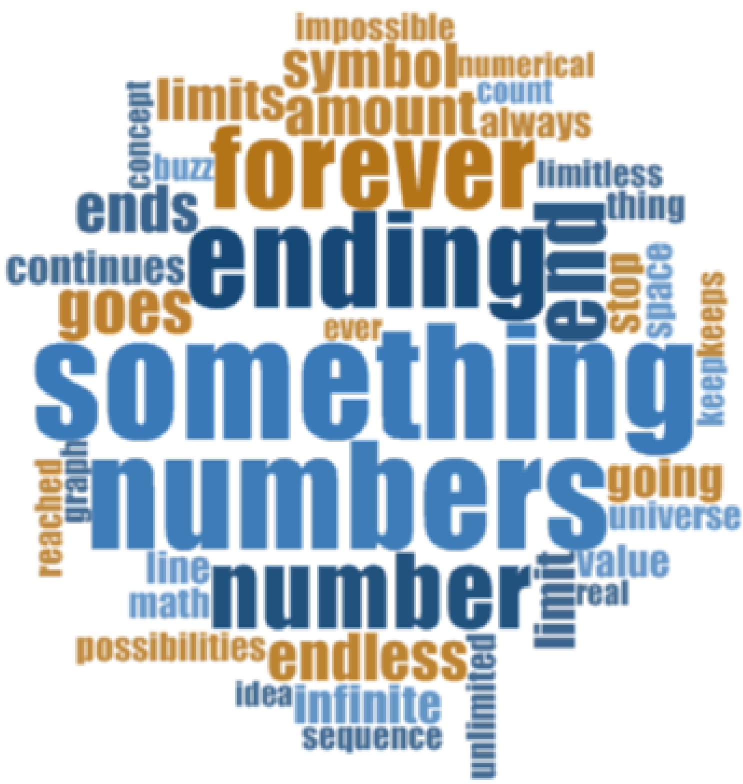


Figure 4.1: Word cloud of process and object terms used to define infinity Task Q1.

Infinity as "something"

Although Figure 4.1 indicates that "*something*" appeared most often in the students' response to Questionnaire Task Q1 (58 times), participants predominantly used process language to describe infinity. Figure 4.2 is a word tree highlighting an example of student's response using the term "*something*". The use of the expression "*It is something that continues forever*" ensured

for this student, "*something*" was concrete object. Process words "*continues*" and "*forever*" in this case are dominating, then students' view representing Level 2 - PROCESS-object view of infinity.

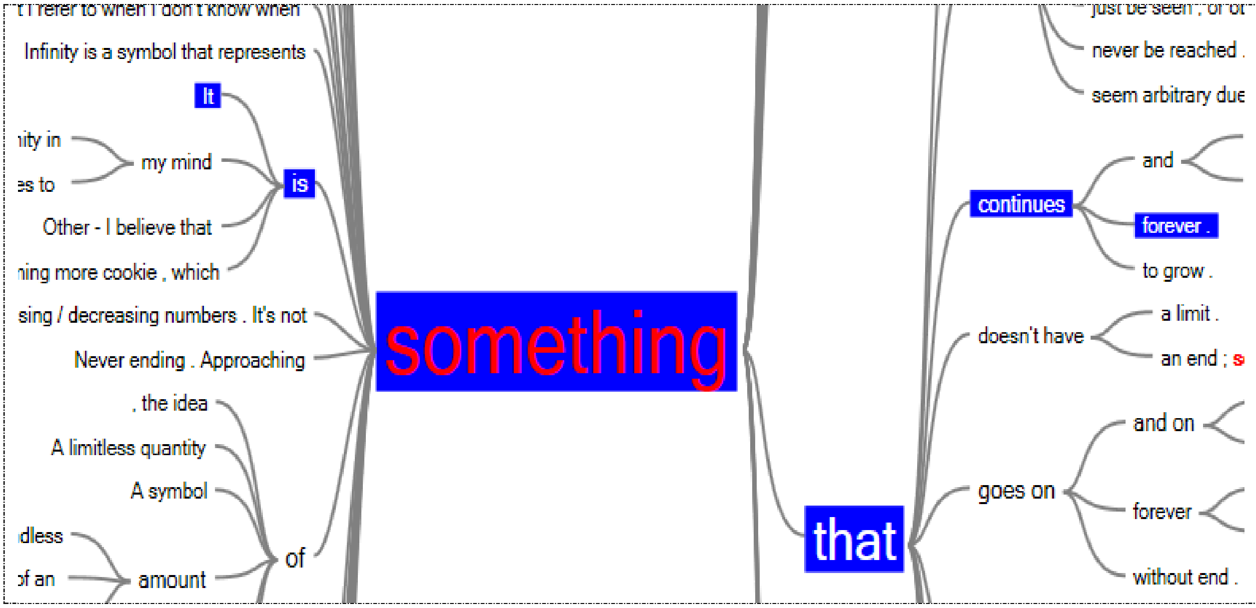


Figure 4.2: Students' use of the word "something" to define infinity in Task Q1 – Level 2.

Illustrated in Figure 4.3 below are other cases where students' view represents Level 1 – Object view of infinity, students' used phrases such as: "*the idea of something*", "*A symbol of something*", and "*lack of an amount of something*". At Level 3 – process-object view, students defined infinity as: "*A limitless quantity of something*", "*a never ending form of something*", "*An endless amount of something*", and "*a unlimited amount of something*". We judge these to be a pretty balance use of process and object language to define infinity.

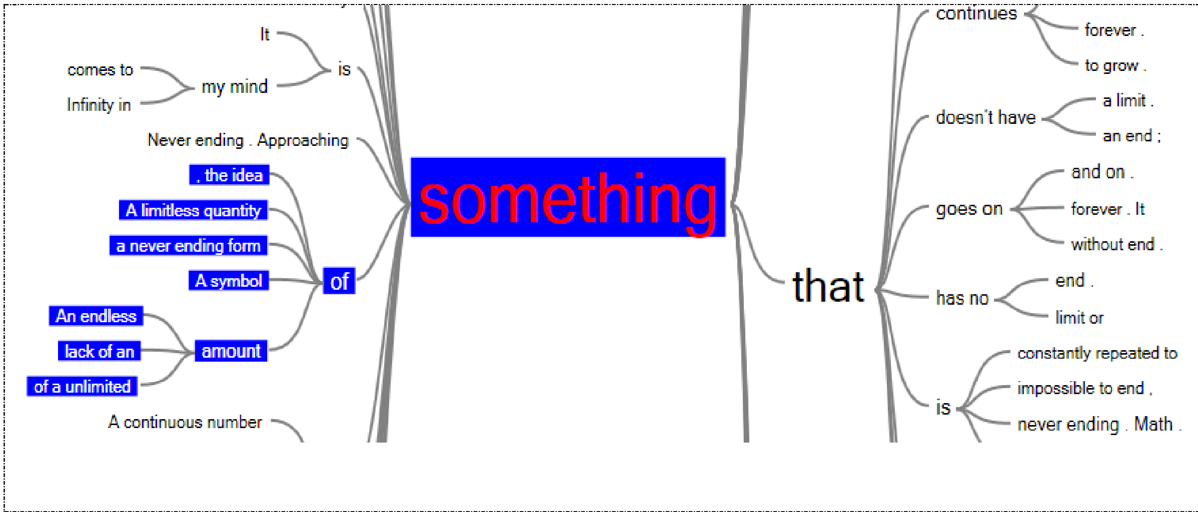


Figure 4.3: Students' use of the word "something" to define infinity in Task Q1 – Levels 1 and 3.

Infinity as a "number"

The fifth most frequent word in students' definition of infinity is "*number*" and it appeared 35 times. Although the ability to conceive of infinity as "*number*" and "*something*" is considered to be an indication of object conception, process language such as "*continuous*", "*continues*" and "*grow*" makes the process view to be dominant. The students' view represents Level 2 - PROCESS-object view of infinity (Figure 4.4).

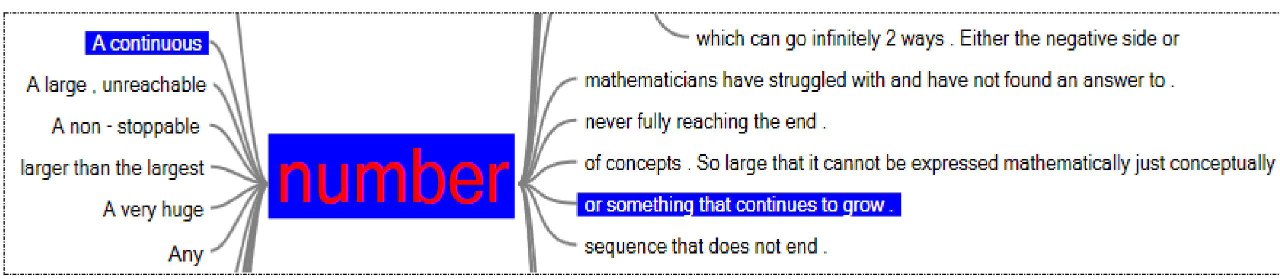


Figure 4.4: Students' use of the word "number" to define infinity in Task Q1 – Levels 2.

In Figure 4.5 the ability to conceive of infinity as "*a large number*" is considered to be indication of object conception, but then the phrase "*sequence that does not end*" highlights the language of process. Hence, the expression "*A large number sequence that does not end*" comprises a pretty balanced Level 3 – process-object view.

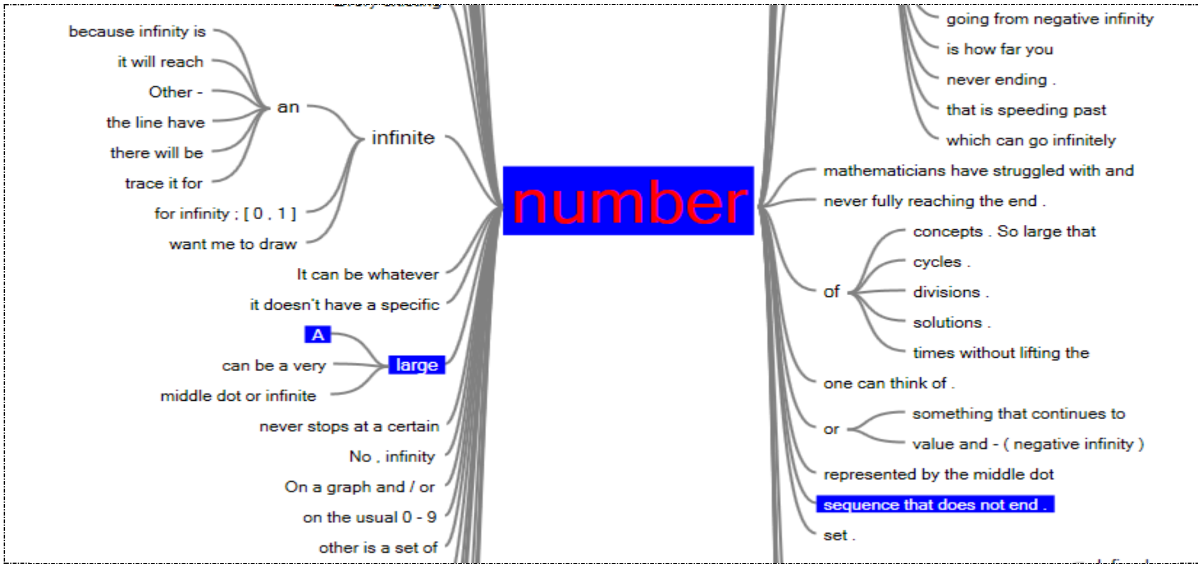


Figure 4.5: Students' use of the word "number" to define infinity in Task Q1 – Levels 3.

Infinity as "endless"

The Word Tree in Figure 4.6 establishes how students used the word "*endless*" to define infinity in Task Q1. "*Endless*" was the ninth most frequent word used to define infinity and it appeared 17 times. Students used phrases such as: "*an endless process*". It is clear that such a student has a distinct singular Process view of infinity at Level 1. At Level 3 – process-object view, students defined infinity as: "*an endless answer; No limit*", and "*an endless amount of possibilities*". At Level 2, students used phrases such as "*An endless quantity that supposedly "never ends"*". This particular participant put emphasis on the term "never end" with the quotation mark and that suggests process language is dominating.

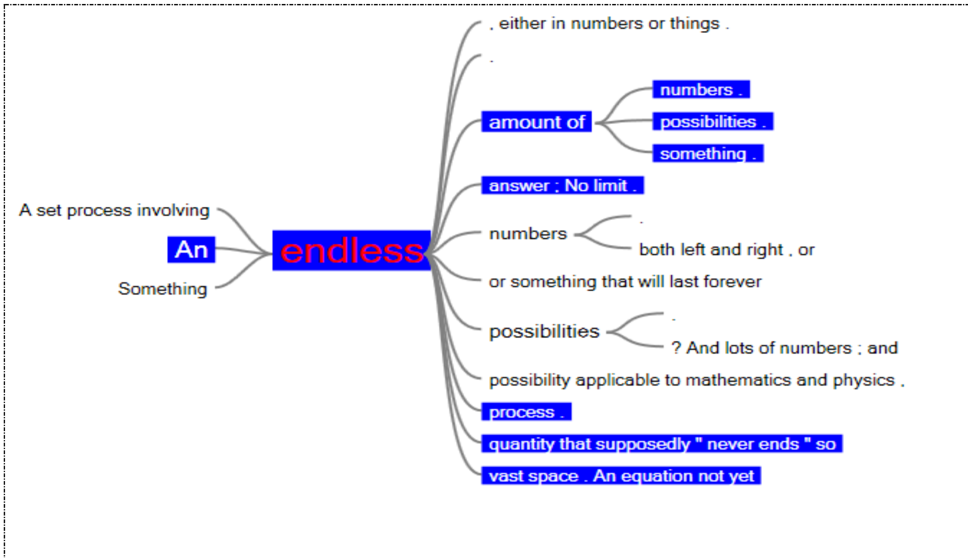


Figure 4.6: Students' use of the word "endless" to define infinity in Task Q1 – Levels 1, 2 and 3.

4.1.2 Analysis of Responses to Questionnaire Task Q2

Task Q2: The cookie monster sneaks into the kitchen and eats half of a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day.

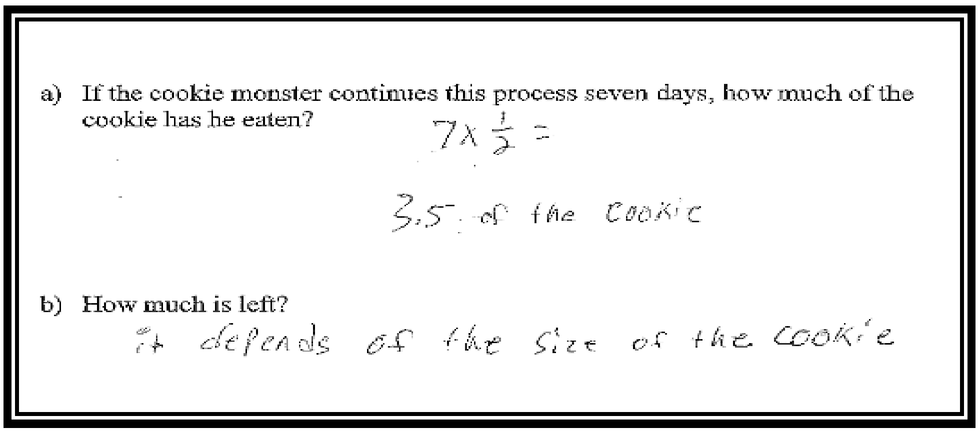
a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

b) How much is left?

c) If the process continues, will he ever eat the entire cookie?

This scenario-based task includes three sub-tasks that are aimed at examining college students' perception of duality, whether or not the college students possess a dual process-object view of infinity. In order to respond to this question, it was important for the students first to realize that the representation of this paradox involved an infinite sequence. Sub-tasks (a) and (b) were meant to support students' interiorization by helping them to reflect on the action of writing out the terms of the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots . -\right\}$. As students perform arithmetic operations on the terms, they are able to make encapsulation – to transform the infinite series conceived as process into a mental object. (i.e.) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots \ldots + \frac{1}{128} = \frac{127}{128}$ to which further actions could still

be applied. On average, less than a quarter of respondents showed a correct response to sub-tasks (a) and (b) by performing mathematical calculation of summing up the terms of the sequence. Some were unable to represent the problem mathematically (Illustration 4.1 and 4.2). Among those that were considered incorrect responses are partially correct responses as well as incorrect responses and no responses. Fine details will not be provided on these subtasks since they do not report on participants' concept images. However, they suggest an underdeveloped concept of the partial sum of a series of participants. Examples of student's responses are illustrated below (Illustration 4.1 - 4.4).


a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

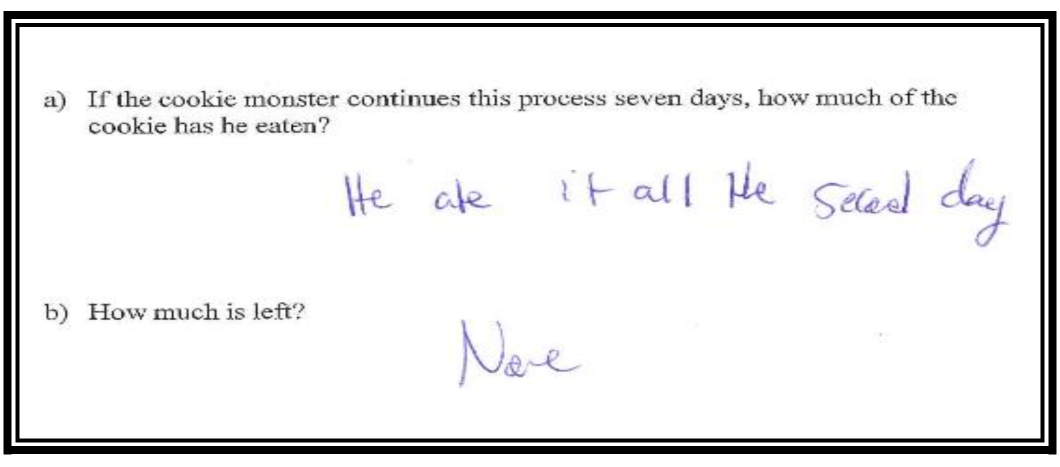
$7 \times \frac{1}{2} =$

3.5 of the cookie

b) How much is left?

it depends of the size of the cookie

Illustration 4.1: Student C1082 incorrect response to Task Q2.

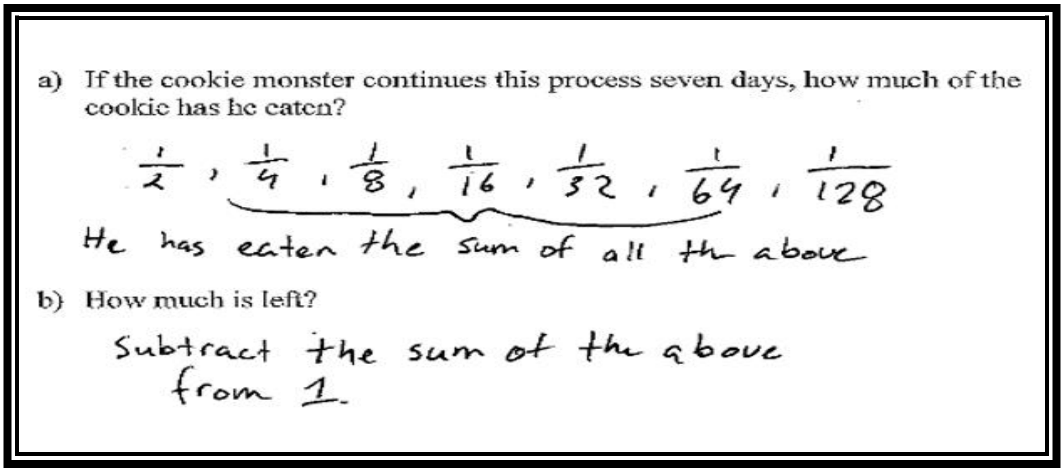

a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

He ate it all the second day

b) How much is left?

None

Illustration 4.2: Student incorrect response to Task Q2.



a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

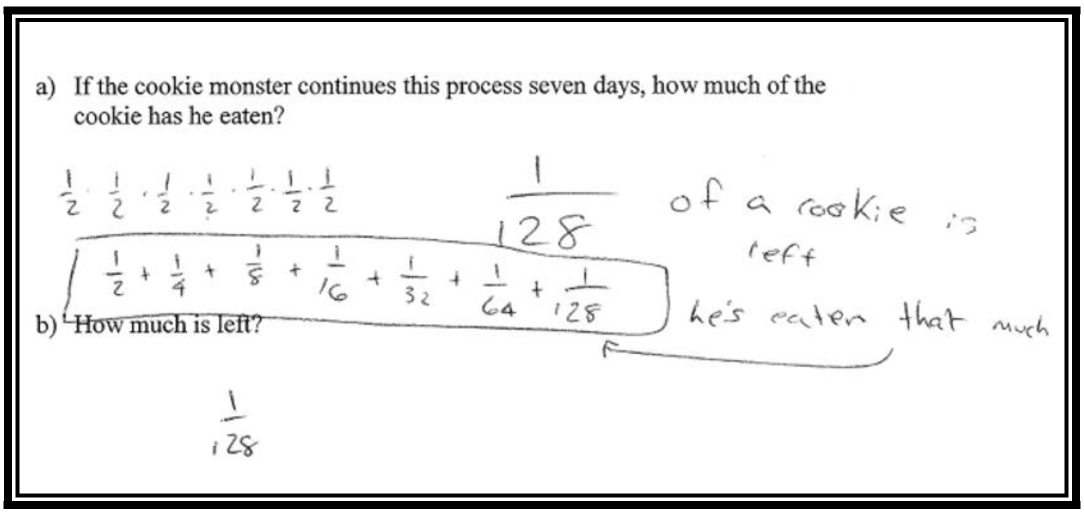
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

He has eaten the sum of all th above

b) How much is left?

Subtract the sum of the above from 1.

Illustration 4.3: Student C1080 partially correct response to Task Q2.



a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$ $\frac{1}{128}$ of a cookie is left

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$ he's eaten that much

b) How much is left?

$\frac{1}{128}$

Illustration 4.4: Student PC040 correct response to Task Q2.

Student C1082 multiplying seven days by half cookie in Illustration 4.1 is an indication of an underdeveloped conception of partial sum of series and student C1080 in Illustration 4.3 sub-task (b) played avoidance from resolving the problem mathematically as in Mamolo & Zazkis (2008). This study did not use sub-tasks (a) and (b) to assess students' conception of duality.

Sub-task (c) which asks that if the process continues, will the cookie monster ever eat the entire cookie was meant to determine students' having the process-object view of the phenomenon. Participants responding "No" to the Cookie Monster problem have a

predominantly 'process' view of infinity. The action of eating half of the cookie remaining can be imagined to continue indefinitely. This type of thinking by students signifies a process conception (potential infinity). Participants explaining their responses from the 'ordinal' perception of infinity are assumed to reason that the partial sums of the series will always be less than 1:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\frac{1}{2^n}+\ldots = \frac{2^n-1}{2^n}<1.$$

Students responding "Yes" have the 'object' view of infinity, the completed infinite process of eating half of the cookie remaining; and that is acknowledging that the last crumb of cookie is been swallowed. This type of thinking by students could be based on a 'cardinal' perception of the above series as an object (a sum of an infinite geometric series) which is:

$$S=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1$$

Duality conception entails participants synthesizing these two views. Table 4.3 illustrates the results to students' responses to this sub-task (c)

Table 4.3: Results of responses to Questionnaire Task Q2c.

Conception	Response	View	Precal	%	Cal 1	%	Cal 2	%	Cal 3	%	TOTAL	%
Not determinable	ND	ND	0	**0**	4	**3**	1	**4**	0	**0**	5	**2**
Singularity conception	NO	P	40	**57**	84	**69**	18	**67**	13	**68**	155	**66**
	YES	O	20	**29**	26	**21**	6	**22**	4	**21**	56	**25**
Duality conception	YES and NO	PO	10	**14**	8	**7**	2	**7**	2	**11**	22	**7**

A majority (91%) of the 238 responses to this task were categorized as singularity conceptions while 7% of the responses were categorized as duality conceptions. The remaining five responses (2%) produced either naïve or incomplete statements that made it difficult to determine their views. Example of such responses are:

C1054: Depends on how many cookies are in the cookie jar. Also if they are whole cookie.

C1065: Cookie monster doesn’t exist.

C1115: You can keep eating half of the remaining cookie, but eventually it will be infinitesimally small.

C1122: It depends, if how big will he eat the cookie and if he can bite the cookie into very small pieces, very tiny, tiny pieces.

C2020: It goes into infinity.

About three quarter (73%) of the students categorized as having singularity conception considered the potentiality of infinity (process view) using explanations such as:

PC004: He will never finish eating the cookie.

C1031: No, because there are infinitely many halves in between, technically the cookie will not finish.

C2011: No, because he is always eating half of half of half and continues forever like infinity.

C3006: No, because it is an infinite process. He is eating half of the half infinite times.

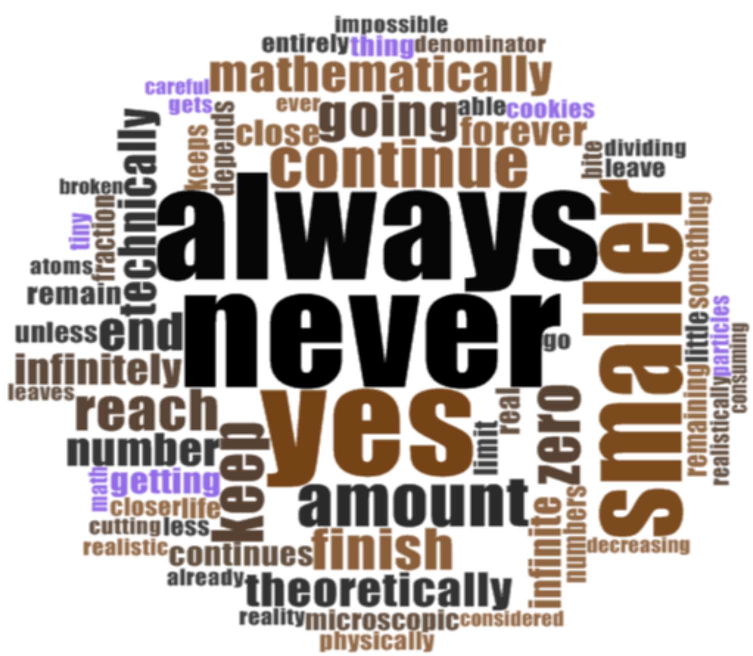


Figure 4.7: Word cloud of the most frequent words used for the Questionnaire Task Q2.

The reasoning for the participants who responded “NO” could be summarized from the word cloud in Figure 4.7 as: “The cookie monster would **never** finish eating the entire cookie but would **always continue** cutting **infinitely**”. This type of thinking by students signifies a process conception (potential infinity). Intuitions of potential infinity emerged, for example, in participants' descriptions of the **Cookie Monster** eating habit with “always continue to eat” what was left, and in their resistance to the idea of ‘completely eating ' the entire cookie. Our study showed that the Cookie Monster problem elicited a predominantly ‘process’ view of infinity, where the action of eating half of the cookie remaining can be imagined to continue indefinitely.

The majority of the students could not appreciate the cardinality of infinity in the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\frac{1}{2^n}+\ldots$ as an object. Only a quarter of the respondents (25%) considered the actual perception (object view). Students who were able to encapsulate the continuous process of eating half and half of cookie used expressions such as:

PC038: Eventually he will eat the entire cookie because of the process he has.

C1023: Eventually yes, since the cookie can be seen as a whole number. Unless you look at the crumbs that fell.

C2015: Yes, because even though you are eating the cookie by half, there would come a time where that half will be so small that can be considered as zero or in this case no more cookie.

C3004: Yes, the amount left will be too small to be considered anything.

A few (7%) considered the dual process-object view of infinity. Students who were able to synthesize these two views used expressions such as:

PC036: It's possible but the fact that while doing this process it will become smaller, thus given the illusion that's gone.

C1099: Eventually he’ll get to quantities too small to split in two, where he'll decide to just eat the crumb. But for the concept, the cookie should be able to infinitely get smaller and smaller.

C2012: The cookie still remains even if it is entirely eaten.

C3015: If it were a perfect world, No. Because the cookie would just get smaller but since the cookie is only so big, there comes a point where the cookie is no longer there.

The common thread identified in this task is that students used different contexts to imply infinity. Majority of students with duality perception used more than one context. Such contexts includes: mathematically/theoretically/ technically, and practically/realistically. Following are other examples of responses to sub-task (c).

PC007: Theoretically he can always divide it further until he has reached the last atom, but practically he will finish the cookie one day.

C1010: Technically never because you can continue to divide infinitely. But in reality the halves will become so small that it will be hard to divide unless nanotechnology becomes available. Besides, cookie monster can't stop at just one cookie.

C1014: Theoretically, Yes he will. But mathematically he won't. In math when you take away a percentage of a whole every day you will never get to "0".

C2002: It will eventually come to an end if talking realistically. However, if talking mathematically it will get really, really close for the cookie to be done but there would be an infinitely numbers in between.

C3011: The physical cookie will be finished when he eats the final crumb. If cookie is n/2, mathematically the cookie will last forever.

Considering all of the examples and other responses in the questionnaire, it can be concluded that the students who responded "YES" used the realistic, physical and practical context to elicit their object schema. The majority of the students' indefinite process schema is elicited using the mathematical, technical and theoretical contexts. Emergent themes in this result analysis include: "contextualization" which indicates reducing abstraction level, "context-dependency", and "holding on to reality". These will be discussed in Chapter 5. The context to which a participant perceives a given task or situation and which he/she considered to represent his/her responses determines the conception that is elicited.

This task also yielded many interesting responses, but since we could not have the responders to be interviewed to elicit further on their statements, we will not discuss them. However, an interesting statement of a student who possess dual process-object view of infinity was the following:

C1024: Theoretically, Yes he will. But mathematically he won't. In math when you take away a percentage of a whole every day you will never get to "0".

This suggests an alternative conception hailing from an "intuitive rule". The four emergent themes from this task will be discussed later in Chapter 5.

4.1.3 Analysis of Responses to Questionnaire Task Q3

Task Q3: "Draw infinity in the space provided. Explain your drawing below."

The third task of the questionnaire is a 'draw infinity' task with students explaining their drawings. This task was meant to elicit college students' concept image of infinity because it relates to individual's cognitive structures associated with the concept, exposing the particular conception of infinity and the associated misconceptions of infinity that the college students may hold.

Figure 4.8: Word cloud of students' drawing of infinity in Questionnaire Task Q3.

Table 4.4: Students' drawings of infinity.

Drawings	N	Pct
SYMBOL	162	68%
GRAPH	33	14%
BLANK	14	6%
CIRCLE	8	3%
ARROW	10	4%
LINE	24	10%
DISTANCE	1	0%
POINTS	1	0%
NUMBERS	3	1%
PROGRESSIVE DOTS	2	1%

Table 4.4 illustrates students' drawing of infinity including graph (14%), circle (3%), arrow (3%), number line (10%) and blank space (6%). A majority 162 out of 238 (68%) of the students drew an infinity symbol (∞) as can also be seen on the word cloud in Figure 4.8.

Results of the response to questionnaire Task Q3 (Table 4.5) indicates that 5% of responses were categorized as duality conception at Levels 3 and 4. None of these were given by the Calculus 3 students. The most widely held conception of the draw infinity task is the singularity conception (63%), which is predominantly the process view of infinity (70%) at Level 1. Precalculus and Calculus 1 students constitute the 15% who were categorized as having the object view of infinity at Level 1, and the remaining 15% semi-isolated view (Level 2). The conception of 76 (32%) participants out of the 238 participants could not be determined. Either the participants did not provide an explanation to their drawings or the explanation was not sufficient to determine the participants' views. Moreover, most of the drawings that do not have an explanation are the drawing of the infinity symbol. Examples of explanations that were categorized as ND include:

PC002: The math symbol for infinity.

PC008: The drawing is a flipped eight.

C1001: The drawing above is an eight but sideways. It symbolizes infinity.

C1106: It is the sign taught to me during my career in school.

C2023: The simple infinity.

C3016: It model stip. It has only one side and edge.

Table 4.5: Results of responses to Questionnaire Task Q3.

Conception	Views	Pre Cal N	Pre Cal %	Cal 1 N	Cal 1 %	Cal 2 N	Cal 2 %	Cal 3 N	Cal 3 %	Total	%
Not determinable	ND	18	**26**	43	**35**	11	**41**	4	**21**	76	**32**
Singularity conception	P	35	**50**	45	**37**	13	**48**	13	**68**	106	**45**
	O	7	**10**	15	**12**	0	**0**	0	**0**	22	**9**
	Po	5	**7**	11	**9**	1	**4**	2	**11**	19	**8**
	Op	0	**0**	3	**2**	0	**0**	0	**0**	3	**1**
Duality conception	po	3	**4**	3	**2**	2	**7**	0	**0**	8	**3**
	PO	2	**3**	2	**2**	0	**0**	0	**0**	4	**2**

A majority of the students who drew an infinity symbol used the language of process (potential infinity) to explain their drawings. In what follows are examples of students' explanations to the drawings of an infinity symbol.

PC012: The drawing is in a swirl type of form that has no corners, symbolizing it will never stop. Hence the word "infinite" in "infinity".

PC041: I drew that sign because in math it represents infinity. Meaning that it goes on forever.

PC059: This is the infinity sign we use in my Math class. It kind of makes sense because it has no ending, it continues forever and ever.

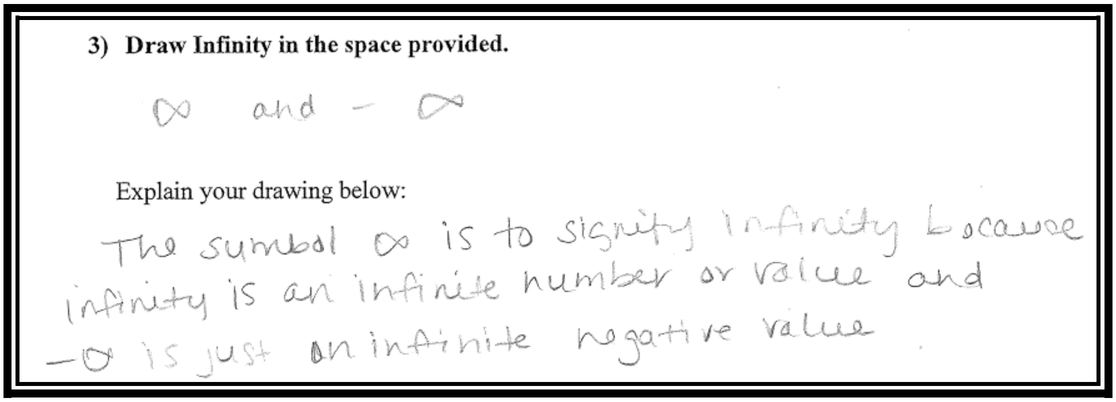
C1042: This is what I have known to be infinity for all of my life. An endless cycle, since if you trace the shape over with a pencil, you can trace it for an infinite number of times without lifting the pencil.

C1074: It is a symbol that just represents never ending, if you follow it, it has no beginning and no end.

C2017: Never ending.

C3011: It is a symbol that has no starting point and no ending point.

Illustration 4.5 is an example of student C1035 who used object (actual infinity) language to explain the drawing of an infinity symbol. To this student, infinity is an infinite number or value.


3) **Draw Infinity in the space provided.**

∞ and $-\infty$

Explain your drawing below:

The symbol ∞ is to signify infinity because infinity is an infinite number or value and $-\infty$ is just an infinite negative value

Illustration 4.5: Example 2 of students' drawing of symbol as infinity.

Illustration 4.6 presents examples of students drawing of an infinity symbol whose explanation of drawings include both process (potential Infinity) and object (actual infinity) language, with process language being dominant in all cases – Level 2. Process language included: it goes on forever, continues forever, no beginning and no end, never ending, never end and there will never be an end. Object languages used was: a realm, circuit, a loop, and cycle of numbers. Some of the students used more than one drawing to represent infinity, in which one of the drawings included the infinity symbol, which a few called infinity sign. Examples of students' explanations include:

C1064: It is a sign that shows a cycle that never ends and goes on forever.

C1075: Because it is like a circle, but in two parts that continues and continues the cycle.

C1120: It is a loop meaning it goes on and on, has no end.

PC004: Infinity continues forever. It goes into a realm that no one has gone to before.

PC011: The drawing represents some kind of circuit that goes on forever.

PC017: It shows the never ending cycle of numbers.

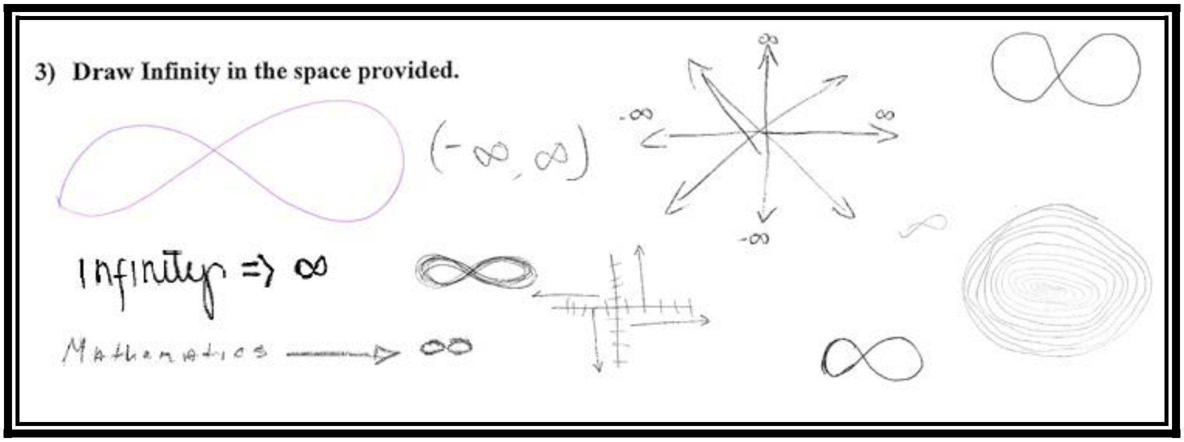


Illustration 4.6: Examples of students' drawings of infinity symbol.

A few students (3%) drew a circle to represent infinity. Student PC016 in Illustration 4.7 drew a circle. The explanation to the circle is that "A circle doesn't have corners and if follows will never find an end." These students' explanation is an indication of process conception of infinity – Level 1.

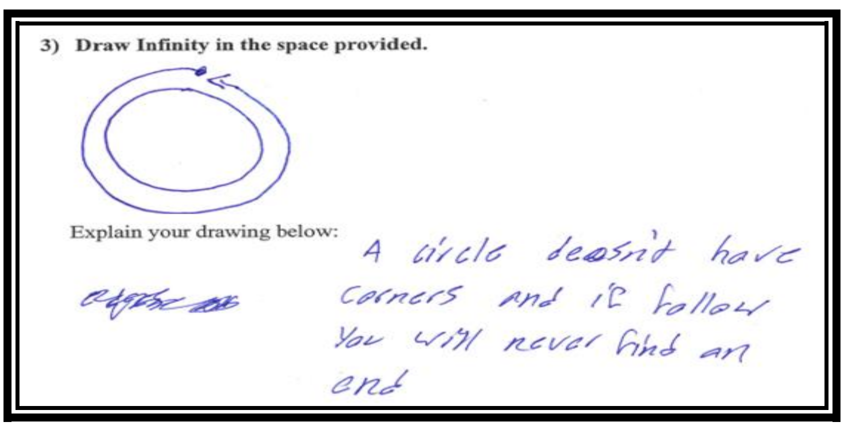


Illustration 4.7: Example of students' drawing of circle as infinity.

A few students (7%) drew a graph to represent infinity. Student C1081 in Illustration 4.8 drew both a graph and the infinity symbol. The explanation to the drawing is that "the process can continue to go on and beyond the set with no limit". These students' explanation is an indication of process conception of infinity – Level 1.

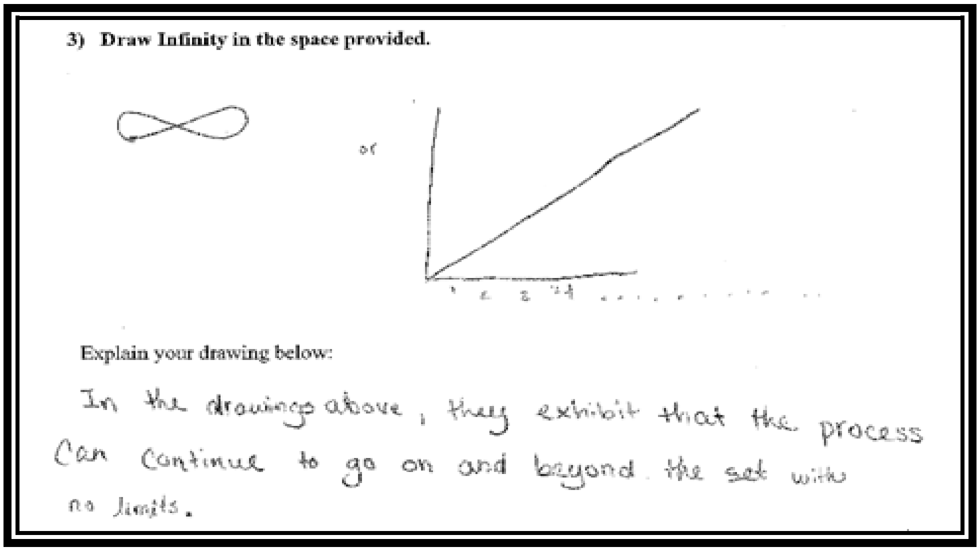


Illustration 4.8: Example of students' drawing of graph as infinity.

Another example of students who drew a graph to represent infinity is student C1029 in Illustration 4.9. Student explained that "In a Cartesian plane, infinity might be expressed as a line with a domain and range of all numbers. (In simple words, a line that goes forever and never stops). Meaning that it always define and continuous in a range of (-∞, ∞) and a domain of (-∞, ∞)". This explanation indicates a process (potential Infinity) view – Level 1.

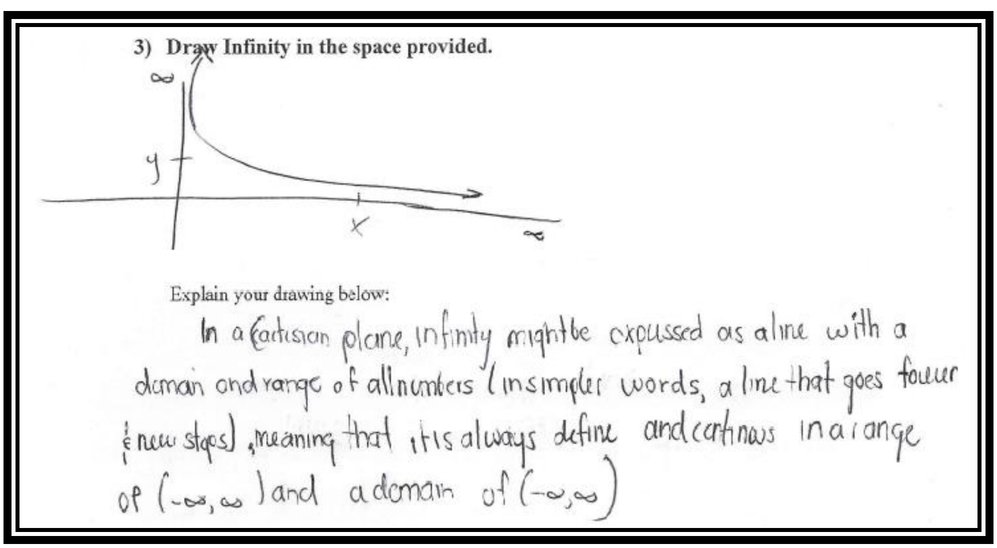


Illustration 4.9: Example 2 of students' drawing of graph as infinity.

A few students (10%) drew a line to represent infinity. Student C3009 in Illustration 4.10 is an example of students who drew a line to illustrate infinity. The explanation to the drawing is that “infinity can’t really be drawn as this is the best I can do, it is a never ending line”. Students’ explanation is an indication of process conception of infinity – Level 1.

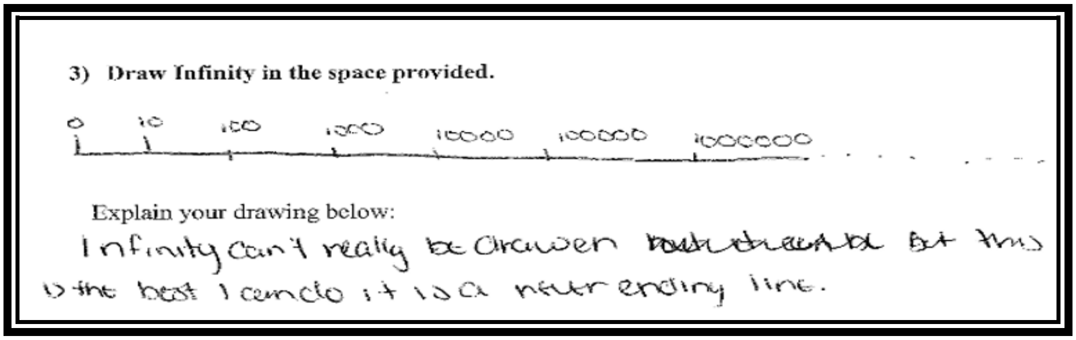


Illustration 4.10: Example of students’ drawing of line as infinity.

A few students (3%) drew an arrow to represent infinity. Student PC007 in Illustration 4.11 is an example of students who drew arrows. Student’s explanation of the drawing is that “the arrows show that it keeps on going further in every direction without an end”. This explanation indicates a process (potential Infinity) view – Level 1.

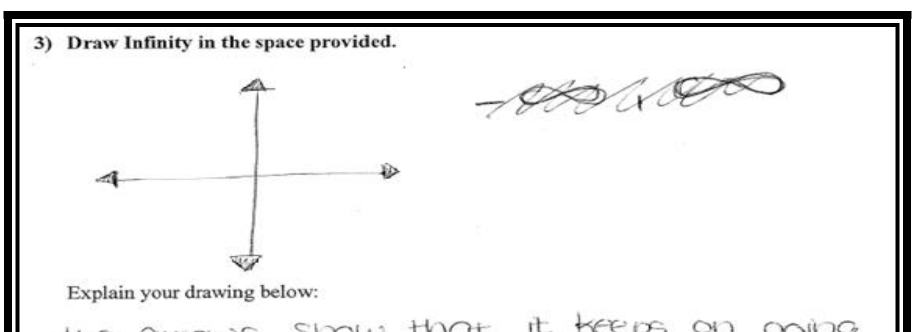


Illustration 4.11 Example 2 of students’ drawing of arrow as infinity.

A few students (6%) used blank space to represent infinity. Examples of students’ explanations are as follows:

PC009: The blank area has an infinite amount of possible uses and interpretations (i.e. it could mean death, life, everything, and nothing. You could write, paint, draw, etc.)

PC013: No need to draw infinity, It's there. There are "infinitely many" possible ways to draw, so it's best to let infinity be emptiness, which means a set of all possibilities, an empty void is a set of infinity.

C1069: It is impossible to draw infinity because by drawing it you are putting limits to it.

C2013: I need an infinite amount of paper. The drawing would never end.

C3010: You can't, it goes on forever. Infinity is a number that will never be reached. That number you solve for will always have a number larger than itself.

Of peculiarity is the response of PC013 who was able to conceive of the many ways to draw infinity as a totality. The student began the action of drawing infinity by drawing repeated dots as indicated in Illustration 4.12. In the process, the student stopped and stated that "no need to draw infinity". We assume student at this point was able to encapsulate the process into a mental object, thereby stating that it "is a set of all possibilities". He was able to imagine the infinitely many ways as a totality. This is an indication of object conception. This student was categorized as having the duality conception of infinity – Level 4. By the same reasoning PC009 was also categorized as having an object view of infinity (Level 1). The expression that relate to drawing that would never end as in the case of C1069 and C2013 suggests that they will be limited by the paper is an indication of process conception (Level 1). C3010 was categorized as semi-isolated dominant view (Po) at Level 2 because he was able to use the language of object (seeing infinity as a number) and the language of process. The idea that infinity always generate another number larger than itself and that it goes forever (potential infinity) suggest the participant has a dominant process view.

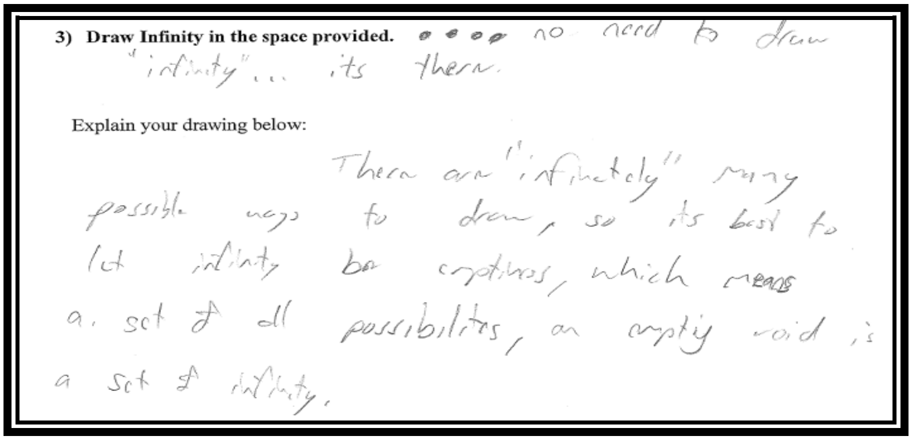


Illustration 4.12: Example of students' drawing of blank space as infinity.

4.1.4 Analysis of Responses to Questionnaire Task Q4

Task Q4: **I feel that my conception of infinity is as (check one):**

a) A process, e.g. something that goes on and on.

b) An object, e.g. Set of natural numbers is infinite.

c) Both a process and an object.

d) Other: ____

The multiple-choice Task 4 is a self-report task that gave respondents an example of illustrations that is considered to be a process and an object view of infinity. Respondents are to identify their conception of infinity, whether it is a process (e.g. something that goes on and on), or it is an object (e.g. a set of natural numbers is infinite) or both a process and object or to identify other.

Table 4.6: Results of responses to Questionnaire Task Q4.

Conception	Views	**Pre Cal**		**Cal 1**		**Cal 2**		**Cal 3**		**Total**	
		N	**%**	**N**	**%**	**N**	**%**	**N**	**%**	**N**	**%**
Singularity conception	P	24	**34**	49	**40**	8	**30**	6	**32**	87	**37**
	O	4	**6**	7	**6**	2	**7**	0	**0**	13	**5**
Duality conception	PO	40	**57**	57	**47**	16	**59**	10	**53**	123	**52**
No Response	NR	1	**1**	1	**1**	1	**4**	0	**0**	3	**1**
	OTHER	1	**1**	8	**7**	0	**0**	3	**16**	12	**5**

As seen in the Table 4.6, 52% of the respondents believe they have the dual process-object conception of infinity, 37% agree to have just the process conception, 5% agree to have the object conception, three gave no response and twelve “Other”. Some of the respondents in the category “Other” gave explanations while some did not. The explanations were coded into process (P), object (O) and process-object (PO), and those who did not give explanation were categorized with no response (NR). Examples to this category included the following responses:

Process view:

C1074: a symbol that represents no beginning and no end.

Object view:

PC024: a vast amount of space which is unlimited and has no exclusions.

C1008: Something that can be explained theoretically but never shown.

C1013: An infinite number that when found will set limit for set of natural numbers.

C1070: unknown because our minds cannot perceive the end of infinity.

Process-Object view:

PC009: Everything and nothing, too vast for any mortal mind to comprehend.

C1010: All of the above and everything and nothing at the same time.

C1033: I believe that ∞ is something that goes on and on, but it's not a process, it’s a concept.

It could be observed that participants PC009 and C1010 were among those who struggled with the idea of infinity being a process and an object. Their responses of infinity being “all of the above and everything” suggests they consent with the choices (a), (b) and (c) but the “nothing” suggests the strength of their convictions/views is recessive. These indicated the participants as having the dual idiosyncratic view (po) at Level 3.

Participant C1033 also from our APOD framework could be classified as having the dual idiosyncratic view (po) at Level 3. Even though she used the language of process and object, she did not consent with any of the choices (a), (b) or (c) and it was not clear what she meant by “∞ is something that goes on and on, but it's not a process”. The participant seems to be having misconceptions about the language used to describe the infinite process.

In order to check the disconnection of this self-report multiple-choice Task Q4 from the first three tasks, this research study employed the Chi square statistics. The expected outcome in this comparison is that students' response to the self-report multiple-choice Task Q4 will not be different from the raters' reported result.

Table 4.7: Raters results vs students' Self-report responses to Questionnaire Task Q4.

CONCEPTION	SINGULARITY		DUALITY	
Course	Raters	Self	Raters	Self
Precal	69%	40%	31%	57%
Cal 1	69%	46%	31%	47%
Cal 2	76%	37%	25%	59%
Cal 3	88%	32%	12%	53%

The question I want to explore with this task is if there is a gap between the raters overall rating of questionnaire Tasks Q1-Q3 and the students' self-reports questionnaire Task Q4? Do the Raters' result of Task Q1-Q3 differ significantly from the students' self-reported result?

Table 4.8: Raters results vs students' Self-report responses to Questionnaire Task Q4.

	RATERS	**SELF RESPONSE**	***Marginal Row Totals***
SINGULARITY	169 (133.5) [9.44]	98 (133.5) [9.44]	267
DUALITY	51 (86.5) [14.57]	122 (86.5) [14.57]	173
Marginal Column Totals	220	220	440 (Grand Total)

The contingency Table 4.8 indicated that using a 0.01 alpha level of significance, the Chi square statistic is 48.0189, $p < 0.01$ so we reject the null hypothesis and conclude that some factors other than chance is operating for the deviation to be so great ($p < 0.01$ means that the deviation is not by chance). Therefore, other factors must be involved. This explains that the students' conception of infinity is different between the raters' finding and what the students' report in Task 4. This is expected though, since a majority reported they possess the process-object conception of infinity without the knowledge of their dominant and recessive views, which the researcher used to categorize students' responses to the infinity questionnaire tasks.

Following are sections addressing the research questions that guided the study.

4.1.5 Results of Research Question 1

1. How is the duality conception externalized by college students at each course in the Calculus sequence?

This research question was developed by the need to investigate how the process-object duality conception is expressed by college students and if the traditional Calculus coursework support the development of the students' duality conception as they progress through the Calculus coursework sequence. Both qualitative and quantitative data were collected in order to explicitly answer this research question. Three main tasks were used to engage students in externalization of their duality conception of infinity. The first task was 'define infinity' task with the following statement "When you think of infinity what comes to your mind?" This task was used to analyze students' concept definitions of infinity. The second task was a scenario-based task in form of a paradox and the third was a "draw infinity" task with the direct statement "Draw infinity in the space provided". This task was used to assess college students' concept-image of infinity.

After performing a level analysis of the questionnaire responses to Tasks 1 through 4 (Appendix B) using the APOD theoretical framework' to determine the overall conception of the 238 participants of the study, Table 4.9 shows the levels of college students' conception of infinity. Levels 1 and 2 indicate a singularity conception and Levels 3 and 4 indicate a duality conception. Beginning from Calculus 1 the percentage of participants at the Level 1 gradually increases throughout the Calculus coursework sequence (from 17% in Precal. to 37% in Cal 3), and the percentage of students at Level 3 gradually decreases (from 31% in Precal. to 12% in Cal. 3). The percentage of students at Level 2 shows no substantial difference from the beginning of course - Precal (52%) to the last – Cal 3 (51%).

Table 4.9: Distribution of college students' infinity conception between levels

Conception	Levels	Pre Cal	Cal 1	Cal 2	Cal 3

		N	%	N	%	N	%	N	%
Singularity conception	Level 1	12	17	30	25	8	30	7	37
	Level 2	37	52	54	44	12	46	10	51
Duality conception	Level 3	22	31	37	30	7	25	2	12
	Level 4	0	0	1	1	0	0	0	0
Total		70		122		27		19	

Figure 4.9 illustrates that 69% of Precalculus and Calculus 1 students were at Levels 1 and 2 representing the isolated singular view of the infinity concept. Of the 70 Precalculus participants in the study, 31% were categorized as having the duality conception of infinity (Level 3). Their views of infinity as a process and object were not strong, indicating the 22 are at dual-idiosyncratic level. 37 out of 49 Precalculus participants having the singularity conception of infinity possess semi-isolated dominating views and the remaining 12 (75%) have the isolated singular view of infinity.

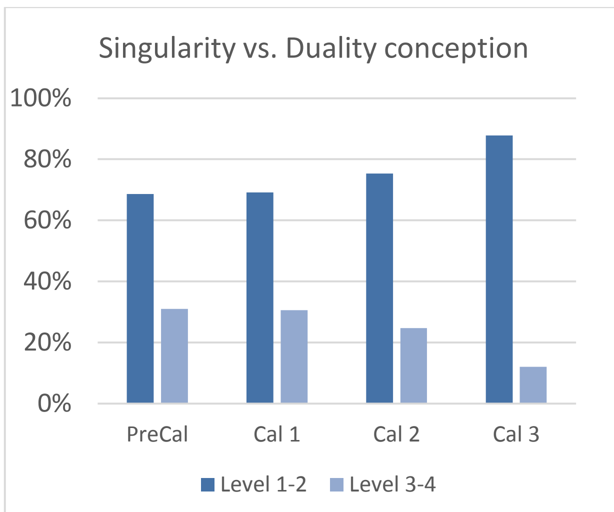


Figure 4.9: Singularity vs. Duality conception

Of the 122 Calculus 1 participants that participated in the study, only 1% actually possess the duality conception of infinity (Level 4), which limited the other 30% to the dual-idiosyncratic level (Level 3). This explains that although these 37 were able to conceptualize infinity as a process and as an object, their convictions are not strong but recessive. Of the 84 participants having the singularity conception of infinity, a quarter were categorized as having the isolated

singular view of infinity while the remaining 54 were exhibiting semi-isolated dominating views of infinity. As shown in Figure 4.9 the gap between the singularity and duality conception is smallest in the Precalculus and Calculus 1 courses (from 69 to 31) compared to Calculus 2 (from 75 to 25) and Calculus 3 (from 88 to 12). This is an indication that there is an opportunity to promote duality conception in the early Calculus courses.

We analyzed the data further to examine students' conception within each level, Levels 1 and 2 in particular. The data presented in Table 4.8 below show that majority of Calculus participant possess the process conception of infinity. At Level 1 were about three quarters of the Precalculus participants, 90% of Calculus 1, 88% and 86% of Calculus 2 and 3 participants respectively. Even at Level 2, among those categorized as having a dominating process view and recessive object view were all of Calculus 2 and 3 participants, about three quarters of the Calculus 1 participants and 79% of Precalculus participants. The table also shows that half of the Calculus 1 participants in the study were categorized as possessing a dual-idiosyncratic view of infinity. This is considered an emergent trend.

Table 4.10: Distribution of college students' infinity conception within levels

Levels Views	Level 1 'P'	or 'O'	Level 2 'Po'	or 'Op'	Level 3 'p' and 'o'	Level 4 'P' and 'O'
N Precal	8 73%	3 27%	31 79%	8 21%	20 35%	0 0%
N Cal 1	27 90%	3 10%	46 74%	16 26%	29 51%	1 100%
N Cal 2	7 88%	1 13%	13 100%	0 0%	6 11%	0 0%
N Cal 3	6 86%	1 14%	10 100%	0 0%	2 4%	0 0%

It can also be observed in Figure 4.10 that singularity conception of infinity was prevalent among the research participants and the gap between those having a singularity conception and those having the duality conception continued to grow as they advance in the Calculus coursework. Of those categorized as having a duality conception were 12% of the

Calculus 3 participants, 31% of the 70 Precalculus and 30% of the 122 Calculus I participants, and a quarter of the 27 Calculus 2 participants. This is an indication that the traditional Calculus sequence promotes a singularity conception as opposed the duality conception.

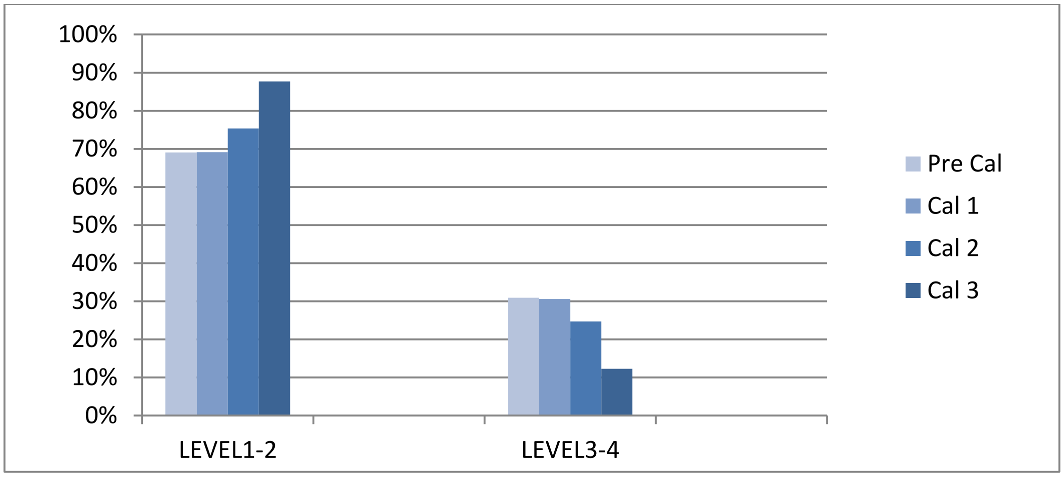


Figure 4.10: College students' duality conception.

4.2 Analysis of the Interview Tasks

After a qualitative and quantitative analysis of students' responses to the questionnaire tasks was conducted to identify the language students used to express or describe infinity and to categorize students' views into levels of conceptions, the results from this analysis were used to select interviewees for the interview phase of the data collection. Table 4.11 summarizes the distribution of the participants. To protect the privacy of the participants, pseudonyms were assigned to individual interviewees. Participant C1087 was given the name Emma, participant C1092 was given Susseth, participant C1089 was given Robin, participant C1014 was given Vanessa and participant C1121 was given Jose. These five participants who volunteered were interviewed using a semi- structured individual task-based interview protocol. Each of the five interviewee were Calculus I students based on the reasons discussed earlier in Chapter 3. Three of the interviewees were categorized as having a singularity conception. At Level 1 (isolated-singular view) was Jose and at Level 2 (semi-isolated singular view) was Vanessa and Susseth.

The remaining two interviewees were categorized as having the duality conception. Robin at Level 3 (dual-idiosyncratic view) and Emma at Level 4 (duality view).

Table 4.11: Distribution of Interview Participants.

Conception	Levels	Codes	Interviewee
Singularity Conception	1	L1	Jose
	2	L2	Vanessa and Susseth
Duality Conception	3	L3	Robin
	4	L4	Emma

The aim of the interview phase was to gain a more detailed clarification of interviewee's responses to be able to categorize their views into the levels of conception, and to help me answer my other research questions. Students were presented the same four tasks in the self-reporting infinity questionnaire and their responses during the interview was compared to the responses written down during the survey. The results of students' written responses to the four questionnaire tasks and responses to the same tasks during interview is presented below (Table 4.12).

Table 4.12: Questionnaire tasks responses during survey and interviews.

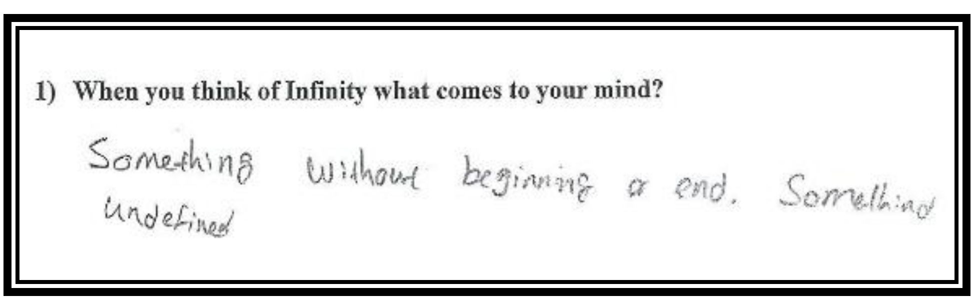
Tasks	Interviewees	Jose	Vanessa	Susseth	Robin	Emma
Q1	Survey	1	1	3	3	3
	Interview	3	2	2	3	1
Q2	Survey	1	3	3	1	3
	Interview	1	3	3	1	3
Q3	Survey	1	0	0	2	3
	Interview	2	0	0	2	1
Q4	Survey	3	1	3	3	3
	Interview	3	3	3	3	3

4.2.1 Results of Research Question 2

2. To what extent does the type of a task impact the college students' external representation of infinity?

Kolar and Cadez (2012) suggest the use of different types of tasks to investigate the concept of infinity. As can be observed in the Table 4.10, for every task presented students used different views to express their understanding of infinity which elicit different conceptions in different levels. Robin for example in Task Q1 and Task Q4 was categorized as having a duality conception of infinity at Level 3. In Task Q2, he was categorized as having a singularity conception of infinity at Level 1 and in Task Q3, he was also categorized as having a singularity conception of infinity at Level 2. Since Robin was consistent in the views that he used to express his written responses and to discuss during interview, I will present the analysis of his coding of the four questionnaire tasks to illustrate how type of task impact students' external representation of infinity concept, which is portrayed through variations in students' view from one task to the other. It should be noted that this part of the interview took place immediately after his discussion of the Tasks-based interview protocol.

Task Q1. Robin's response represents a dual idiosyncratic view of infinity concept at Level 3. He used recessive process and recessive object views (po) to address the concept-definition of infinity. Illustration 4.13 was Robin's written response. Referring to infinity as "something or something undefined" suggests an object view of infinity and the dynamic phrase "without beginning or end" suggests a process view of infinity. Robin represents an object which he was trying to describe by its properties "without beginning or end" (process view). Robin was rated at Level 3 (po). When asked to throw more light on his explanation during interview, he said the following:


1) When you think of Infinity what comes to your mind?

Something without beginning or end. Something undefined

Illustration 4.13: Recessive process and object view (concept-definition task).

24. R: Yes! The reason I said undefined is, often times I think people think infinity is actual concept because of the sign. But it represents something that is like an actual number. Say for instance we have x that is set to equal to something. However, infinity or negative infinity, you can't really set that, because it keeps going on and on. Just like numerical value, you can actually place towards infinity, someone just realize it keeps going on and on. That's infinity.

25. G: So it's like a numerical value?

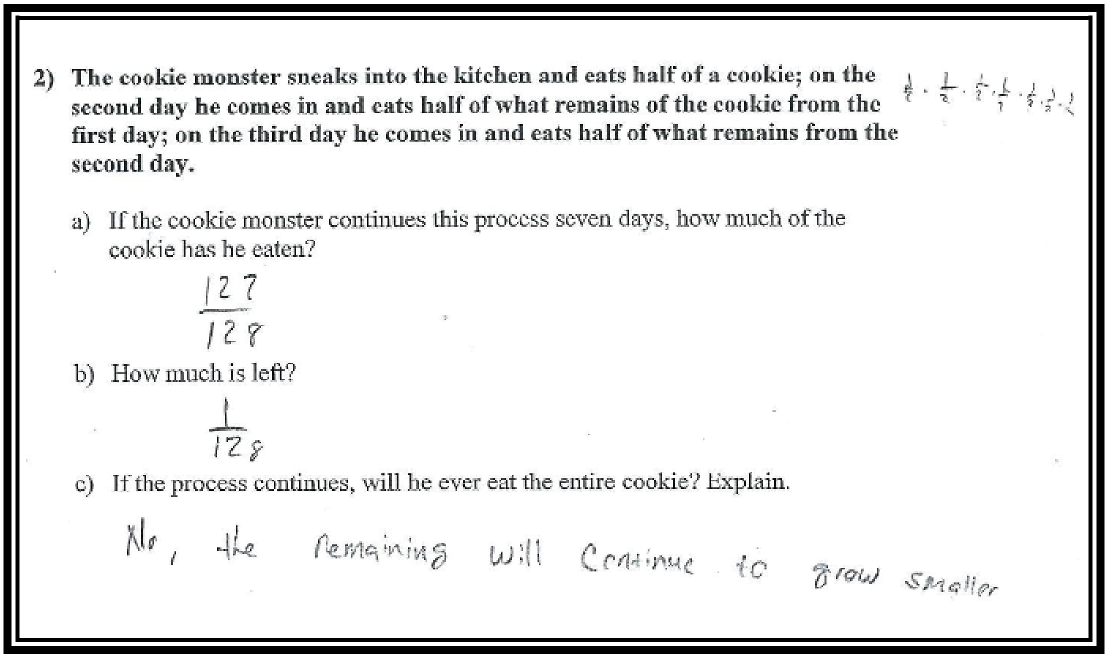
26. R: No, not numerical value. I think it just represent something that keeps going on and on. I think people would sometimes get that kind of confused. They think infinity is actual numbers but infinity is not number. It's just keeps going and going. It's never gonna stop.

27. G: Ok. So when you say something undefined…

28. R: Yes. Undefined because it's not an actual number per se, so we can't really define it. We just know it just keeps going and going. Like say for instance, I have 2 plus infinity. That number will be undefined because the infinity is not actual number we can add. That just means 2 plus what will keep going on and on. So the answer will kind of be undefined. Well, will be undefined.

Robin was able to use arguments based on both potential and actual infinity. His ability to define infinity as "it keeps going on and on" and "keeps going and going" is an indication of process conception and his ability to define infinity as an "actual number" and "numerical number" indicated an object conception. Even though he later rejected infinity being a number, the fact he could apprehend the concept of infinity being a number (Monaghan, 1986, p. 290) indicates he could still view infinity as an object. As seen from his response, Robin used balanced process-object language to express his conception of infinity but his duality conception lacks internal consistency. This indicates his process-object duality conception of infinity is not fully formed, hence recessive.

Task Q2. Robin's response "continue to grow smaller" represents a process view of infinity at Level 1. Illustration 4.14 was Robin's written response.



2) The cookie monster sneaks into the kitchen and eats half of a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day.

a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

$\frac{127}{128}$

b) How much is left?

$\frac{1}{128}$

c) If the process continues, will he ever eat the entire cookie? Explain.

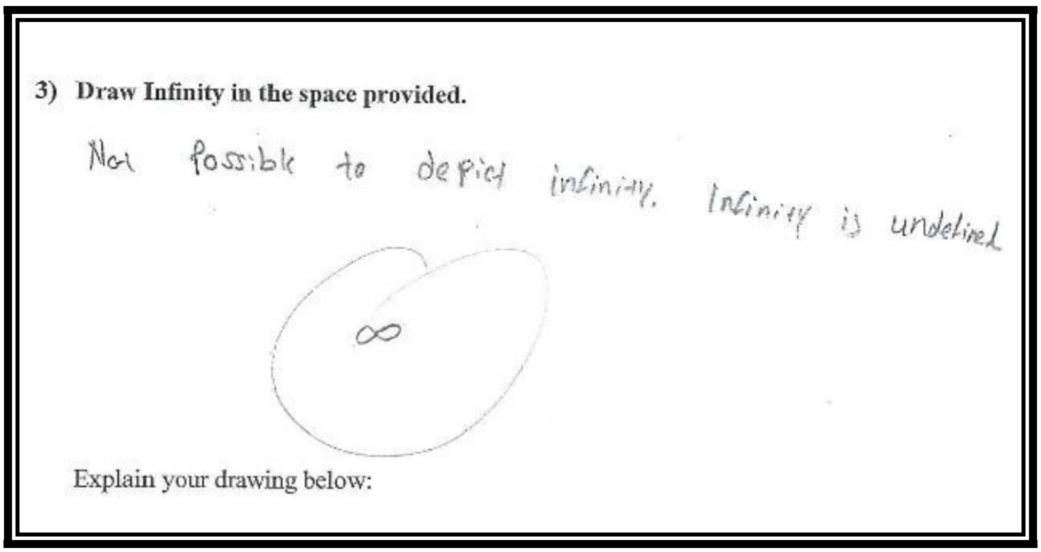
No, the remaining will continue to grow smaller

Illustration 4.14: Process view of infinity (scenario-based task).

32. R: Yeah! You never… you will see it. It just keeps getting smaller and smaller. Cause of the half of that half. Then, half of half will be one-fourth and half of that will be one-eight. Then half will be one-sixteenth. Then just because we don't see it. It just keep going and going and going. So until everything will grow smaller. Which I think goes back to my first point which says; each number has an infinite number of numbers between each point. So I don't think he will ever finish it because it will keep going at that rate. The number will just be smaller and smaller. Would be one over … like some large … large number that will just be getting really, really small, but you really can't see it, it's still there, technically speaking or mathematically speaking.

As seen above, Robin states the Cookie monster will never finish eating the cookie with the following arguments: “It just keeps getting smaller and smaller”, and “It just keep going and going and going”. Robin’s singularity conception is clearly process view.

Task Q3. Robin’s response represents a semi-isolated dominant view of infinity concept at Level 2. He used dominating process view with recessive object view (Po) to address concept-image task. Illustration 4.15 was Robin’s written response. Prior to the interview, Robin had no drawing to illustrate his concept image of infinity, the symbol was added during the interview and he stated the following:


3) Draw Infinity in the space provided.

Not possible to depict infinity. Infinity is undefined

∞

Explain your drawing below:

Illustration 4.15: Dominating process view with recessive object view (concept-image task).

42. R: Oh! Ehm! Well. I guess you can draw aa-mm! Like a sign which we can stand for infinity. But the actual infinity you can’t really draw because it’s without beginning or end. It’s hard to put something without beginning or end on a piece of paper.

43. G: When you say, draw the sign… Ehm! Do I…let me see the sign you are talking about?

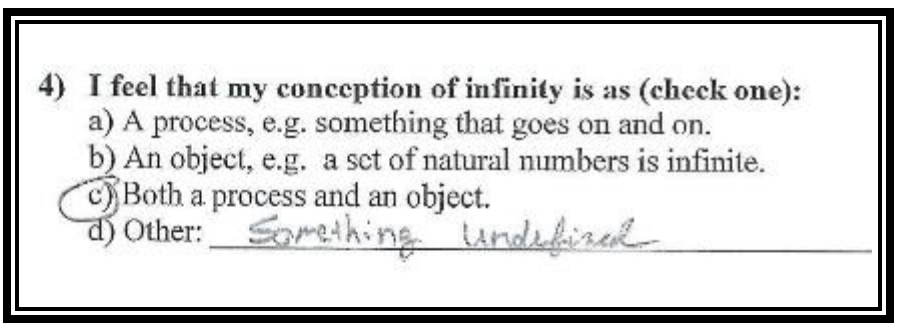
44. R: [Drew the infinity symbol] Yes!

45. G: Oh! Oh. Ok.

46. R: O yes! That represent infinity, but I don't think we can actually draw infinity. I guess that would be kind of akin to trying to represent a four dimensional, I guess object in three dimensional way. It just wouldn't work. Something like that!

As can be inferred from his responses, Robin's concept image of infinity is actual infinity (object conception) which is clearly not related to the sign/symbol. Drawing the sign was an attempt to objectify infinity.

Task Q4. Robin's choice indicated that his conception of infinity is a process as well as an object, which is represented as dual idiosyncratic view of infinity concept at Level 3 in this case. Illustration 4.16 was Robin's choice. When asked to explain why he made that choice, the following discussion took place:



4) I feel that my conception of infinity is as (check one):
a) A process, e.g. something that goes on and on.
b) An object, e.g. a set of natural numbers is infinite.
c) Both a process and an object.
d) Other: Something undefined

Illustration 4.16: Recessive process and object view (multiple-choice task).

49. G: … You chose both… process and an object. Can you explain?

50. R: Yes! I think because… Well in math, I think there's an actual infinity and potential infinity, in which with an understanding in both somehow well. Like I guess in, when we use limit we can actually use the concept of infinity to actually… Ehm! Get actual concrete numbers. But the actual infinity is not really defined. And I guess we all still use infinity when we're trying to calculate aa-mm! Interest in something like aah! Re… ahm! Compounds who move continuously, we really can't do it ourselves to actual infinity, but we use math,

you know the theorem can actually state it's gonna be approaching e, which is an actual number. So this I guess is kind of like a process-object. But still, ehm! Infinity is still not something that we understand completely, because it's not really defined. Like a concept or something. We can't really say infinity is this number right here, because it's not. Because then it wouldn't be going. It wouldn't be infinity.

51. G: Ok! If… I want you to explain your view, your own understanding of the process view and then the object view. Can you do that for me?

52. R: Ehm! Well, I think we could actually use the process view to actually get concrete numbers, like I stated previously. Like we use it, when we use limits to get horizontal asymptote and also we use that with ahm! We can count continuously with actual real numbers. However, the actual infinity is not something that can be understood is … well… without understanding the math I know. …It just fold over itself, you know. But I don't think we can actually define it because it just keep going and going and going. We just understand, since it doesn't stop or ends, it will just tend to infinity.

53. G: So, that's your process…?

54. R: Yes!

55. G: Then, what of your object…?

56. R: Well. I'm sorry. I think the process would have been… I really reverse that pert. Yeah!

57. G: Oh. Ok ha.ahm!

58. R: Yeah! The object aa-mm! You can't really define, but the process you can actually use infinity to get the concrete numbers, is what am saying.

The results from the Table 4.10 also show that there exists fluidity in 3 out of the 5 interviewees' view of infinity expressed in their written responses to the questionnaire tasks and

during the interview. This inconsistency also prevailed with two of these interviewee's responses to the protocol tasks. Shifting from one conception level to another is an indication of a fluid or an unformed concept of infinity. For instance,

Case of Jose:

Jose (J) in responding to the define infinity **Task Q1** wrote "Numbers that goes beyond what we can count on a daily basis". The idea that we are counting makes this a predominantly process view, and that caused him to be rated at Level 1. Below is an excerpt of the interview which explains the shift from Level 1 to 3.

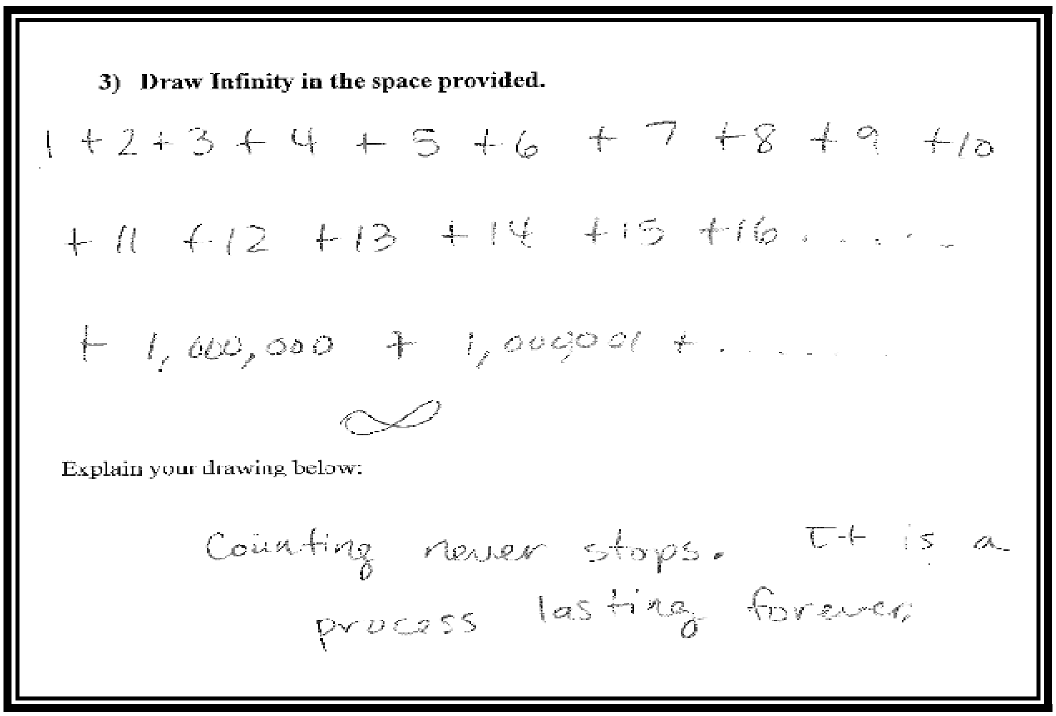
111. I: Do you have anything to elaborate on that or that's still…?
112. J: That still holds true. And when I wrote this, I didn't think of even life. I just thought of… This is the first thing that came to my mind. It was numbers…
113. I: Huh-un! Yeah!
114. J: … And numbers, you know you… you can get one number. Let say 1, and add decimals and decimals. You can put 1.1, 1.13, 1.134 …
115. I: Huh-un!
116. J: … and so forth. So if… you can have an infinite number, all the way up until, let's say for example 1.99999…
117. I: Huh-un!
118. J: … and take that number to infinity.

Thinking of infinity as "life" in line 112 brings his attempt to encapsulate the process into an object. Also the idea of adding repeating decimals non-terminating and taking that number to infinity in lines 114 – 118 suggests infinity as a destination, as a place and thus an attempt of encapsulating the process as a totality, and clearly a conceptual object. But he also says "you can have an infinite number", so using the adjective of infinity, referring to a "number", "infinite

number", this is an object language. His ability to draw on object language in addition to the process language during interview put him on Level 3 (po).

For the **Task Q2**, all of the participants confirmed their reasoning to the various responses written on the survey.

The figure below (Illustration 4.17) illustrated Jose's response to the infinity questionnaire **Task Q3**. He used the counting context to respond to task and this ensued for him a process view at Level 1. As can be inferred from his responses, his understanding of infinity is basically in terms of numbers, counting numbers, adding to numbers which is process dominant. Following the drawing is an excerpt from the interview, which indicated that he maintained the same view.



3) Draw Infinity in the space provided.

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10

+ 11 + 12 + 13 + 14 + 15 + 16 . . . . .

+ 1,000,000 + 1,000,001 + ...

∞

Explain your drawing below:

Counting never stops. It is a process lasting forever.

Illustration 4.17: Jose's response to the infinity questionnaire Task Q3.

164. J: OK. My basic concept of infinity here in this space is… this is just one, when you can take numbers…

165. I: Huh-un!

166. J: … and add them to each other. Getting to a million, billion, trillion and beyond.

167. I: Huh-un!

168. J: And… or you can take fractions….

169. I: Huh-un!

170. J: … and add and add and add and add you gonna get to infinity. You can take, you know, symbols such as the infinity symbol…

171. I: Huh-un!

172. J: … and add infinity, and add infinity, you know, to infinity and beyond.

173. I: Huh-un!

174. J: … or negative infinity or you can even take negative numbers. So whether you have negative numbers or positive numbers, it doesn't make really a big difference. You just gonna add infinity and beyond.

When he was asked what the symbol mean, he said that "It's a mathematical symbol" and continued explanation using the concept of limit in Calculus:

178. J: … that I see. So if we take… Let's say for example in calculus you we take the limit…

179. I: Huh-un!

180. J: … as in approaches infinity…

181. I: Huh-un!

182. J: … then, you get close to the answer, but you're not gonna get to the answer. It's an infinity answer.

Interpreting his responses simply means taking the limit of natural numbers as it approaches the answer, which is infinity, and which according to him is unreachable because "you're never gonna get to the answer". This is a common misconception amongst students. Jose's understanding of limit concept is not well developed. Interestingly, while this discussion

was still going on, this student came up with the idea of taking an interval 0 to 3. This is what he said:

200. J: I mean you think ah! We can only get three numbers out of there, 'cause we are looking at whole numbers?

201. I: I see!

202. J: That's the first thing that comes to at least my mind.

203. I: Huh-un!

204. J: But we can take from 0 to 3 out of the infinite many numbers between 0 and 3.

205. I: Huh-un!

206. J: So we can, whether we are looking at this specific interval, whether you are looking from negative infinity to infinity, you're always gonna have infinity.

Jose, in an attempt to objectify infinity, used infinity to represent the cardinality of the infinitely many numbers between 0 and 3. This ensued for him a process conception as well with a dominating object view (Po) at Level 2.

Case of Vanessa:

Vanessa (V) responding to the **Task Q1** wrote: "A long list of never ending numbers". We interpreted "a long list" as a sequence, hence a process (Level 1). When asked in an interview, this was what transpired:

107. I: Ok. Alright, let me take you back to the, [interviewee laughing as I opened to her survey response] take your mind back to this. It's says when you think of infinity, what comes to your mind? You said a long list of numbers…

108. V: Never ending numbers

109. I: Do you still want to stick to it or you still have more…

110. V: No, no

111. I: … explanation you want to give to that, or you want to explain better to me? When you say long list…

She was specifically asked to clarify the phrase "long list" and she said "I guess I will add just something that never ends, like that's infinity". That sounds different than just saying "a long list". So "something" brings the language of object. Also saying "like that's infinity" is trying to impose the idea of cardinality on "something that never ends". And that's where we interpreted it as a set for example, as compared to "a long list". But still the process view is dominating, hence, her response here was rated at Level 2 (Po).

All of the participants confirmed their reasoning to the various responses written on the survey except for Vanessa in **Task Q4**. Below is an excerpt of the interview with Vanessa:

129. I: Now let's look at this number 4. You're talking about your conception of infinity that is it a process or an object, and you choose a process!

130. V: I think I wanna change my answer to that one. 'Cause I think it can be anything a process… or an object. Like a number like Pi, that goes on forever. And that's a number… an object. And a process is like running a race when we're doing halves or even cookie when we're only eating half every day. So I think infinity can be anything as long as it goes on and on and on and on and on forever.

131. I: Oh! So that's a process? And then the object part is, you said… you give an example of …

132. V: Like Pi, the object could be like Pi. Like a number that never ends or a song like never… like that song that sang never ends or anything really. Just something that never ends. It doesn't matter what it is.

133. I: Okay. Now look at this [the definitions on the multiple choice question 4] so you're good, you're cool with this? The object definition…

134. V: I want to say maybe not like object but for sure like numbers, or infinite. Cause even when you count, that's infinite too, but counting is a process.

135. I: Counting is a process. So, the set of natural numbers is a... is infinite, so you see that as a process also or as an object?

136. V: Hu-mm! No I wanna say it's a process 'cause it's the process of counting. [Pause] Yea! Okay! Never mind. I'm sticking with my answer (a). Yea! It's just something that goes on and on and on.

137. I: So, just… It's a process?

138. V: Yes! Just the process [laughing]

139. I: Well, okay! No o…

140. V: I know. I'm confusing it. It's because that's how I think in my head when I think about the stuff.

Vanessa wrote in her survey response that her conception of infinity was a process but when asked during interview, she kept changing her mind, floating from Levels 1 to 3 and back to 1. From process view to process-object and then back to process. It is obvious she has a dominating process view even though she used a strong object example of pi (π). Her arguments lack internal consistency. This is an indication of a not well formed process-object duality conception. Not accepting it can be both views posits cognitive conflict for Vanessa. She thinks it has to be one or the either.

Case of Susseth:

In responding to the **Task Q1** Susseth (S) wrote: "A large number of concepts, so large that it cannot be expressed mathematically, just conceptually". Although by "a large number", she used the language of object, but similarly to the idea of Vanessa, we interpreted "a large number of concepts" as a sequence of concepts, hence a process. She also said "conceptually" to emphasize the object view and that kind of encapsulate the mathematical process. Clearly, her

perception of infinity as a concept comprises a pretty balanced process-object view (po) at Level 3. When asked to elaborate more on this during interview, this was what she said:

14. S: Um! Well I guess what I meant more was that when it's used mathematically, it's not necessarily used as a concrete number. It's Um! When you're at infinity, its, that's just it. It's like a very, very large number. It's not anything Um… concrete that we can actually use like when we're doing calculations. When you have infinity, well then, it's the end. When you have infinity, you take away small amount you still have infinity. So Um! and then conceptually I think it's … it's difficult to imagine infinity conceptually, but when I hear infinity I first think of it as an idea not as like when use mathematically.

What was fascinating about Susseth's response were the different ways linguistically that she used infinity. She described infinity as something that can be used like a tool (when it's used mathematically), as a location, a place, a point (When you're at infinity, it's the end), as a very large number, as a possession (When you have infinity) and as an idea. All these clearly expressed a dominating object view of infinity. She was also able to perform subtractive operation (action) on infinity. Her interview response was therefore rated at Level 2 (Op).

Case of Emma:

In responding to the **Task Q1** Emma (E) wrote: "Infinity is forever, an amount that cannot be reached or counted". There is an indication of both process and object. "Infinity is forever" is process language. "An amount" captures the object view and "can never be reached or counted" the process-object view. So the statement suggests anything with boundless amount, and clearly encapsulated process-object language rated as Level 3 ('po'). She was later asked to look at her response during interview:

3. I: It talks about when you think of infinity, what comes to your mind. And I like your response, saying "infinity is forever, an amount that cannot be reached or counted". I want you to really throw more light into that for me.

4. E: Well if you count to numbers like a hundred and that sort of thing. And if you keep going, you can actually count to a million, but you can never really get to infinity because it just goes on and on and on. So in that way you can never count to infinity, you can only count to a million. But even then you'll be really tired. So in that sense, infinity goes on forever because you'll never be able to count it, because by the time you get there, you might be dead. [Both laughing]

5. I: Interesting! [Laughing] Ok, so when you say an amount, so does it mean that when you're given a certain amount, you can't reach it or what?

6. E: Yes! Like if I have 5 something that have 5, that I can count to 5. But if someone says am going to give you infinite number of apples, then you can't ever have that many apples because it's too many.

As seen in her responses, she went clearly to process language of counting as a process (e.g., "goes on and on and on"), even with an example when she was asked to elaborate on "an amount", in order to bring her back to that object language that she used in the written response. In a way she plays the object language of "amount" that encapsulates the cardinality by process perception of not been able to reach it. So that's why we see her interview response as clearly a process view at Level 1.

For the **Task Q3**, all of the participants also confirmed their reasoning to the various responses written on the survey except for Emma that shifted from Level 3 to Level 1 again. It should be noted that Emma was posed this task just after her responding to the Task-based interview protocol. Emma drew a line with an arrow on both ends and her explanation to the drawing is that "It's like a line that never reaches a destination much like the actual infinity". This is clearly an object view. Below is the interview excerpt:

59. I: Let's look at… one I like… what you wrote about your drawing; because I saw you draw a line for number (3).

60. E: Uh-hum!

61. I: And I saw the arrows going in both directions. Right? Then it says… Your explanation says “It’s a line than never reaches a destination, but, much like the actual infinity”. What do you mean by that?

62. E: Because, when you’re adding arrow to a line like this, it means the arrow just keeps going and it has no self-stop. So, I believe it’s just like infinity because there is no real end to infinity, it just keeps going.

63. I: Infinity just keeps going? Then what do you mean by actual infinity? Let me just know your understanding about actual infinity.

64. E: I think what I meant by actual infinity is just the, thought of infinity going on forever, and never reaching an end.

65. I: Ok. Going on forever?

66. E: Uh-hum!

The idea of infinity that keeps “going on forever” and “never reaching an end” is predominantly a process view of infinity (Level 1). So, Emma’s conception of duality is fluid: it shifted from Level 3 (as assessed in the questionnaire) to Level 1 (as assessed in the interview). This inconsistency in response suggests the presence of cognitive conflict on the part of Emma. There seems to be a misconception about actual infinity. Emma confused potential infinity with actual infinity when she defined actual infinity as the “thought of infinity going on forever and never reaching an end”.

As can be observed in all of these cases, the interviews revealed more than the participants’ infinity questionnaire responses. (Monaghan, 2001). In summary, the participants’ understandings about infinity seemed to vary from one interviewee to another, as well as from one task to another. Different task corresponds to different types of view by the respondents. There were variations in what tasks the participants considered as infinite tasks and how they used either their process or object views or even both views and ideas to respond to the tasks. Robin was a typical example of this variation.

4.2.2 Results of Research Question 3

3. To what extent does the context of a task impact the college students' conception of duality?

To further investigate the influence of context of a task on college students' conception of duality, this section will present interviewees' responses to the interview protocol tasks. Two tasks were used for the interview to elucidate more on students' duality conception and to answer my Research Question 3. These tasks were similar to the Scenario-Based Task Q2 (The Cookie Monster Problem) but presented in different contexts. The first task on the Task-based Interview Protocol (Appendix C) was Zeno's dichotomy paradox and the second task were five statements about sums of infinite geometric series. The responses to data collected were inductively analyzed using the four mental constructions from APOD framework presented in Chapter 3, section 3.4.4 to categorize student's conceptions into levels and the results obtained was compared to the infinity questionnaire Task Q2 response.

Table 4.13: Task-based interview protocol result.

	Jose	Vanessa	Susseth	Robin	Emma
Interview Task A	1	3	3	1	1
Interview Task B	1	3	1	1	3
Survey Task 2	1	3	3	1	3

As can be observed in Table 4.13, three out of the five interviewees maintained their views used to express their understanding of infinity, with the exception of Susseth and Emma. However, only Vanessa was categorized as having a duality conception of infinity in considering these tasks and whose conception remains consistent in all the different contexts (mathematical and practical) of the problems. The analysis of their responses are presented next with excerpts from the students' interviews:

Task A: *Zeno's Dichotomy Paradox*. If a runner is to complete a race course, he/she must first traverse ½ the distance, then the next ¼ of the distance, then the next 1/8 of the

distance, etc. If all these intervals are traversed, will the runner complete the course? Explain your reasoning.

Task B: Which statement below is true? Explain your reasoning.

A. $1/2 + 1/4 + 1/8 + \ldots < 1$

B. $1/2 + 1/4 + 1/8 + \ldots = 1$

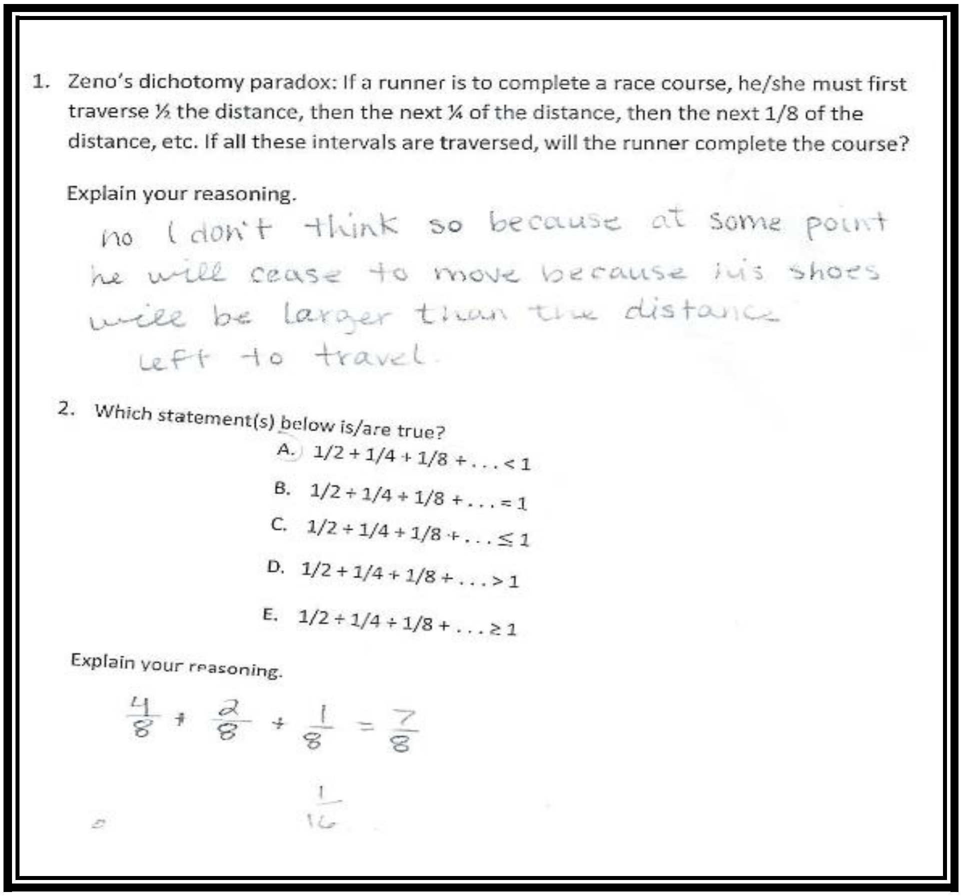
C. $1/2 + 1/4 + 1/8 + \ldots \leq 1$

D. $1/2 + 1/4 + 1/8 + \ldots > 1$

E. $1/2 + 1/4 + 1/8 + \ldots \geq 1$

Case of Susseth:

The interview protocol Task A was presented to Susseth following the discussion of the questionnaire Tasks Q1 and Q2 and the interview protocol Task B was presented to her following the discussion of questionnaire Tasks Q4. These variations were a result of responses by the interviewee. Susseth's responses are provided in Illustration 4.17 followed by an excerpt from the discussion.



1. Zeno's dichotomy paradox: If a runner is to complete a race course, he/she must first traverse ½ the distance, then the next ¼ of the distance, then the next 1/8 of the distance, etc. If all these intervals are traversed, will the runner complete the course?

Explain your reasoning.

no I don't think so because at some point he will cease to move because his shoes will be larger than the distance left to travel.

2. Which statement(s) below is/are true?

A. $1/2 + 1/4 + 1/8 + \ldots < 1$

B. $1/2 + 1/4 + 1/8 + \ldots = 1$

C. $1/2 + 1/4 + 1/8 + \ldots \leq 1$

D. $1/2 + 1/4 + 1/8 + \ldots > 1$

E. $1/2 + 1/4 + 1/8 + \ldots \geq 1$

Explain your reasoning.

$\frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$

$\frac{1}{16}$

Illustration 4.17: Susseth's response to interview protocol Tasks A and B.

Susseth was presented the interview protocol tasks and asked to look at Task B first (Zeno dichotomy paradox). After a long pause of reading and writing down response, the following discussion ensued:

34. S: Ok Um! I don't think that the runner will ever complete the course because Um! So half a distance is pretty long, a quarter of the distance is small and then, um! At some point he'll, Um! I guess if it continued from my understanding of the problem, well I guess his shoes might end up been larger than what's left to travel say one-sixth (1/6), one… one… one hundredth, and one –one hundredth of the distance whatever, and he won't... If he moves his foot to run, he would cover more than he was technically supposed to cover. So I don't think he'll, he'll ever

complete the course because at some point he'll just have to stop moving because the distance left, one, whatever, one thousandth, one-one thousandth of the distance. It will be too large for him to like physically move his foot over.

35. G: Will be too large?

36. S: Well, too small I mean. I'm sorry. So, but if it's just one-half (1/2) one-fourth (1/4) one-eighth (1/8) well then, you know 'cause he'll have to stop. But if it's like continued, I guess the fraction gets smaller and smaller and smaller. Yea at some point he won't be able to physically cover the distance. [Laughing]

37. G: So physically, he won't cover the distance?

38. S: Yea! [Laughing]

39. G: Oh OK. Ok! If that is physically, is there any other reasoning you have?

40. S: Um! Well I guess even mathematically, I just guess the interval will just continue to get smaller and smaller and smaller. Um! I guess. Yeah! If you approach an inf… an… a limit where definitely you might find a point where it's like ok, well the limit equal to zero, so he won't be able to move because … like… that's the limit. [Laughing] I don't know. I'm sorry.

41. G: So in essence he won't complete the course?

Case of Emma:

The interview protocol Task A and Task B were presented to Emma following the discussion of the questionnaire Tasks Q1 and Q2, after which the discussion of the remaining questionnaire Tasks Q3 and Q4 took place. These variations was as a result of responses by interviewee. Following is an excerpt from Emma's responses to Task A:

16. E: I believe that the runner will never complete the course because as you keep running and running, you'll still going to be adding distance, so as you keep reaching smaller and smaller number, the smaller the distance but you're still

running it. So, I think that you're never going to finish the course, 'cause you're still adding more and more distance, as you continue along the fractions.

17. G: Oh ok. So as the pace continues, it would still continue. It doesn't reach an end?

18. E: Yes!

Emma's response is consistent with the characteristics of potential infinity, hence categorized as a process view at Level 1. Next, Emma was asked what she thinks about the second interview protocol Task B. Emma's response is provided in Illustration 4.18 and her explanations are as follows:

20. E: I believe that (B) and (C) are true because as you keep adding the fractions, you eventually get very close to 1 if not equal to 1. And if you keep adding the fractions, then you eventually get a number that's greater than 1. Even if it's just a little bit, it will still be bigger than 1.

21. G: Ok! Wait. Let's take it one at a time. The (B)?

22. E: Uh-hum!

23. G: What do you say?

24. E: I think it's true, because if you keep adding the fractions you will eventually get somewhere, very close, if equal to 1.

25. G: That will be equal to 1?

26. E: Uh-hum!

27. G: Is there any way you can do that, on the paper?

28. E: Well not very close, but I'll run out of fractions.

29. G: You'll run out of fractions?

30. E: Yea! I can't even figure it out any more. I got up to about point ninety seven (0.975). But I believe that if you keep going and going eventually you'll get to 1.

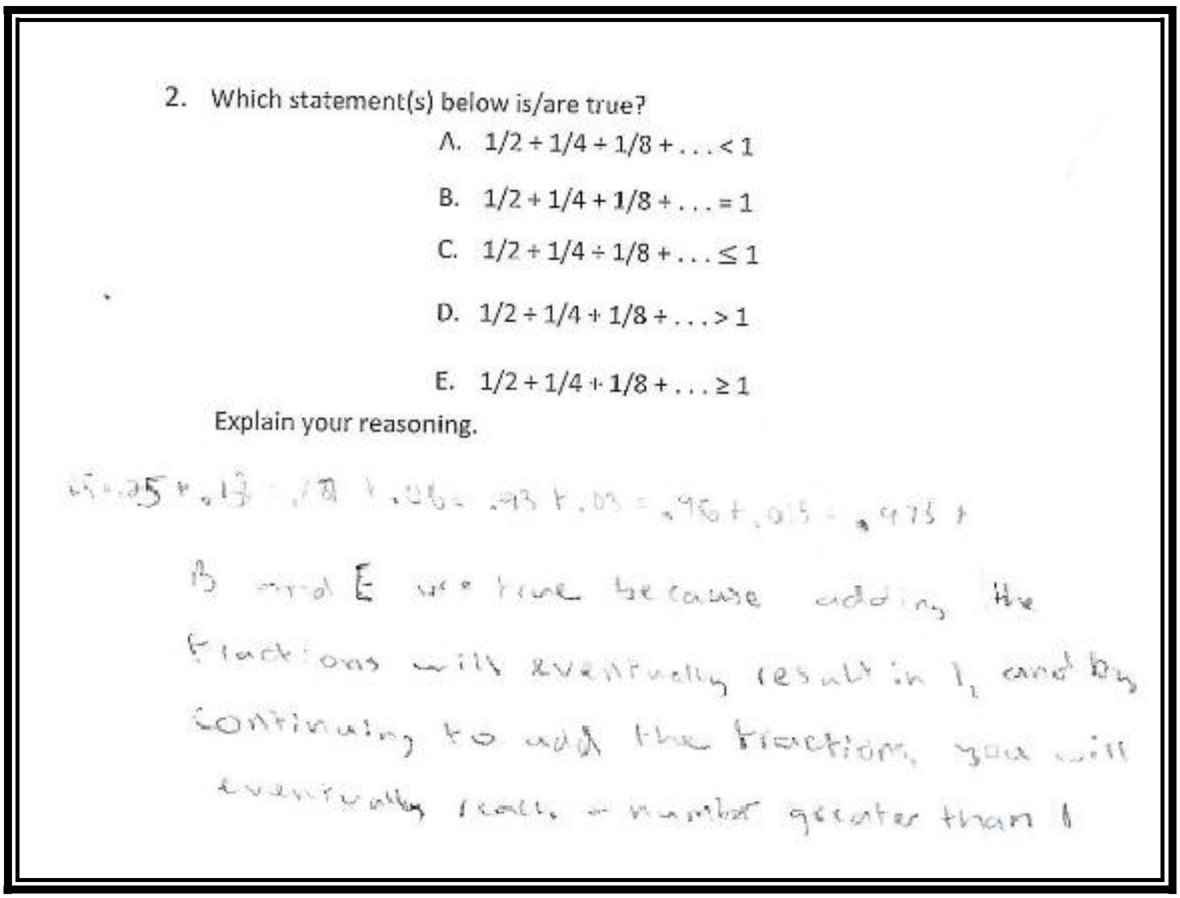
31. G: To one?

32. E: Yes!

33. G: Ok! So that's for (B)? Then what about the (C)? You said (C) also.

34. E: I think that once you get to 1, you'll still have fractions to add, even if it would be like one more millionth. But it would still be another number that you can add, so it'll be number bigger than 1.

Emma responding (B) indicates ability to think about the partial sum as a totality and encapsulate into an object of 1 (object view).



2. Which statement(s) below is/are true?

A. $1/2 + 1/4 + 1/8 + \ldots < 1$

B. $1/2 + 1/4 + 1/8 + \ldots = 1$

C. $1/2 + 1/4 + 1/8 + \ldots \leq 1$

D. $1/2 + 1/4 + 1/8 + \ldots > 1$

E. $1/2 + 1/4 + 1/8 + \ldots \geq 1$

Explain your reasoning.

.5 + .25 + .13 = .78 + .06 = .93 + .03 = .96 + .015 = .975 +

B and E are true because adding the fractions will eventually result in 1, and by continuing to add the fractions, you will eventually reach a number greater than 1

Illustration 4.18: Emma's response – Protocol Task B.

When Emma was asked if the mathematical sign (≤) in (C) means bigger than I, she admitted she looked at the symbol in a wrong way and that she meant to say (E) instead. "It will be greater than or equal to 1". Emma responding (E) indicates limited understanding of partial

sums of series. After Emma settled for those two options, she was asked to look at option (D) and her response continued as follows:

51. G: Ok! Now, do you look at (D) from what you said now?

52. E: (D) could be true because it's in between. It's greater than, but it's not equal to. So it could be true.

53. G: I want you to look at everything very well. [Both laughed] It's because of what you said, you know. It's because of what you said in (E). So am thinking, do you mean that inclusive or not? So let's know. Which are your options? Which ones do you think would be true?

54. E: (B), (D) and (E).

55. G: (B), (D),

56. E: And (E).

57. G: (E). Oh ok! So it will be equal to 1, it will be bigger than 1, and it will be bigger or equal to 1 like you said?

58. E: Uh-hum!

After Emma took a thorough look at the five statements again, she decided on options (B), (D) and (E). Again, Emma responding (D) indicates limited understanding of partial sums of series. Emma's ability to think about the cardinal perception of the given series in option (B) to be equal to 1 made her to be classified as having an object - singularity conception of infinity at Level 1.

4.4.2 Analysis of Responses to Interview Protocol Task B

The second task on the Task-based Interview Protocol (Appendix C) was stated as follows:

Case of Jose:

Jose was asked to take a look at these five statements and explain his reasoning, their discussion goes thus:

8. J: OK. Just as in the runner's case you're also adding fractions here.

9. B: Huh-un!

10. J: And the way I got to reason this was I first took… Um! I found the one-half plus one-fourth plus one-eight …

11. B: Huh-un!

12. J: I found the common denominator to be eight. So the one-half would… equal four-eighth

13. B: Huh-un!

14. J: the one-fourth, two-eighth. And one-eighth, that would equal of course seven-eighth. So that doesn't equal to one? It we add now let say one-sixteenth, then the common denominator would be sixteen. So, one-sixteenth plus two-sixteenth from the one-eighth, plus from the one-fourth you get four-sixteenth. And from the one-half you get eight-sixteenth. This will eventually equal fifteen-sixteenth.

15. B: Ok.

16. J: So this will never equal to 1. And this would be true if you keep adding one-half, just let's say, you know, we go from, you know one-half to one-fourth, to one-eighth, one-sixteenth, one-thirty-two, one-sixty-four. So every time you're adding – or you're multiplying by one-half.

17. B: OK.

18. J: So, you never… just as the runner's case you never equal it to 1. So the statements here that are listed ah! One-half plus one-fourth plus one-eight equaling to 1…

19. B: Huh-un!

20. J: … will never happen.

21. B: OK.

22. J: one-eight… one-eight plus one-fourth plus one-half here is lesser or equal to one. Oh! That statement wouldn't be true since you have an equal sign there. So

that throws it out. The one-half plus one-fourth plus one-eighth greater than 1, that's not true because it's like saying you're getting one whole piece…

23. B: Right!

24. J: … and you're cutting it up. So you're not gonna add to already what you already have as a whole which is 1, so you're only cutting up pieces if you're doing anything. That's why.

25. B: I see!

26. J: And then the last statement, one-half plus one-fourth plus one-eighth greater than or equal to 1 of course, that won't happen because; first of all it's not gonna equal 1…

27. B: Huh-un!

28. J: … and it's not gonna be greater than 1.

29. B: OK.

30. J: So the only statement true here is (A) which is one-half plus one-fourth plus one-eighth plus whatever on, less than 1.

As can be inferred from his responses, he was able to perform every steps of the iteration to compute the sums of the series and his conclusion is that (A) is the only statement, which means the statement is going to be less than one. This indicated clear process view.

Chapter 5: Discussion and Conclusions

In this chapter the methodological contribution, discussion of the results obtained in Chapter 4, summary of the study, implications of the study, limitations of the study and some recommendations for future research are presented.

5.1 Methodological Contribution

The data analysis from this research study contributes methodologically to the field of mathematics education by its ways of modifying the APOS framework developed mostly from Dubinsky's explanation of Piaget's notion of Reflective Abstraction (Dubinsky & McDonald, 2001) and applied by Dubinsky et al. (2005), to explain how the concept of infinity may be conceptualized. Their explanations expressed in terms of the mental mechanisms of interiorization and encapsulation. Besides using the grounded theory, this study also used the modified Action-Process-Object-Duality (APOD) framework adapted from APOS theory to analyze students' responses to the infinity questionnaire tasks and the interview protocol tasks. It enabled an improved interpretation of students' perceptions of infinity as either a process or an object, and especially to organize the PROCESS-OBJECT view into two major views – the dominant views and the recessive views, providing levels of a student's duality conception while examining in depth the students' conception of infinity for particular tasks. More specifically, the APOD framework enabled me to elucidate a better strategy that could guide researchers in categorizing students' views of infinity into different levels to assess students' duality conception of infinity. However, the modified APOD framework is just an additional dimension to the APOS theory and by no means exhaustive of all the dimensions of views students elicited in their responses. More research should be conducted to further extend this framework (e.g. via technology).

5.2 Discussion of the Results

As outlined in Chapter 1, the purpose of this study is to examine college students' conception of duality in representing infinity - to determine whether or not the college students

possess a dual process-object view of infinity. More specifically, how could students' conception of duality be assessed? To investigate college students' conception of duality, the research questions guiding this study are addressed in this section.

1. How is the duality conception externalized and expressed by college students at each course in the Calculus sequence?
2. To what extent do the types of task impact the college students' conception of duality?
3. To what extent do the contexts of task impact the college students' external representation of infinity?

These research questions were developed by the need to examine college students' conception of duality in representing infinity with the intent to elucidate strategy that could guide researchers in categorizing students' views of infinity into different levels. From reviews of the literature, it is known that concept of duality as any other fundamental ideas of mathematics are "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 151). This research study examined the process-object duality conception of infinity of the students registered at different level of the college Calculus sequence course, using different tasks. Three major tasks were used to assess college students' process-object duality conception of infinity in order to determine their perception towards the nature of infinity. Tasks Q1 and Task Q2 were used to engage college students in externalization of their concept images and concept definitions of infinity as duality because they relate to individual's cognitive structure associated with the concept, and they have the potential to reveal the particular conception of infinity and the associated misconceptions of infinity that the college students may hold. As indicated by the participants' responses to these two tasks, students' conception of infinity is predominantly singularity conception, and a majority of the college students in the Calculus sequence course comprehend infinity as a process. This finding is in accordance with the work of many researchers who studied students' conception of infinity (Monaghan, 2001; Tall & Tirosh, 2001; Kattou et al., 2010) who contend that the individuals'

intuitive interpretation of infinity as potential posits a cognitive conflict, leading to many students having incomplete and inconsistent concepts of infinity "and individual written responses showed a wide variety of evoked concept images riddled with conflicts and inconsistencies" (Tall, 1992). The individual intuitive interpretation also hinders the schematization of process-object duality of infinity.

In the light of the findings of this research study associated with the students' process-object duality conception of infinity, the following can be deduced:

a. There is diversity and variation among students' process-object perceptions.
b. Students used different context to imply infinity.
c. Students were subject to contextualization in an attempt to reduce abstraction.
d. Students used idiosyncratic views to externalize infinity.
e. Inconsistency in students' responses posits cognitive conflicts and suggests they have limited schema of infinity.
f. Students' conception of infinity is predominantly singularity conception.
g. Students' perception of infinity as a number may be seen as a process or as an object.
h. The traditional Calculus sequence promotes a singularity conception as opposed the duality conception.
i. Students' arguments lack internal consistency.
j. Students' duality conception is task and context dependent.
k. Students used alternative conception hailing from an "intuitive rule".
l. Students demonstrated underdeveloped concept of partial sum of series.
m. Students experience challenge to physically represent geometrical problems on the concept of infinity, and misconceptions about infinity symbol.

These findings are organized into three subheadings for discussion. Each of these findings will be discussed.

5.2.1 Assessing Students' Conception of Duality

Diversity and variation among students' process-object perceptions. Results revealed that students who demonstrated a balanced view of both process and object mostly used idiosyncratic views to externalize their infinity concept. Among the college students that elicited the process-object conception, I observed that either the process or the object view was dominating. Some students tend to have a stronger view of object than the process, or a stronger view of process than the object, or rather both the process and the object views are not strong to elicit the notion of infinity. This resulted in three recessive sub-categories of the process-object views of infinity: PROCESS-object view (Po - case where the process view is dominant and the object view recessive), OBJECT-process view (Op – case where the object view is dominant and the process view recessive), and process-object view (po – case where both views are recessive). The variations in the students' process-object views were observed during the analysis of their questionnaire response, and it was furthermore revealed during the interview by the way students focused on using a particular view more to explain their responses to the questionnaire tasks. This study found that while students were interiorizing action into process, some of the students' emergent process view was strong while some was not. The not strong emergent process view represents the recessive process view at this stage, and is referred to as the students' idiosyncratic process view (p), which when encapsulated, becomes the students' idiosyncratic object view ('o'). The moment students perceive the process as a totality and perform an action on the process, the process is then said to have been encapsulated "into a cognitive object" (Dubinsky et al., 2005a, p. 339).

Dubinsky et al. (2005b) suggest that instructional strategies "should focus on helping students to interiorize actions repeated without end, to reflect on seeing an infinite process as a completed totality, and to encapsulate the process to construct the state at infinity, with an understanding that the resulting object transcends the process" (p. 264). Selden (2002) posits that "in order to be able to deal with mathematics flexibly, students need both the process and object views of many concepts, as well as the ability to move between the two views when appropriate"

(p. 10). This study found that there is diversity and variations in the college students' duality conceptions. There were a variety of process-object views being reported indicating students' as having underdeveloped conceptual understanding of infinity concept and a limited duality conception. Dubinsky et al. (2005a) proposed that the process-object duality conception of infinity and the relationship between the process conception and object conception can contribute to an individual's infinity schema. I hereby make the same claim that there is need for college students to understand and develop the process-object duality conception which relate to both potential infinity and actual infinity in order to improve their infinity schema and deal with other concepts of mathematics.

Conception of infinity is predominantly singularity conception. As discussed in the previous section, it was observed that most of the students who used both the process and object views to discuss their responses to the infinity tasks tend to use one view more than the other, and the dynamic notion of infinity seems to be prevalent. These dynamic notions were strongly associated with singularity conception of infinity in the infinity questionnaire. It can also be observed in Figure 4.10 that singularity conception of infinity was prevalent among the research participants, and the gap between those having a singularity conception and those having the duality conception continued to grow as they advanced in the Calculus coursework. This suggests that the traditional Calculus sequence promotes a singularity conception as opposed to a duality conception. However, as shown in Figure 4.9 there is an indication that there is an opportunity to promote duality conception in the early Calculus courses.

The students' explanation of what comes to their mind when they think of infinity is their personal concept definition of infinity, and is a product of their personal experiences with the concept of infinity and partly or fully the explanation of their concept image as well (Vinner, 1991). The define infinity task generated a large variety of responses by the students. Results of responses to Questionnaire Task Q1 and levels of students' conception are displayed in Table 4.1 and 4.2. One of the main purpose of this research study was to examine how college students understand infinity as a process and as an object by using the framework of APOS. Overall

across all tasks, it was observed that a majority of the responses were categorized as singularity conception at Levels 1 and 2 from Table 4.1 and Table 4.2 whereas one tenth were categorized as duality conception at Levels 3 and 4.

Students demonstrated underdeveloped concept of partial sum of series. With regards to working sub-tasks Q2a and Q2b correctly, it was observed that less than a quarter of the college Calculus sequence students that participated in the study could complete the first two sub-tasks of Task Q2, which ask how much of cookie the Cookie monster has eaten in seven days and how much was left? These questions are to provide the introductory knowledge required to construct further knowledge, as the ability to complete these tasks validates that students are able to complete the rudimentary exercises associated with the significance of finding an infinite sum. Sub-task Q2c which asks that if the process continues, will cookie monster ever eat the entire cookie was meant to determine students' having the process-object view of the phenomenon. Participants responding "No" to the Cookie Monster problem have a predominantly 'process' view of infinity. The action of eating half of cookie remaining can be imagined to continue indefinitely. This type of singularity conception of infinity by students signifies potential infinity. Among the five interviewee, only one person got the correct answer to the sub-tasks (a) and (b). Many of the participants did not fully come to understand the sub-tasks Q2a and Q2b, and many played avoidance in resolving the mathematics by providing textual answers. This study revealed that students demonstrated an underdeveloped concept of partial sum of series and limited understanding of word problems involving fractions. Students experienced difficulties in retrieving textual information from Tasks Q2a and Q2b. They had not developed the concepts-in-action necessary for conceptualizing and to connect the situation to a formal mathematical structure, in order to perform the relational calculation. It can be concluded that the problem solving skills of some of these college students are not well developed and there is need to develop a cognitive tool that can help students identify and organize given information in a word problem. Bautista and Mulligan (2013) reaffirmed that before solving problems, students should be able to (1) state what is asked, (2) state what is given, (3) identify word clues, and (4) specify

the correct operation to be used (p. 224). This result is in line with the findings by other researchers (Monaghan, 1986, 2001) who contend that the first year of calculus course does transfer limited understanding of the nature of convergent series, real numbers, infinity and the concept of limits to students.

Students were subject to contextualization in an attempt to reduce abstraction. The ability to abstract is a very vital tool for engaging in meaningful mathematics (Hazzan, 1999). It was observed that students engage in reducing the level of abstraction when solving the Cookie monster problem Task Q2c. Students are reducing abstraction when they can use familiar procedures to make sense of unfamiliar problems and make abstract concepts more concrete. While reducing the level of abstraction, students tend to ignore the meaning of the stated circumstances in the problem and cling to the familiar mathematics entities. By reducing abstraction, students connect mathematical concepts to real life situations (Wijeratne & Zazkis 2013). Students' in responding to the task as to whether Cookie monster will eat the entire cookie if the process continues reduced the level of abstraction. More evidence of this was seen in Susseth who made use of the familiar procedures (splitting elements and atoms) from chemistry class to make sense of the unfamiliar situation of the cookie monster. During the interview, instead of her referring to the unfamiliar action of eating half of cookie remaining and the completed infinite process of eating half of cookie remaining, the procedures which seem difficult to conceive, she attempted to reduce the level of abstraction by considering (familiar) flour and butter and whatever items that make up the cookie. Susseth's responses to Zeno's dichotomy indicated a clear contextualization. (See Section 4.2.2, Illustration 4.17 and Appendix F, Lines 17-18, 34). The scenario-based task (paradox) revealed the naïve and emerging duality conceptions of infinity of the students (Wijeratne & Zazkis 2013). Although the student was able to express her dual view in two contexts, (mathematically – No, he will not eat the entire cookie and physically – Yes he will eat the entire cookie) there was no strong supporting argument except within the physical context.

Students experience challenges to physically represent geometrical problems on the concept of infinity. Another finding is that representing a physical geometric problem concerning the concept of infinity posits a challenge. In the draw infinity Task Q3, some students did not draw anything and made statements such as "it cannot be drawn", "undefined", and "not enough space" (e.g. PC032, C1069 and C2013). This result resonates with the findings by other researchers. For example, in a study on finding out how primary teacher students who received no in-depth instruction on abstract mathematical content understand different types of infinity such as: infinitely large, infinitely many and infinitely close, Kolar and Cadez (2012) found out that a majority (85%) of the students accepted the fact that the sequence of ever expanding squares is finite while 29% of the participants believed that the sequence of ever shrinking squares is infinite, reason being that "if we draw a square on a sheet of paper and continue drawing smaller and smaller squares within it, the process will end at the point where we are no longer able to draw even smaller square within an existing one" (p. 402). Their result is supported by the findings of Tall (1999) and they contend that it may be "due to the intrinsic limitations of the physical world – paper" and indicate "difficulties related to the physical representation of geometrical problems on the concept of infinity" (p. 403).

Misconceptions about the infinity symbol: The result to Task Q3 seems to yield a large percentage of responses that could not be determined. The reasons for this has been explained in Chapter 4. Moreover, most of the drawings that do not have explanations are the drawing of the infinity symbol that are not supported by any explanation. There are lots of misconceptions about the infinity symbol. Many students have a vague understanding of the concept of infinity and its relative symbol, and this ambiguous misconception has formed their concept image. This research study found that one of the epistemological obstacles that college students need to overcome is the idea of infinity as symbol. Sfard (1991) argues that most researchers who suggested the notion of duality "rarely gave much attention to the question of tacit philosophical assumptions underlying any mathematical activity; rather, they referred either to certain more obvious aspects of the subject-matter…or to the cognitive processes involved in handling the

knowledge" (p. 8). We believe that the examination of the process-object conception of infinity presented by Monaghan (2001, p. 245-246) does not fully address the complex nature of the infinity concept. Duality as a fundamental hidden idea is not explicitly presented. Monaghan takes for granted the explicit representations in determining students' view of infinity by using obvious cases. While coding the students' responses to Task Q1 and especially Task Q3, it was realized that students' idea of representing infinity with the symbol is predominantly a process conception. Their explanations to the drawing of the infinity symbol is mainly associated to the idea of tracing the line of the symbol or following the drawing of the symbol, it is never ending but continues forever (e.g., Section 4.1.3, C1074, C1042, C3011, PC012, PC034). Some other students say "it has no beginning and no end". It is clear that these students have distinct singular process view of infinity. All the explanations students gave about this symbol indicated process conception. The only exceptions are those that say that's how they know it in their math class (PC059, C1106). If Kolar and Cadez's (2012) interpretation of the symbol ∞ as representing the concept of actual infinity is blindly followed, then one can conclude that this implied for the students to have an object view of infinity. Kolar and Cadez (2012) stated that "We believe that it represents the concept of actual infinity and indicated the awareness of the respondents about the infinite amount of numbers" (p. 404). I claim that the symbol itself does not represent or explain students' view of infinity; the explanation to the symbol or the narratives provided by the students to elucidate their views determines the view of conception. Tall (1992) reported the outcome of researches (Wheeler and Martin, 1987, 1988) conducted with elementary pre-service teachers in a large university enrolled in an upper-division course in mathematics methods. These teachers were asked to explain what the symbol ∞ and the final three dots in the expression "1, 5, 52, 125, 625, …" mean. The results showed that half of the pre-service teachers were not familiar with the symbolism and their responses to the meaning of the three dots predominantly evoked potential infinity. Their responses include: "unending process", "the numbers go on without stopping", or "no matter what number you say, there is always one greater simply by adding one to it". Several students in my study gave the same types of

explanations. Ueno (2004) used the 'basic metaphor of infinity' to present Lakoff's cognitive method of explaining several concepts of infinity in mathematics and also to assess its meaning in mathematics education. He explained how people make wrong use of the infinity symbol to interpret an infinite sum as the limit of partial sums. They take infinite sum as a "result of adding an infinite number of terms', a sum 'up to the ∞th term'" (p. 56) the ∞th term being the limit. This is very impracticable as the ∞th term' does not exist.

5.2.2 Linguistic Analysis in understanding the Duality Concept

Monaghan (2001) warns about the methods researchers use in assessing young people's ideas of infinity. She stated that care needs to be taken when interpreting the "forms of words people use that may go beyond the concepts they have" (p. 246). Understanding that meaning and language are intertwined (Kvale & Brinkmann, 2009), besides the APOD framework, the mode of interview analysis employed by this research study was the meaning coding which is supported by linguistic analysis to fully capture the intent and meaning represented by college students in their written responses to the infinity questionnaire tasks and their responses and arguments presented during the interview and to categorize them appropriately as either process, object or process-object. After a thorough linguistic analysis of students' responses by way of checking for the use of grammar, personal and impersonal pronouns, nouns and use of metaphor to better understand students' object of encapsulation, this study revealed some strong metaphors that college students use to externalize their duality conceptions of infinity: infinity as a number, infinity as something, infinity as undefined, and infinity as a mathematics symbol. These metaphors posit challenge in understanding students' conception to successfully categorize them appropriately as process, object or process-object. Monaghan's (1986) states that "The best way to examine subjects' ideas on infinity as a process is to examine their responses to questions in interviews" (p. 199). The second phase of data collection which employed interviewing helped to overcome this challenge, by checking the consistency and frequency of participants' process and

object language put to play to elicit their views to a particular task. In what follows, some of these shall be explored.

Infinity as a number: According to Lakoff and Núnez (2000) it is meaningless to think of infinity as a number because a number n equal to ∞ in the equation $n = \infty$ "means nothing". They argue that the "symbol ∞ means nothing at all except in the phrase "tends to infinity" and "approaches infinity" and that since there are three different cognitive uses of numbers, ∞ as a number is used in enumeration and comparison and not in calculation. For example in their study, ∞ is assumed to be an endpoint in an enumeration 1, 2, 3, …, ∞, meaning "larger than any finite number and beyond all of them". Jose in my study gave a similar illustration in Figure 4.11. He used ∞ as the endpoint to his numerous additions of natural numbers, saying "*you take the limit as in approaches infinity*" and "*you get close to the answer*, *but you're not gonna get to the answer. It's an infinity answer*". Considering the phrases "approaches infinity" and "It's an infinity answer", this then means Jose consents to the addition of terms at every step of iteration Figure 4.11 and also by the concept of actual infinity encapsulated the process as an object to be ∞. Jose was able to think of the series as a totality or as a complete whole (Dubinsky, et al. 2005b; Kattou, et al., 2009; Monaghan, 2001). The idea of taking the limit of natural numbers as they approach the answer, which is infinity, and which according to him is unreachable because "you're never gonna get to the answer" constitutes a cognitive conflict and indicates actual infinity contradicts many of the students' intuitive ideas of infinity (Jirotková and Littler (2004).

Jirotková and Littler (2004) investigated 44 Czech and 54 English students' understanding of infinity in geometric context using a series of seven tasks to explore the mental processes students used when they are thinking about infinity. These students were 11-15 years of age. After describing each child's hypothetical statements in the task their results claimed that more students 75 % of Czech and 59 % of English students considered the idea of two infinities choosing Adam, and 38% of students did not observe any contradiction between actual and potential infinity, indicating their understanding of infinity was not clear. They found out that most pupils are more comfortable discoursing about infinity in the context of numbers, which is

also evident in Monaghan's (2001) research. The results also showed that from age 12 onward the students tested did not show stability of intuition of infinity when the contexts were changed. This is contradictory to Fischbein, Tirosh & Hess's (1979) research. They suggest the use of different contexts to better help students in their understanding of the infinity concept. This study identified infinity as a "*number*" or "*numbers*" as the fifth most frequent word in students' definition of infinity. This indicates that the result of this study resonates with these and other studies (Fischbein, Tirosh & Hess, 1979; Jirotková & Littler, 2004). Students used expressions such as "infinity is a very large number" or "a very huge number". This metaphor is consistent with the findings by other researchers (Fischbein, Tirosh & Hess, 1979; Monaghan, 1986, 2001; Oehrtman, 2003), who considered the ability to conceive of infinity as "*number*" to be an indication of object conception. Take the example in my study, as discussed in Chapter 4. When Jose was asked to elaborate more on his response to Task Q1, he said "This is the first thing that came to my mind. It was numbers" but then he went further to buttress his idea of numbers by taking a number and adding decimals and decimals repeating, "and take that number to infinity". This non-terminating nature of an infinite operation indicate a process view and the idea of taking the number to infinity could be interpreted as a language of encapsulating process to object. This result suggests that "infinity as a number" may be seen as a process or as an object, depending on the views students expressed in their narrative or explanation for working the task. This means that even though students may say "infinity is a number", this does not rule out 'infinity as a process' coloring their minds (Monaghan, 1986). Prior to this question, Jose was started with the interview protocols and so was able to explain his reasoning and to the question of what his first encounter with the concept of infinity was. He posited "anything can be infinity, we are infinity", also using the example of "a reproductive kind of system". His use of process and object language, suggests from this study that irrespective of how the students perceived infinity as an object, most of them also saw infinity as a process. This is also evident in (Monaghan, 1986, p. 199) when the subjects were asked to compare the cardinality of the set of natural numbers with that of the even numbers.

Another example is Susseth who stated that infinity is “a large number of concepts, so large that it cannot be expressed mathematically, just conceptually”. As mentioned in Chapter 4 (Case of Susseth, Appendix F, Line 14), I was intrigued by the way Susseth linguistically expressed her views of infinity. She described infinity as something that can be used like a tool by her statement “*when it’s used mathematically.* She described infinity as a location, a place, a point and a final resultant state (Lakoff & Nunez, 2000) by her statement “*when you’re at infinity, it’s the end”.* She described infinity as a very large number or as a possession by her statement “*when you have infinity*”, and she also described infinity as an idea. All these clearly expressed a dominating object view of infinity. She was also able to use the subtractive verbal operation (action) on infinity to express how infinity cannot be used mathematically as an object (something concrete). She said: “It’s not anything Um… concrete that we can actually use like when we’re doing calculations”. She used negation to affirm that she did not agree to the object conception of infinity in the context of mathematics calculations, but that still does not rule out infinity as a process being in her mind (Monaghan, 2001). She said further, “When you have infinity, well then, it’s the end.” This means according to BMI she was able to conceive infinity as the final resultant state. Also, “When you have infinity, you take away small amount you still have infinity”. This calculation can be expressed as $\infty - a = \infty$. Her sophisticated way of renouncing the “senseless expression” according to Lakoff and Nunez (2000, p. 165) clarified her views and positioned her at Level 2 (Op). Meaning “a very, very large number” does not suggest infinity as a mathematical object but infinity as a very long number, as in the case of Vanessa (a lost list).

Infinity as something: Ueno (2004) differentiates between Aristotle’s potential infinity and actual infinity. He used several dynamic words to define potential infinity: process that is ongoing, conditions that something has no end or some indefinite repetitions of action, “motions without end” “a situation which continues endlessly” (p. 55). Actual infinity he refers to as a ‘thing’ or that kind of infinity that we feel is position in space as a ‘real thing’. Sfard (1991) also states that “Seeing mathematical entity as an object means being capable of referring to it as if it

was a real thing – a static structure, existing somewhere in space and time." (p. 4). The mechanism of metaphor permits us to conceptualize the result of an infinite process. BMI adds metaphorical completion to an ongoing process, so that it may be conceptualized as having a result. That is becoming "an infinite thing" (Lakoff & Nunez, 2000). In this research study, Figure 4.1 in Chapter 4 indicates a majority of the participants refer to infinity as something. The use of the expression "*It is something that continues forever*" indicates "*something*" was concrete object. The words "*continues*" and "*forever*" in this case are process language and dominating, thus the student's view represents Level 2 - PROCESS-object view of infinity. This strong metaphor was exemplified in Robin, analyzed in Chapter 4, Illustration 4.13, and Lines 24-28. This study showed that participants who used the phrase "*something*" predominantly used process language to describe infinity. This indicates that students think more in terms of potential infinity (Jirotková & Littler, 2004).

Infinity is a concept: Gray and Tall (1994) define the symbolism that intrinsically represents the amalgam of process and concept ambiguity a '*procept*'. The procept theory defines concepts as an object by reason of encapsulation and thinks of mathematical entities in terms of process and objects (Tall, 1991). Figure 5.1 is an example of response 'other' to Task Q4. This is what student C1033 stated in this figure: "*I believe that infinity (∞) is something that goes on and on, but it is not a process, it's a concept*". This idiosyncratic statement is very problematic and contradictory, at the same time reveals a strong misconception of infinity. The student believes infinity goes on and on but yet assumed it not to be a process, hence he did not chose option (a). However, because this student could draw on the language of process and object, taking infinity to be a concept and not object, we found analyzing this student's response by APOS framework as duality conception problematic (Kim, Sfard & Ferrini-Mundy, 2005). Moreover, analyzing the response using the modified APOD framework, this student fits perfectly in Level 3. This level represents the dual-idiosyncratic view of infinity, where both process and object views are recessive (i.e. not strong or convincing). Therefore, we are convinced that the modified ADOP framework is a better instrument for analyzing and

categorizing students' immature views of infinity into different levels to assess whether students' actually possess a duality conception of infinity.

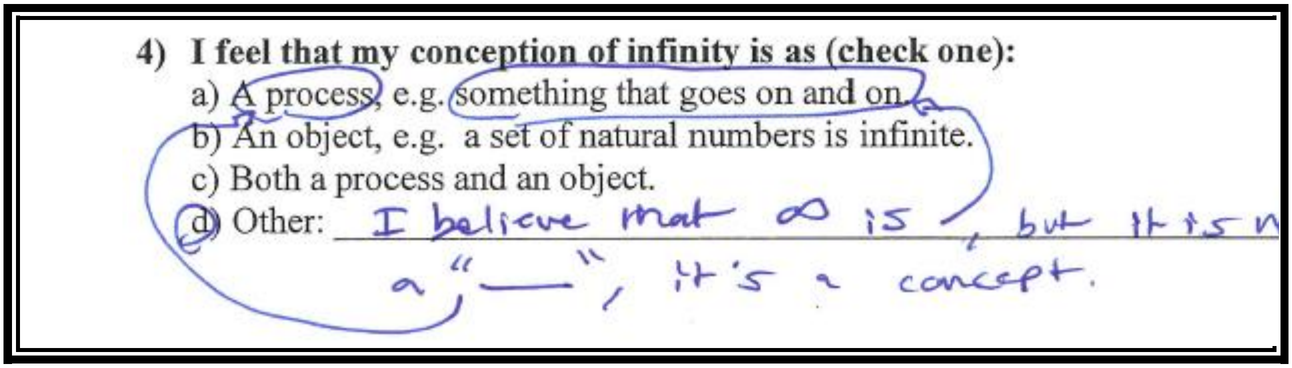

4) I feel that my conception of infinity is as (check one):
a) A process, e.g. something that goes on and on.
b) An object, e.g. a set of natural numbers is infinite.
c) Both a process and an object.
d) Other: I believe that ∞ is , but it is n
a "—", it's a concept.

Figure 5.1: C1033's Response to Task Q4.

Influence of language, intuitions and everyday knowledge: The influence of language on everyday experience has been discussed by many researchers (Cornu, 1992; Davis & Vinner, 1986). Students in Jirotková and Littler's (2003) study used the noun infinity without any clear differences between the two words 'infinity' and 'infinite'. Researchers found it difficult to determine whether or not the students could actually differentiate between which is actual infinity and which is potential infinity. Monaghan (2001) argues that students mostly think of infinity as a process, and even when they use the expression such as 'going towards infinity', it does not necessarily mean they are thinking of infinity as an object or as actual infinity. Among the common metaphors students use when discussing infinity which evolving in this study are: *actual infinity, something undefined, to infinity and beyond, approach infinity, taking the number to infinity, tend to infinity,* and *towards infinity*. Students often use these phrases and statements without actually knowing how to express themselves or how to convey what they mean by them. Also, recognizing students' conception of these statements is difficult because of their nature. These statements could be considered from the point of view of actual infinity or potential infinity, as is evident in many studies (Kolar & Cadez, 2013; Monaghan, 2001). Some of these statements are as a result of everyday experience with social media and they appear to interfere with students' understanding of infinity, because of the concept image they create in students

(Davis & Vinner, 1986). Jirotková & Littler, (2004) claim that lack of development of college students' intuitive ideas of infinity is a contributing factor to the difficulties that students experience with the concept of infinity. They believe that students' intuitive concepts of infinity are gained from personal experiences and that often the tacit models the students build up are inconsistent.

Infinity is undefined: Of particular interest to this study is "*infinity is undefined*", which resulted in another conflict for students. Robin for example in the define infinity task - Task Q1 began by saying infinity is "something undefined". When asked to clarify what he meant by "undefined" his response was "undefined because it's not an actual number per se" and again he said "infinity is not actual number we can add" (Chapter 4, Section 4.2.1, Appendix G, Line 28). This result is compatible with the findings of Monaghan (1986) who when asked one of his subjects "What is 1/0?" said ∞ in the first interview and 'undefined' the second time. This is the explanation by the subject (DGM): "Well again, if you think of it as the highest number you can get, then you can add one to it and get a higher number. So there's no numeric answer to it." Colloquial use (Kim, Sfard, & Ferrini-Mundy, 2005) of the word "actual" seems to hinder students' understanding of the concept of actual infinity, as seen in Emma's responses: "what I meant by actual infinity is just the, thought of infinity going on forever, and never reaching an end" (Appendix H, Line 64). This in not actual infinity but potential infinity. This is also strongly evident in Robin's responses analyzed in Chapter 4. (Appendix G, Lines 42-52).

5.1.3 Challenges in Assessing Students' Conception of Duality

This study revealed some challenges in assessing the college students' duality conception of infinity, some of which are in line with what other studies have found. Students' duality conception is task and context dependent. Students used different context to imply infinity. The Cookie monster problem was presented in a real life or practical context, which makes it more fascinating for the students to engage with. The context in which a student perceives a given task or situation and in which he/she considers to represent his/her responses determines the

conception that is elicited. Students used practical context more in discussing infinity in the Cookie Monster problem subtask Q2c, especially to convey their object conception. Students with duality perception used more than one context. Such contexts include: mathematically, theoretically, technically, practically and realistically. A student majority used real life experiences to provide responses. Such responses are listed in Section 4.1.2 (C1054, C1065, and C1122). This again, as revealed in other studies, may be due to the fact that the students' experiences are linked to a finite reality and students often depend on the finite world to tackle infinity problems (Kolar & Cadez, 2012). Even among those who claimed mathematical context, only a few used the concept of convergence of series and limit to elucidate their responses, and not even explicitly. Again, students avoided resolving the problem mathematically. The competing notions of infinity through the influence of context observed in Ruby's responses present a potential cognitive conflict and suggest the conceptual understanding of the infinity concept depends on developed and dominant process-object duality conception.

As analyzed and discussed in Chapter 4, this study found that depending on the task of the survey instrument, a different view was being expressed by the participants. This posits challenges for interpreting students' perceptions of infinity as either a process or an object, and especially in determining the students' process-object duality conception. Some researchers have found that the type of tasks given to students can significantly influence student responses (Kolar & Cadez, 2012; Monaghan, 2001). The type of tasks given to students may activate different aspects of a student's concept image resulting in inconsistent responses. Tall and Vinner (1981) affirmed that different aspects of a student's concept image may be invoked based on the task presented. This study found that the type of tasks given to the college students triggered different facets of their concept images resulting in inconsistent responses. Cognitive conflicts emerged between the idea of having entirely eaten the cookie and never ending process conception of eating half of cookie remaining. This inconsistency in students' responses posits cognitive conflicts and suggests that they have limited schema of infinity. I argue that infinity schema that organize the process and object duality conception of infinity would enable the college students

to identify that the paradox and partial sums of infinite series are somewhat normatively recognized as cases of both actual infinity and potential infinity, rather than just potential infinity.

5.3 Summary of the Study

This research study examined the conception of duality in understanding infinity of 238 college students enrolled in the Calculus sequence courses (Pre-Calculus, Calculus I through Calculus III) at one of the southwestern universities in the U.S. who volunteered to complete infinity questionnaire tasks, with the intent to elucidate strategies that could guide researchers in categorizing students' views of infinity into different levels. The study employed two instruments to gather both quantitative and qualitative data that were collected in two distinct phases: (a) a self-reporting questionnaire given to college calculus students during the sampling stage and (b) semi-structured individual task-based interview protocol, analyzed in Chapter 4. After the infinity questionnaires were collected and analyzed by three independent experts using a self-designed coding scheme to assess how students externalized their conception of infinity, twenty three students (N=23) were selected and contacted to participate in a semi-structured individual task-based interview (Appendix C) based on the fluidity of their view, coursework, the four categorical levels used to determine students' positioning toward duality of infinity concept, students' responses to the multiple-choice Task Q4 and its disconnection from the first three tasks, or a statement or drawing that needed more clarification. Five of the selected students, all from Calculus I agreed to participate in the interview, and each participant was a representative of a category level of duality conception of infinity.

The interviews with students were conducted to gain additional insight into their written responses to the infinity questionnaire tasks and related tasks given during the interview. Since most of the participants provided relatively short and simple responses to the open-ended questions, interview provided opportunity for students to explicitly talk about their conception of infinity as a process, object or process-object, and I was able to check for consistency in the

language the students used to describe infinity categorize their views appropriately. The semi-structured interview protocol consisted of two questions related to the Cookie monster task but presented in a different context. In order to analyze students' responses and determine their duality conception level, particularly because of the fluctuations in the students' views from process to object and vice versa, the students' responses were coded and organized into two major views – the dominant views and the recessive views which were further categorized into the singularity conception and duality conception, based on the strength of students' responses/views. The quantitative and qualitative data obtained were analyzed and used to answer the four research questions in this study.

The aim of this study is to examine college students' conception of duality in understanding infinity with the intent to elucidate strategies that could guide researchers in categorizing students' views of infinity into different levels. This study is grounded in APOS theory (Dubinsky, Weller, McDonald, & Brown, 2005) to interpret students' responses to the questionnaire and interview data. Results from this research study confirmed the findings of other studies that revealed that students' conception of infinity is predominantly process conception. Most of the students exhibited a singularity conception of infinity, which indicated undeveloped understanding of the infinity concept. This study also reports that coding and assessing college students' conception of duality is a challenging and complex process due to the dynamic nature of the conception that is task-dependent and context-dependent. As expected, there exists fluctuation in students' views of infinity; and the type of task and context students used to present their reasoning posit challenges for researchers in interpreting students' conceptions of infinity as either a process or an object, and for students in recognizing the dual nature of infinity. These students' lack of understanding in recognizing the dual process-object view nature of infinity seemed to be affected by their limited schema of infinity, especially actual infinity, which contradict many of the students' intuitive ideas of infinity.

It was found that the traditional Calculus sequence promotes a singularity conception as opposed to a duality conception. Inconsistencies exist in the views that students use to elicit their

infinity conception, depending on the type of tasks and context. Most of the participants responded to the tasks using a single view, and anytime they want to draw on another view conflict results. This cognitive conflict seemed to inhibit the students from having both views balanced, although there were some exceptions to this pattern. Some students seem to accept the dominance of one view over the other, or in few cases, when a student loses confidence, a pretty balanced recessive view emerges. Students' abilities to synthesize both the process and object views to perceive their understanding of mathematical concept will result in a well formed conceptual understanding of infinity. Understanding the process-object duality of the infinity concept is crucial for the learning of many concepts in mathematics, especially in Calculus. The findings of this study, on the other hand suggest that students have limited infinity schema. More research is needed to further study students' ways of reasoning on this fundamental concept (infinity concept) in mathematics using the framework of APOS and the modified APOD, to improve pedagogy and especially investigating Pre-calculus and Calculus teachers' duality conception of infinity and how it might suggest how professional development programs might improve the pedagogical content knowledge of these teachers and improve practice. Finally, this research study contributes to research in mathematics education on the use of paradoxes and other tasks to investigate college students' conception of duality in understanding infinity.

5.4 Implications of the Study

This study has implications on ways researchers interpret college students' view of infinity. By gaining additional understanding of how college students externalized their conception of duality of infinity and by the variations among students' conception of infinity, educators and researchers will be able to better interpret students' representations of infinity.

An actual implication of the study is that it helps to recognize misconceptions and starts addressing them so that college students will have a more comprehensive view of fundamental mathematical ideas as they progress through the Calculus coursework sequence. If pre-or-miss-conceptions are not timely recognized and addressed, then students' traditional experiences could

be easily built on strong 'narrow-minded' mental scripts that could be later transferred to "immature" understanding of mathematical concepts. It appears that both the structure of the formal mathematics curriculum which does not directly address the concept of infinity and the traditional pedagogical strategies for the Calculus sequence coursework strengthen students' processional singularity conception. Understanding the dual nature of mathematics concepts is crucial for the learning of mathematics. In order for college students to develop a deep conceptual understanding of the process-object duality of infinity, mathematics educators need to pay particular attention to the way that mathematics concepts are being taught. Instructors should restructure the formal Calculus sequence curriculum and deliberately plan teaching activities for Calculus courses that will cause the students to reflect about infinity in order to reduce and remove misconceptions that students have about infinity as well as the concept images that may hinder their understanding of the concept.

Types of tasks given to college students significantly influence their responses. Instructors need to carefully organize tasks that enable college students to perceive infinity operationally as process and structurally as object in one context and to exhibit flexibility as they move between both views and simultaneously schematize both views to elucidate their understanding of important mathematical concept. I suggest also to introduce paradoxical tasks to students to elicit their duality conception. Furthermore, having the students explore different tasks in different contexts, will also support in activating different views of infinity from multiple perspectives and enable comprehension duality conception. The results of this study could be used as a springboard to further analyze cognitive obstacles in college students' understanding of the infinity concept.

5.5 Recommendations for Future Research

Understanding the process-object duality conception of mathematical concepts could help students become more knowledgeable and flexible in learning abstract and complex mathematical ideas. Gray and Tall (1994) describe concepts that could be viewed both as a

process and an object as procept. Based on the findings of this research study, some directions for future research are suggested:

- I recommend a proceptual perspective as a tool to help college students at their earlier stages of learning to understand and overcome the contradictory and counterintuitive nature of the infinity concept.
- Just like may studies have been conducted about students' understanding of the infinity concept, implications of APOD framework can be used to conduct similar studies with college, high, middle and elementary school students to understand those students' duality conception of infinity.
- Similar studies can be conducted to examine students' conception of other concepts built on the notion of infinity, such as limit, or limit of a sequence, series and infinite sums.
- Further study can be conducted to examine students' duality conception of other mathematical concepts.
- A study that examines preservice middle and high school mathematics teachers' duality conception in understanding infinity can also be conducted.
- A longitudinal study that permits following up the same students to determine the development of duality concept among college students from Precalculus through Calculus 3 could be the basis of further research.

5.6 Limitations of the study

Among the limitations of the study are:

- The research samples were convenience samples, especially for the interviews. There were categories of responses which do not have interviewees represented because only Calculus I students were interviewed.

- Interviewees were selected shortly (two weeks) after the questionnaire was administered, and thus before a thorough and in-depth analysis of the questionnaire was accomplished.
- Lack of time for a further follow up interview to allow researcher to probe the interviewees further on some other interesting responses during the interview. Some of the interviewees were cautious about their time since they had another class to attend immediately after their scheduled interview period.
- The scenario-based task prompted process oriented responses and conceptions because of the language used, which may lead participants to think that Cookie monster will continue the process of eating half of cookie endlessly.
- English is a second language to most of the participants, and the questionnaire and interviews were written and conducted in English. This may have influenced the way or the language participants used to express their views and the interpretation of what they intended to say.
- As observed throughout the data presentation of this study, there were differences between the samples (Precalculus through Calculus 3) and within the samples. The findings from the results may not be generalizable to all students in the Calculus sequences. The result might have be different if same sample sizes of students in the Calculus sequences were studied and if the students were studied from their starting Precalculus through Calculus 3, to determine if coursework actually supports the development of the duality concept of these students.
- Reliability and validity of the study could not be considered outright. Several factors may have contributed to the findings. The instructional strategies, experiences, cultural background, prior knowledge, beliefs and values of the researcher and each participant of the study differ, and which may have influence in the idiosyncratic interpretations of views.

5.7 Conclusions

This research study found that coding and assessing college students' conception of duality is a challenging and complex process due to the dynamic nature of the conception that is task-dependent and context-dependent. Interpreting students' views of infinity posits a challenge for researchers due to the dynamic nature of the conception. There is diversity and variation among students' process-object perceptions. The fluidity in students' view is an indication that students may possess both process and object views of the concept to various degrees, which we indicated could be dependent on the dominance of one view over the other, and equality of views will be the case of duality conception. Triangulating the result from the questionnaire with the interview confirms that the categorization into levels is a more appropriate tool for assessing students' duality conception of infinity. The development of Action-Process-Object-Duality (APOD) framework by this study elucidated an effective strategy that could guide researchers in categorizing students' views of infinity into different levels to assess students' duality conception of infinity.

The fluctuations between students' views however suggest an undeveloped duality conception. Results of our study reveal that college students' experiences in the traditional Calculus course are not supportive of the development of a duality conception. On the contrary, it strengthens the singularity perspective on fundamental ideas of mathematics such as potential infinity. Table 4.1 presents the data supportive of this claim. It is important to provide college students with relevant experiences to build the concept of duality, which will help them to understand mathematical concepts (e.g., infinity) at a more rigorous level. Both the process and object conceptions complement one another, and they are crucial for effective problem solving to ensue. The results of this study could serve as a facilitating instrument to further analyze cognitive obstacles in college students' understanding of infinity concept.

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Appendix A: Informed Consent Form

University of Texas at El Paso (UTEP) Institutional Review Board
Informed Consent Form for Research Involving Human Subjects

Protocol Title: College Students' Conception of Duality: Case of Infinity
Principal Investigator: Grace Babarinsa
Co-Investigator: Mourat Tchoshanov
UTEP College of Education: Teacher Education

In this consent form, "you" always means the study subject. If you are a legally authorized representative (such as a parent or guardian), please remember that "you" refers to the study subject.

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1. Introduction

You are being asked to take part voluntarily in the research project described below. Please take your time making a decision and feel free to discuss it with your friends and family. Before agreeing to take part in this research study, it is important that you read the consent form that describes the study. Please ask the study researcher or the study staff to explain any words or information that you do not clearly understand.

2. Why is this study being done?

You have been asked to take part in a research study on "College Students' Conception of Duality: Case of Infinity". Approximately, 200 students will be enrolling in this study at UTEP. You are being asked to be in the study because you are a UTEP student who is faced with the challenge of understanding the concept of infinity. If you decide to enroll in this study, your involvement will last about three months.

3. What is involved in the study?

If you agree to take part in this study, the research team will give a survey to complete. Participants will be asked to complete this survey that will take approximately 20 minutes to complete.

4. What are the risks and discomforts of the study?

There are no known risks associated with this research

5. What will happen if I am injured in this study?

Not applicable

6. Are there benefits to taking part in this study?

There will be no direct benefits to you for taking part in this study. This research may help us to understand why college students have certain misconceptions in learning and understanding the idea of infinity.

7. What other options are there?

You have the option not to take part in this study. There will be no penalties involved if you choose not to take part in this study.

8. Who is paying for this study?

Internal in-kind funding: Funding for this study is provided by UTEP, Department of Education in the form of in-kind services, such as, copies of surveys.

9. What are my costs?

There are no direct costs.

10. Will I be paid to participate in this study?

You will not be paid for taking part in this study.

11. What if I want to withdraw, or am asked to withdraw from this study?

Taking part in this study is voluntary. You have the right to choose not to take part in this study. If you do not take part in the study, there will be no penalty.

If you choose to take part, you have the right to stop at any time. However, we encourage you to talk to a member of the research group so that they know why you are leaving the study. If there are any new findings during the study that may affect whether you want to continue to take part, you will be told about them.

The researcher may decide to stop your participation without your permission, if he or she thinks that being in the study may cause you harm.

12. Who do I call if I have questions or problems?

You may ask any questions you have now. If you have questions later, you may call Dr. Mourat Tchoshanov or Ms. Grace Babarinsa at 915-747-7668 or mouratt@utep.edu.

If you have questions or concerns about your participation as a research subject, please contact the UTEP Institutional Review Board (IRB) at (915-747-8841) or irb.orsp@utep.edu.

13. What about confidentiality?

Your part in this study is confidential. None of the information will identify you by name. All records will be coded to maintain anonymity. Surveys will be accessible to the research team during the time of this study, as well as, follow-up studies that may be generated as a result of data analysis. Surveys will be permanently deleted or destroyed when all studies are completed. All records will be kept by assigning a number to each student. Every student's identity will be kept anonymous.

14. Mandatory reporting

If information is revealed about child abuse or neglect, or potentially dangerous future behavior to others, the law requires that this information be reported to the proper authorities.

15. Authorization Statement

I have read each page of this paper about the study (or it was read to me). I know that being in this study is voluntary and I choose to be in this study. I know I can stop being in this study without penalty. I will get a copy of this consent form now and can get information on results of the study later if I wish.

Participant Name: ______________________________ Date: ____________

Participant Signature: ______________________________ Time: ____________

Consent form explained/witnessed by: ______________________________

Signature

Printed name: ______________________________

Date: ____________ Time: ____________

Appendix B: Infinity Questionnaire Tasks

Number Code ______________________ Date _______________________________

Gender ____________________________ Course _____________________________

Ethnicity ___________________________ Total Math GPA _____________________

1) **When you think of Infinity what comes to your mind?**

2) **The cookie monster sneaks into the kitchen and eats half of a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day.**

   a) If the cookie monster continues this process seven days, how much of the cookie has he eaten?

   b) How much is left?

   c) If the process continues, will he ever eat the entire cookie?

3) Draw Infinity in the space provided.

Explain your drawing below:

4) I feel that my conception of infinity is as (check one):

a) A process, e.g. something that goes on and on.
b) An object, e.g. set of natural numbers is infinite.
c) Both a process and an object.
d) Other: ____

Appendix C: Tasks-based Interview Protocol

Number-Code: __________________

2. What math courses have you taken in high school? Where?
3. What was your first encounter with the concept of infinity? Was it before taking the survey?
4. Have you had any encounters with the concept of infinity after taking the survey?
5. Tasks

    A. Zeno's dichotomy paradox: If a runner is to complete a race course, he/she must first traverse ½ the distance, then the next ¼ of the distance, then the next 1/8 of the distance, etc. If all these intervals are traversed, will the runner complete the course?

    Explain your reasoning.

    B. Which statement below is true?

    A. $1/2 + 1/4 + 1/8 + \ldots < 1$

    B. $1/2 + 1/4 + 1/8 + \ldots = 1$

    C. $1/2 + 1/4 + 1/8 + \ldots \leq 1$

    D. $1/2 + 1/4 + 1/8 + \ldots > 1$

    E. $1/2 + 1/4 + 1/8 + \ldots \geq 1$

    Explain your reasoning.

6. Show the participant their response to Q4 on the survey and ask them to elaborate on their response.

Appendix D: Transcript for Jose

Interview with Jose, April 30, 2013 at 12:00 pm in the Education Building

Duration: 21.12minutes.

1. B: OK. Today is Tuesday April 30^{th}, 2013. I'm here with student 88405015. Now, I guess the first think I'll like to talk to you about is this first… This first Um! Task you did. Today, would you be willing to kind of elaborate on your… your reason for the...
2. J: Sure! Sure! Um! If the runner is completing the race course, and she first gets to half the distance, and then she goes another fourth the distance, and another eighth of the distance, then this runner won't complete the course. Um! She'll get very close to the end, but never to the finish line. So as you're adding distance to each fraction, you're only getting closer to 1, not actually equaling it to 1.
3. B: Huh-un!
4. J: So, if you're gonna in this case find ah… the intervals or see what, how much he's travelling, all you gonna do is add to the distance… the distance of course is gonna get smaller cause you're going from one-half, to one-fourth, to one-eighth. But… that's… that's what it is.
5. B: Huh-un! OK. Um! Let me see here! So you mentioned adding pieces over and over and over again?
6. J: Right! Yes!
7. B: OK. Wonderful. Well yes! Thank you for the explanation. Now, I'll like you to take a look at the second task here, you completed, and explain your reasoning in this instance.
8. J: OK. Just as in the runner's case you're also adding fractions here.
9. B: Huh-un!
10. J: And the way I got to reason this was I first took… Um! I found the one-half plus one-fourth plus one-eight …
11. B: Huh-un!

12. J: I found the common denominator to be eight. So the one-half would… equal four-eighth

13. B: Huh-un!

14. J: the one-fourth, two-eighth. And one-eighth, that would equal of course seven-eighth. So that doesn't equal to one? It we add now let say one-sixteenth, then the common denominator would be sixteen. So, one-sixteenth plus two-sixteenth from the one-eighth, plus from the one-fourth you get four-sixteenth. And from the one-half you get eight-sixteenth. This will eventually equal fifteen-sixteenth.

15. B: Ok.

16. J: So this will never equal to 1. And this would be true if you keep adding one-half, just let's say, you know, we go from, you know one-half to one-fourth, to one-eighth, one-sixteenth, one-thirty-two, one-sixty-four. So every time you're adding – or you're multiplying by one-half.

17. B: OK.

18. J: So, you never… just as the runner's case you never equal it to 1. So the statements here that are listed ah! One-half plus one-fourth plus one-eight equaling to 1…

19. B: Huh-un!

20. J: … will never happen.

21. B: OK.

22. J: one-eight… one-eight plus one-fourth plus one-half here is lesser or equal to one. Oh! That statement wouldn't be true since you have an equal sign there. So that throws it out. The one-half plus one-fourth plus one-eighth greater than 1, that's not true because it's like saying you're getting one whole piece…

23. B: Right!

24. J: … and you're cutting it up. So you're not gonna add to already what you already have as a whole which is 1, so you're only cutting up pieces if you're doing anything. That's why.

25. B: I see!

26. J: And then the last statement, one-half plus one-fourth plus one-eighth greater than or equal to 1 of course, that won't happen because; first of all it's not gonna equal 1…

27. B: Huh-un!

28. J: … and it's not gonna be greater than 1.

29. B: OK.

30. J: So the only statement true here is (A) which is one-half plus one-fourth plus one-eighth plus whatever on, less than 1.

31. B: OK! That answers it. Thank you. OK. So this Um! I also have a copy of the Um! Of the survey that you … that you completed. But before we get in that I'll like to ask you kind of about your Um! Your mathematics background. What ah! What math courses have you taken?

32. J: Ok!

33. B: And starting let's say in high school.

34. J: In high school! Ok I took ah! Algebra 1, Algebra 2, Geometry…

35. B: Huh-un!

36. J: Um! And then I took Precal 1, and well . . . Precall overall.

37. B: Huh-un!

38. J: … and then I took, ah! I'm taking Calculus 1.

39. B: Ok. And did you take any math in the Fall?

40. J: I took… I had to take this class over again. Calculus 1…

41. B: Oh! I see ok.

42. J: … since I got a D in the class.

43. B: Oh! Alright. OK. So, how do you? I mean. How do you feel about it, taking it the second time?
44. J: The second time… well first, the…the biggest challenge for the fall was the instructor that I had.
45. B: OK.
46. J: Um! I could… she was available during her office hours…
47. B: Huh-un!
48. J: But even if you do go to her office hours, it's not very clear as to what, you know, assignment she's assigning them.
49. B: I see!
50. J: Of what not! So I mean I have to find tutors. Or you know studying myself the book or even like that, the book was too much for me to of… you know, study on my own.
51. B: Right!
52. J: And I do know, you know! Um! A friend of the family that is a …
53. B: Huh-un!
54. J: … a math instructor at the high school. And he was stumped. And some of the questions itself he doesn't even remember from long ago when he took it.
55. B: Huh-un!
56. J: So everywhere I would go there was a dead end to say.
57. B: Right
58. J: So now taking it the second time. I'm very pleased with the instructor now.
59. B: Ok.
60. J: He actually explained it step by step.
61. B: Huh-un!
62. J: More examples. Things like that. So, like makes me… You know, understand the concept of calculus.
63. B: Huh-un!

64. J: … what calculus is all about.
65. B: Ok. So what is calculus all about?
66. J: Calculus is trying to find a… you're not truly… the way I see. You are not truly finding an answer.
67. B: OK.
68. J: It's the process as to how close to the answer you can get.
69. B: I see.
70. J: And calculus is a spectrum from the negative infinity…
71. B: Huh-un!
72. J: … to positive infinity. Anything in between that is what calculus is all about. Whether it's finding areas of triangles, whether its areas of rectangles, derivatives, anti-derivatives, integrals, you know. Same thing of course. But that's spectrum. In trying to get there.
73. B: Interesting!
74. J: And that… and Calculus also incorporates everything also, I've already learned, which is…
75. B: O! I see.
76. J: … Algebra 1, Algebra 2, Geometry,
77. B: Huh-un!
78. J: Precal, it incorporates all into that. I mean you just… adding to that.
79. B: OK. Marvelous! So, Um! What was your first encounter with... with the concept of infinity?
80. J: My first encounter was [Pause] I will probably say, the way I will use everyday life.
81. B: OK.
82. J: The concept of infinity. I see infinity as beyond. You know something we can't really understand truly. We can get to it, there's a process to get to it, but we'll never get to that. That… that answer you can say, or that particular finding what you're trying to get to. Infinity can be anything from your everyday life, to math, to the sciences…

83. B: Huh-un!

84. J: Anything can be infinity. We are infinity. I mean. Of course, when we… we're born at a certain point and we'll die at a certain point. But anything in between, I consider an infinity for us. 'Cause, we are always told that our expectations can lead us anywhere. Goals and all that.

85. B: O! I see.

86. J: So we have an infinity process too. We can choose track A or track B. And track A has an infinity number of sources that we can use. Track B another infinity number of resources we can use. And that's how we get to where we are today. Then we grow from there on.

87. B: Fantastic! Interesting! And how does it… you said it's related to sciences as well. How?

88. J: Sciences…

89. B: How is it relating to science?

90. J: The sciences a… I connected to a reproductive kind of, system.

91. B: I see.

92. J: Or you can even say, ay… Um! You know. W… we… I'm taking Chemistry.

93. B: Huh-un!

94. J: And in chemistry, we have bonds. We have ah... different levels of ionization. We're trying to find, you know, different aspects of how… how one. You know one certain atom relate to the other, and how we can link both of them. Do bonds, all sorts of those things. And I see that concept there as well. We can always take a simple atom…

95. B: Huh-un!

96. J: … and we can just add to that atom over and over an infinite number of times.

97. B: OK.

98. J: We can grow from one small particle.

99. B: Huh-un!

100. J: And we are… Just as us humans. You know as humans we are… we are from baby and we are… when our mom will feed us …

101. B: Huh-un!

102. J: And now we are this… you know what we are today. And we still have more to learn.

103. B: I see! Excellent! So have you? Um! Had any encounters or thought about the concept of infinity after doing the survey?

104. J: Encounter? Uh-umm! No, actually I didn't take them into account. I didn't really…

105. B: No, no, no! It's ok. That's ok.

106. J: I didn't really think about that.

107. B: Huh-un!

108. J: I actually just, you know, I look at infinity and a spectrum, and that's it. That's about it.

109. B: Ok. Yeah! No problem. Ok. Ok. Now I'm gonna take a look at the Um! These responses you gave to the survey. And now in looking at ah… number (1). Ok! You said numbers that go… what's this? Beyond what we can…

110. J: Count on a daily basis.

111. B: … count on a daily basis. Do you have anything to elaborate on that or that's still…?

112. J: That still holds true. And when I wrote this, I didn't think of even life. I just thought of… This is the first thing that came to my mind. It was numbers…

113. B: Huh-un! Yeah!

114. J: … And numbers, you know you… you can get one number. Let say 1, and add decimals and decimals. You can put 1.1, 1.13, 1.134.

115. B: Huh-un!

116. J: … and so forth. So if… you can have an infinite number, all the way up until, let's say for example 1.99999…

117. B: Huh-un!

118. J: … and take that number to infinity.

119. B: Ok. Marvelous! And so for the… the second problem that was there. The cookie monster question. How did you… I guess how did you arrive… at this?

120. J: Well. I… I first, you know, I saw that this cookie monster eats half the cookie…

121. B: Huh-un!

122. J: … then on the second half, he takes one-half. And on the third day he comes in and takes one-half. But what I didn't take into account was probably one-half, multiplying it by one half which gets to one-fourth.

123. B: Huh-un!

124. J: Multiplying it again by one-half gets to one-eighth…

125. B: Huh-un!

126. J: So, I… I think … you know over seven days… I really don't think that's the correct response<u>. I think I will actually change my response</u>. Now seeing that I took the even both… both these questions…

127. B: Um! Huh-un!

128. J: … and I will way that this answers are actually are closer to 1.

129. B: Ok!

130. J: … and in some way. I mean we're not exceeding 1, but looking at… or it might be a lot… or actually a lot smaller, 'cause, we're… we're… It could be one over one ninth. Instead of multiplying it. Right? I don't know. It's not right either.

131. B: Ok. Ok. Yeah! No, no problem. And so you will say the same thing…

132. J: Yeah! I think.

133. B: … for… for (B)?

134. J: For (B). Yeah! Tha… that's how much is left for not being that.

135. B: OK. Well, Yeah. How would it be different?

136. J: I think. Um! Thi… this would be… you know. One minus whatever are… actually… Let's see how it will be! Um! [Thinking] [Sigh]One over… Yeah! Right! In fact you're big… much bigger number on top. Over large… you know! Something big on the bottom. It would still be something like this…

137. B: Huh-un!

138. J: … but much bigger number.

139. B: Ok. So you think you can go with the…? Is it a smaller number that's left or a big amount that's left?

140. J: Huh! I will say it's… it's a big amount that's left.

141. B: OK.

142. J: Because if you're taking half, I mean you're taking a fourth. So every time…

143. B: Huh-un!

144. J: … you're taking a smaller piece.

145. B: Huh-un!

146. J: So it'll still be a big portion left over.

147. B: Ok! OK. And then for this last.

148. J: No. He wouldn't eat… He wouldn't eat the entire cookie. Because, Um… of course a cookie is one.

149. B: Huh-un!

150. J: The whole is one.

151. B: Right!

152. J: I'm saying that I'm taking the cookie monster and he's eating over seven days. Means that he's eating over seven days, but he's taking every time and each day a smaller piece.

153. B: Huh-un!

154. J: So he’s not really into eat… He’s never ever gonna eat the entire cookie unless you’re adding let’s say, one-half plus one-half, or one-fourth plus one-fourth, plus if all the fractions were equal to each other…

155. B: I see!

156. J: … in one… in one equation. Then he will get to a point of eating the whole cookie.

157. B: Ok. So what… I guess, what circumstances would …, would have to exist for that to happen?

158. J: The… the frac… you’re not gonna get… is… you gonna have to eat half one day or one-fourth or whatever fraction he was trying to eat, but the next day, he’s eating that same fraction.

159. B: I see.

160. J: And if the fraction is smaller and smaller, he’s gonna have to eat more of that same fraction.

161. B: Huh-un!

162. J: He’s not gonna be able to come in and eat half, come in and eat you know, another half. He’s still gonna get that small piece.

163. B: OK. I see! OK. Thank you. And so there’s still the third one.

164. J: OK. My basic concept of infinity here in this space is… this is just one, when you can take numbers…

165. B: Huh-un!

166. J: … and add them to each other. Getting to a million, billion, trillion and beyond.

167. B: Huh-un!

168. J: And… or you can take fractions….

169. B: Huh-un!

170. J: … and add and add and add and add you gonna get to infinity. You can take, you know, symbols such as the infinity symbol…

171. B: Huh-un!

172. J: … and add infinity, and add infinity, you know, to infinity and beyond.

173. B: Huh-un!

174. J: … or negative infinity or you can even take negative numbers. So whether you have negative numbers or positive numbers, it doesn't make really a big difference. You just gonna add infinity and beyond.

175. B: I see. And so in including the symbol… well, I guess, what is that symbol?

176. J: Ah! Would be mathematics. It's a mathematical symbol …

177. B: Huh-un!

178. J: … that I see. So if we take… Let's say for example in calculus you we take the limit…

179. B: Huh-un!

180. J: … as in approaches infinity…

181. B: Huh-un!

182. J: … then, you get close to the answer, but you're not gonna get to the answer. It's an infinity answer.

183. B: OK. Excellent. Thank you. And you're… and your explanation is… is the same as ….

184. J: …. Yeah! Same thing.

185. B: … OK.

186. J: Um… we can never stop counting

187. B: Huh-un!

188. J: … numbers exist. I mean we always saying that infinity … Or I've already said that infinity starts at zero and goes to infinity. Well, I mean just looking at the positive side.

189. B: I see!

190. J: We can also look at the negative side.

191. B: Huh-un!

192. J: We can take from zero to negative infinity.

193. B: Huh-un!

194. J: … and you gonna get negative number. Infinite many numbers.

195. B: Huh-un!

196. J: Or you can you know, some people don't look at it, where you can take an interval …

197. B: Ok.

198. J: … from let's say 0 to 3.

199. B: Yeah!

200. J: I mean you think ah! We can only get three numbers out of there, 'cause we are looking at whole numbers.

201. B: I see!

202. J: That's the first thing that comes to at least my mind.

203. B: Huh-un!

204. J: But we can take from 0 to 3 out of the infinite many numbers between 0 and 3.

205. B: Huh-un!

206. J: So we can, whether we are looking at this specific interval, whether you are looking from negative infinity to infinity, you're always gonna have infinity.

207. B: OK. Spectacular! And now looking at the question (4) the multiple choice question.

208. J: Um! Yes! Um! I do feel that infinity is a process. Something that goes on and on. Or and it can also be an object. A set of natural numbers is infinite, which is going back to what I explained. Um! The process of something goes on and on can be a cycle.

209. B: … Huh-un!

210. J: It could be us counting numbers. It could be us adding numbers. It could be us you know, whatever. It… whatever we're adding, whatever we're explaining, it's gonna be an infinite process. While some things do comes to an end…

211. B:

212. J: … but that doesn't explain in my sense what infinity means. And as an object, the set of natural numbers, while we are deal with numbers on a daily basis.

213. B: Huh-un!

214. J: Whether it's to tell time, whether is to … you know, the, the … you know how fast we're driving on the driving on the freeway…

215. B: Huh-un!

216. J: … or you know, speed limit signs, whatever. That's also sta… set of natural numbers is infinite.

217. B: Ok. So when you think of infinity as an object, you're thinking of it in terms of numbers? Right?

218. J: Right!

219. B: OK.

220. J: Yes!

221. B: Is there anything else that comes to your mind in terms of an infinity being an object?

222. J: [Whispers] Infinity! Um… [Pause] Um… No I'm … I'm not sure about that one. Ahh! [Pause] No. I mean as in… I will speak on you know, us human beings and animals.

223. B: Huh-un!

224. J: We come to an end. We have a starting point. We have an end. Everything in between us is infinite. But then, we … That end comes to an end. There is nothing beyond.

225. B: Huh-un!

226. J: What we perceive, you know. I'm using that as an object. Just…

227. B: Right!

228. J: But I think we come to an end. And that's where we stop, being point. That's our endpoint.

229. B: Huh-un!

230. J: That's where life for us is gonna end.

231. B: Ok.

232. J: Well that's an object. I don't see any infinite, any infinity in objects. In certain objects, of which . . .

233. B: Ok

234. J: Uh-umm! [Long Pause]

235. B: Because I think I remember you saying that Um… That we as human beings, there is infinity you know, within us.

236. J: Oh! There is infinity. [Interruptions] ….. O no … What we can… What we… How we have it is… at a certain points in our lives…

237. B: Huh-un!

238. J: … we, we chose to do certain things.

239. B: Ok.

240. J: We chose to … a certain career.

241. B: Huh-un!

242. J: And in that career, there's another you know, let say, another option.

243. B: Huh-un!

244. J: And in that option another option and it's like a branch.

245. B: I see!

246. J: And the branch has a… a tree.

247. B: OK.

248. J: Has an infinite number of branches.

249. B: Huh-un!

250. J: You know. Yeah! Of course we see a tree and we say, well it has you know X number of branches.

251. B: Huh-un!

252. J: But in within those branches is branches, and within those branches is branches. So we have that option, in life …

253. B: Huh-un!

254. J: … to accept whatever we chose to do. You know we have between saying "Yes" and "No".

255. B: Huh-un!

256. J: You know. And if we say "Yes", then it will open up to this.

257. B: Huh-un!

258. J: And if we say "Yes" to them, and it will open up to something else. If we say "No", then that's our stopping point there.

259. B: Huh-un!

260. J: We don't have anything to go above "No". We can explain why we say "No",

261. B: Huh-un!

262. J: … but when a! ...there's no branches that goes in, in "No".

263. B: I see!

264. J: So only if you chose to do something with yourself …

265. B: Huh-un!

266. J: … with life, with whatever concept you're dealing with or you're talking about. You have an infinite number of things to talk about …

267. B: Huh-un!

268. J: … to experience, to… you know grab ahold of and saying I'll go this track, but then tomorrow, I'm gonna go this track.

269. B: Huh-un!

270. J: So, you experiment and you'll see where you think you fit in life. Where you think you belong. Where you think you … you can help out more or where you gonna prosper.

271. B: Ok. So I guess, I … What I think I'm hearing, is you're saying is that infinite are possibilities …

272. J: Yes! O Yes!

273. B: …or experiences.

274. J: Most definitely!

275. B: OK.

276. J: Because, if, if, if… if we were just set on one, one possibility…

277. B: Huh-un!

278. J: …then I think life would be boring. You know, we would… everyone will be doing the same thing. Well I don't know, driving the same thing, you know, walking the same way.

279. B: Huh-un!

280. J: So that won't make life exciting either.

281. B: Right.

282. J: That's why we're created each, in an individual bases. Because we are here to… for a mission.

283. B: Huh-un!

284. J: Each one of us is given a mission. And that mission is infinity.

285. B: Huh-un!

286. J: Infinity for whatever reason. Whatever we're accomplishing. Whatever we are doing in daily, in our daily lives. Whether its math, you know. Again going back to science.

287. B: Huh-un!

288. J: An evolution, a cycle, or us you know, we go from place to place to place. Hum!

289. B: Yeah!

290. J: Um!

291. B: Ok! Thanks, you answered it. Or do you have any question for me?

292. J: No. I shouldn't!

293. B: O OK!

294. J: No! No! [Both laughing]

295. B: Well I wanna just thank you for taking the time to meet and then talk with me.

296. J: Oh thank you.

297. B: And…

Appendix E: Transcript for Vanessa

Interview with Vanessa, March 1, 2013 at 10:00 am, in the Library

Duration: 21.39minutes. **C1 - 014**

1. G: Alright. So, um, to start with, I just want to find out, um, “What math courses have you taken in high school?”
2. V: In high school I took Precal, Geometry and Cal 1.
3. G: O… ok! Where?
4. V: Oh, I went to mission early high school. So, um that’s where I took them.
5. G: Where is that?
6. V: Um! (Sigh) do you know where Mision de El Paso is? EPCC.
7. G: EPCC
8. V: It’s right after gateway um! (Sigh) east.
9. G: Oh!
10. V: It’s like right next to there.
11. G: Oh! Ok, ok. So that’s where you took all those math credits before you came to UTEP.
12. V: Yes.
13. G: Awesome, awesome! Aw! Good. Where did you take high school?
14. V: Where did I go to high school?
15. G: Yes.
16. V: Mission early college high school.
17. G: Mission early, Oh ok, now I know the place. Now I know, now I know; now I know. I’ve been there once. I’ve been there [phone rings] [Pause]
18. G: Um! If I may ask, what was your first encounter with the concept of infinity?
19. V: At first I didn’t get it. But… That’s because I really don’t remember doing it, to be honest….. But I think at first I didn’t get it, and after a while, I like kind of started understanding it a little bit more.

20. G: So you want to tell me … Was it before the survey that you have an idea or it was after the survey?
21. V: After
22. G: After the survey?
23. V: Yea. I think so.
24. G: So what was your experience with the survey?
25. V: Um! It was different. I wasn't expecting… expecting like the questions. But …… so I guess it was like a little bit hard, so I just tried my best. Based of like I was learning it in class.
26. G: Ok so you've not really come across the concept of infinity in your classroom or any of the math classes you've taken.
27. V: Uh! I did. I just never like understood it.
28. G: Oh ok. When you say you don't understand it. Can you remember if any of the topic, that was treated, that it was mentioned?
29. V: In the survey?
30. G: Ye…No. When you took some… some of the math classes you take. Is there any particular topic that they mentioned infinity, you know. That it was actually taught?
31. V: Uh-umm! I think when we were doing graphs, I remember learning about it. It's really hard for me to remember. [Laughing]
32. G: Oh ok. Tha… that's a long time ago.
33. V: Hmm!
34. G: Yea. So you didn't get it then, and so when you were taking the survey, you were like… Uhg! So you just try… Ok.
35. G: Um! Before... I'll… want us to look at a particular question that, you know, I was really interested in your response to it…
36. V: Ok.

37. G: … you know, in the survey. But before then, I want us to look at this. [Handed out the survey interviewee wrote the last time]… Zeno's Dichotomy. [Long pause - Interviewee reading her responses to survey questions]
38. V: Do you want me to answer it?
39. G: What do you think about it? Yea! I wouldn't mind if you'll… I mean… to explain your reasoning. [Pause]
40. V: And then he has to do a fourth of this half right?
41. G: Uh-hum!
42. V: And then an eight of that fourth?
43. G: You first… it's like you first cover half of the distance…
44. V: Huh-un!
45. G: … then you cover the next half of the half. That is the one-fourth.
46. V: Half of the half! [Wheeze] I wanna say that [Pause] Hmm! [Wheeze] If he's going by like half in half, I don't think he would… complete it, or maybe he would but it would take him a really a long time.
47. G: To complete the …
48. V: Yeah! Like it would take him a lot longer by just going like halves of the halves.
49. G: Ok. So if all these intervals were… all this distances were covered in this… in this procedure one-half, one-fourth, one-eight? You want to say that the runner will… will complete the distance?
50. V: I wanna say… from what I remember by infinity, I wanna say "No" because he's just gonna keep going and going and going and going and going. So, No. He won't.
51. G: So, can you explain the procedure, because you said, going and going. Is there any way you can answer that, you know! In the paper, based on what you have there?
52. V: Like you want me to write it down?
53. G: Yea. Right! Yea, yea!
54. V: Umm! [Long pause - Writing in paper]

55. G: You’ve got very nice hand writing
56. V: [laughed] [Pause] Ok. [Wheeze]
57. G: Ok. So you said the runner will not complete the… the course. So, the distance running will gradually become shorter and shorter. So if it becomes shorter and shorter, so he’ll still not complete it?
58. V: No
59. G: So it continues to get shorter and shorter and what will happen?
60. V: Eventually you don’t run at all, right?
61. G: Why will he not run?
62. V: Well… Ugh! I don’t know. It’s because it’s so confusing. I don’t like infinity. Like… like I said eventually, his distances will be like so short. Like he’ll still run, but they’ll be so short that they won’t’ really be significant.
63. G: Ok. Alright. I want you to take your mind off infinity right now. Just look at this question if you’re given as a math problem.
64. V: Oh well then. Yeah, he will finish. ‘Cause you just gonna add up the distances right?
65. G: What will you do? Let me see what you will do.
66. V: So, I will say like let’s say the course is a mile. So, half of a mile… [Whispered- so what happen now? Oh I can’t think now] …Oh, basically it’s like you will add the half and then you will add another half of that, a fourth, and then you will add the eighth, and then you will… You will just keep adding the halves until you reach the number
67. G: So we gonna have half plus…
68. V: Yea, so we have like half plus a fourth, plus an eighth, plus what’s half of an eighth? Plus one twelve? Like that, and you just keep going and going until you reach your total… of the course.
69. G: Of the course!
70. V: Hmm!
71. G: Ok. Ok. Um! Turn to the side [flipping paper pages]. Let’s look at this together.

72. V: OK.

73. G: Which of these statements do you think is true, or are true? [Pause] Because, it's like it's related to something that you put down there. [Pause] Or does it explain what you're trying to say, which of it?

74. V: Um! I wanna say it'll be this one [B] [Pause] or probably this one [C]

75. G: Can you explain your reasoning or because I don't get it?

76. V: [Laughing] it's because am trying to remember. But… It's because my reasoning is that… I'm so sorry. That you're gonna be continuing to add numbers, right? Regardless, so eventually you gonna get to the number that you're looking for. Like as long as you keep adding numbers.

77. G: So, is this the number we're looking for?

78. V: Yes.

79. G: And what does this mean?

80. V: Less than or equal to.

81. G: Less than or equal to? So, how does that explain your reasoning, or how does it relate to…?

82. V: It's because am like between…

83. G: Or why did you choose…

84. V: I'm like between two answers. No, you know what, I change my mind. I think it's only (B). [Erasing the choice C] Yea. That makes more sense.

85. G: It's only (B)?

86. V: Umm!

87. G: So it will be equal to 1?

88. V: Yes!

89. G: Can you explain to me please?

90. V: [Laughing] Well like how you just keep adding numbers, like you gonna add the half and you add the fourth, and then you add the eighth, and then you add the twelfth. So you'll just keep adding and adding and adding and adding until you get to 1.

91. G: Ok I want you to… Because you were the one that first wrote this series down here. [½ + ¼ + 1/8 + ……. = ____] I want you to… in the concept with this [Zeno's Dichotomy paradox] Explain it.

92. V: I don't know how to explain it.

93. G: [Laughing] you are just wonderful because… [both laughing]

94. V: [Pause] Ugh! I don't know how to explain it.

95. G: So what will the runner, I mean will the runner… what will happen actually? .... Maybe you want to think back again?

96. V: Ok, let me… let me rethink about this because am confusing myself.

97. G: Yea!

98. V: Yea! [Laughed]. Ok. I'm sorry; let me go in my note really quick.

99. G: No problem! [Pause]

100. V: Ok. I got it. He's not gonna complete the course because he's… am gonna go with my first answer [C] that he's just gonna be running very, very, very short distances, so he won't complete the course. He's just gonna continue running for like an infinite amount of time. So… it will be …. It's not gonna be this [E]. It will never be greater than one [D]. Yes! It's gonna have to be… Yeah! This one [Pointed at number(C)]

101. G: So you're choosing (C) now?

102. V: Yes!

103. G: So it's gonna be less, what is that?

104. V: Less than or equal to 1.

105. G: Less than or equal to 1.

106. V: Yes. 'Cause, he's just gonna keep running and running and running and running. 'Cause they are really, really short distances. So he won't finish.

107. G: Ok. Alright, let me take you back to the, [interviewee laughing as I opened to her survey response] take your mind back to this. It's says when you think of infinity, what comes to your mind? You said a long list of numbers…

108. V: Never ending numbers

109. G: Do you still wanna stick to it or you still have more…

110. V: No, no

111. G: … explanation you wanna give to that, or you want to explain better to me? When you say long list…

112. V: I guess, I guess I will add just something that never ends, like that's infinity; never ends.

113. G: Ok. Something that never ends. [Pause] Ok. Then what about the cookie monster now? I'm interested in this part.

114. V: [long pause] Yea, you see that was what I was trying to remember. Now I get it.

115. G: What is that?

116. V: Oh! where it says Um! Like if you're only taking away a percentage every day, you never gonna get to zero. You always just gonna get… since the numbers are infinite, you just get like smaller and smaller like point five, point five nine, nine, nine, nine, eight. You know what I mean? So you'll never actually get to zero. You'll just get smaller and smaller numbers every time. So, like with the cookie monster, like he'll probably never really finish eating the cookie if he's only taking little, little bits. I think he'll keep eating that cookie like forever.

117. G: Forever! Ok. You said something. You said theoretically!

118. V: Like I wanna… the way… the reason I say theoretically is because, like the way I see it in my head is if the number gets so small, it becomes Um! Insignificant. Like it doesn't… It's not really significant because it's really, really small so like I kind of imagine like a cookie and point like… you know like… one of… out of one twenty, that

is really, really small so in real life, it will probably look like really little. So it will seem like he did finish the cookie but really he didn't because he still has that one twentieth. So that's why I said theoretically 'cause it will probably look like he did but he'll still have some cookie left.

119. G: Um! Okay. So now, you gave me two ideas. Theoretically, it will look like because it's very little

120. V: Huh-un! [Agreeing]

121. G: … so it would look as if he has finished it…

122. V: Yes!

123. G: … but mathematically…

124. V: But math… so it's kind of like, like I was trying to explain it to you, to the runner like it would probably look like he finished the race, but if he's only going by halves, by little insignificant numbers. He actually will never really finish. Like really like mathematics on paper, he really won't finish but it will look like it. Because it's such a small amount that he has left to run.

125. G: Oh ok.

126. V: Now I remember it, okay.

127. G: Oh ok. Thank you.

128. V: [laughing and mistakenly knocked off the recorder] Sorry about that.

129. G: Thank you. Now let's look at this number 4. You're talking about your conception of infinity that is it a process or an object, and you choose a process!

130. V: I think I wanna change my answer to that one. 'Cause I think it can be anything a process… or an object. Like a number like Pi, that goes on forever. And that's a number… an object. And a process is like running a race when we're doing halves or even cookie when we're only eating half every day. So I think infinity can be anything as long as it goes on and on and on and on and on forever.

131. G: Oh! So that’s a process? And then the object part is, you said… you give an example of …

132. V: Like Pi, the object could be like Pi. Like a number that never ends or a song like never… like that song that sang never ends or anything really. Just something that never ends. It doesn’t matter what it is.

133. G: Okay. Now look at this [the definitions on the multiple choice question 4] so you’re good, you’re cool with this? The object definition…

134. V: I want to say maybe not like object but for sure like numbers, or infinite. Cause even when you count, that’s infinite too, but counting is a process.

135. G: Counting is a process. So, the set of natural numbers is a... is infinite, so you see that as a process also or as an object?

136. V: Hu-mm! No I wanna say it’s a process ‘cause it’s the process of counting. [Pause] Yea! Okay! Never mind. I’m sticking with my answer (a). Yea! It’s just something that goes on and on and on.

137. G: So, just… It’s a process?

138. V: Yes! Just the process [laughing]

139. G: Well okay! Okay!! No o…

140. V: I know. I’m confusing it. It’s because that’s how I think in my head when I think about the stuff.

141. G: Yea! I know. I’m thinking like maybe you know… maybe after the… the… survey, maybe you know you’ve had another experience…

142. V: Huh-un!

143. G: … with infinity that maybe you want to use to, just to explain further, or maybe…

144. V: I kind of really haven’t. I mean after the survey I took one more math class and then I finished. So I really haven’t experienced infinity like in terms of like a class or

anything. But… so this is my first time actually thinking about it, since I took the survey. [Laughing]

145. G: O okay! What course are you doing?

146. V: Um! Well right now I'm not taking any math classes. So am just taking like Chem… Um… Physics. Ok I'm taking a math class. Physics but that's about it.

147. G: Oh, okay! So you've taken all the math classes you need to take?

148. V: Huh-un!

149. G: So. What do you wanna be? What do you wanna do?

150. V: I want to be a pediatrician.

151. G: [Whispered] Pediatrician!

152. V: Huh-un!

153. G: I'll bring my kids. Ok! [Both laughing] That's cool. Wow! So when do you intend graduating?

154. V: Um! I wanna graduate hopefully within a year.

155. G: You must have taken so many classes then.

156. V: Yea! Yea I'm already almost done.

157. G: Am really impressed with this. Thank you so very much. You really did good. That you could still remember what you did. That's quite about two… almost two years now, right? That's 2009 fall.

158. V: [laughing] I know. I know. I didn't even really remember the survey to be honest. When I got the email, I was thinking, what survey did I take? [laughing]

159. G: Ooh! Yea! Honestly, I really appreciate your… your time.

160. V: No, thank you so much for allowing me to. [Hand shake] Sorry I confused you so much.

161. G: O no! No you didn't confuse me. I just want to, you know, to really understand you because; your responses are so wonderful. I'm like… I need to really bring out what

is inside of you; you need to explain it very well to me. You know! [Interviewee laughing] Thank you. So just take this little gift from us. We really appreciate you.

162. V: O thank you so much…. for that. Thank you so much.

163. G: Have a good day. [Following to the door]

164. V: You too.

Appendix F: Transcript for Susseth

Interview with Susseth, April 29, 2013 at 12:00 pm in the Education Building

Duration: 16.31minutes.

1. G: Hello Susseth. How are you today?
2. S: Fine thank you.
3. G: Good. I want to thank you once again for giving your time to take this interview and also for responding to the survey the other time. So we appreciate it. Um! We call you for interview because we are interested in some of the responses you gave us in the survey, so we really want to know more. I know that the time was so short at that period, so I know you couldn't respond as much as you want to. So we say that maybe when we talk with you…
4. S: OK.
5. G: … you will be able to explain more, that we'll understand you very well. So, Maybe before I ask, let me, before we go in there let me ask Um! That's calculus 1 class right?
6. S: Yes mam!
7. G: What are the math courses you've taken in the previous time?
8. S: Um! Just precalculus
9. G: Oh you've taken precalculus? What about in high school?
10. S: Um, in high school I went up to calculus. I took Um! I guess. Algebra 2, geometry, precalculus and calculus.
11. G: Oh ok, ok. So coming to UTEP you're taking precalculus again. So it's like… you're already…
12. S: Yeah! It's like a repetition.
13. G: … have the foundation ----- Oh! That was good, that was good. Ok, Um look into the survey, I'm really interested in a few questions. Let's look at number 1. It says "when you think of infinity, what comes to your mind?" You said like "A large number of

concepts, so large that it cannot be expressed mathematically, just conceptually". Can you?

14. S: Um! Well I guess what I meant more was that when it's used mathematically, it's not necessarily used as a concrete number. It's Um! When you're at infinity, its, that's just it. It's like a very, very large number. It's not anything Um… concrete that we can actually use like when we're doing calculations. When you have infinity, well then, it's the end. When you have infinity, you take away small amount you still have infinity. So Um! and then conceptually I think it's … it's difficult to imagine infinity conceptually, but when I hear infinity I first think of it as an idea not as like when use mathematically.
15. G: Ok, ok, Um! Let's look at number 2c. It was talking about the cookie monster. Yea! Ok you said if the… the question says if the process continues, will he ever eat the entire cookie? And you should explain. And you said not mathematically but physically there comes a time, I can't read…
16. S: Oh! Yeah.
17. G: … It says there comes a point where you're splitting elements and atoms and that's not viable.
18. S: When I guess you experiment Um! With that, yea! When you're doing mathematically like an equation, then, Um! Just… Yeah! I guess conceptually when you're approaching infinity or something like that or infinitely large, infinitely small, it's… well it makes sense because it's just numbers like on a page. But physically, when you're trying to divide something I guess to a point we can only… well… when you divide it's not where it was anymore. You like a cookie, you… you get to a point where once you divide it, I guess it's not cookie technically it's like flour and butter or whatever items make up the [laugh] what comes in the cookie, so here I guess maybe mathematically he would not eat the cookie but physically well if you come in and you take it, at some point there won't… I don't think that there will be anything left. Just only crumbs and then you come back and you can't physically divide what's left.

19. G: Ok. That's physically.
20. S: Yea
21. G: Then give me the explanation of the mathematically again.
22. S: Um! Mathematically, I guess when you approach like a limit…
23. G: OK.
24. S: … and the limit is expressed it makes sense. Because you have the formula that you followed, you have that, you know you get one-sixteenth (1/16) even if you get one over very, very large number, it still makes sense. It's on a paper. You see the number. That's ...I guess that's what I mean [laugh] Am sorry I don't know if it's …
25. G: O no! No! You're good. Ok. I want you to look at this. The Zeno's dichotomy.
26. S: OK.
27. G: I want you to read it and sss… Let me see how you can… [Handed interview task paper to S].
28. S: Do you want me to fill in my ID?
29. G: Yeah! O no! Don't worry about that.
30. S: OK. [Pause to read the problem] Alright! Do you want me to…?
31. G: Yea!
32. S: Ok! [Long pause to write down response]
33. G: Ok, so…
34. S: Ok Um! I don't think that the runner will ever complete the course because Um! So half a distance is pretty long, a quarter of the distance is small and then, um! At some point he'll, Um! I guess if it continued from my understanding of the problem, well I guess his shoes might end up been larger than what's left to travel say one-sixth (1/6), one… one… one hundredth, and one –one hundredth of the distance whatever, and he won't... If he moves his foot to run, he would cover more than he was technically supposed to cover. So I don't think he'll, he'll ever complete the course because at some point he'll just have to stop moving because the distance left, one, whatever, one

thousandth, one-one thousandth of the distance. It will be too large for him to like physically move his foot over.

35. G: Will be too large?
36. S: Well, too small I mean. I'm sorry. So, but if it's just one-half (1/2) one-fourth (1/4) one-eighth (1/8) well then, you know 'cause he'll have to stop. But if it's like continued, I guess the fraction gets smaller and smaller and smaller. Yea at some point he won't be able to physically cover the distance. [Laughing]
37. G: So physically, he won't cover the distance?
38. S: Yea! [Laughing]
39. G: Oh OK. Ok! If that is physically, is there any other reasoning you have?
40. S: Um! Well I guess even mathematically, I just guess the interval will just continue to get smaller and smaller and smaller. Um! I guess. Yeah! If you approach an inf… an… a limit where definitely you might find a point where it's like ok, well the limit equal to zero, so he won't be able to move because … like… that's the limit. [laughing] I don't know. I'm sorry.
41. G: So in essence he won't complete the course?
42. S: No
43. G: Ok. Look at the (b) part [Interview protocol number 2]. It says which of these statements is true.
44. S: Um! I think less than 1. But am I? Can I Go ahead?
45. G: Hun-un! Hun-un!
46. S: Yea! Ok. So yea! Less than 1. Um! Because when you start adding the smaller fraction, that's why the number at the bottom had to get bigger, because… Um! Like you have to convert like one half. When you're adding the one-eighth (1/8) you want to change it to four-eighth (4/8) so way you can add them easier. I guess and then, when… as it continues, like they won't cease, and then the number at the bottom gets smaller and

smaller, I mean it gets larger. That means the fraction is smaller and what you're adding gets smaller and smaller and smaller. So I don't think that it will ever reach 1.

47. G: So it will be less than 1?
48. S: Yea, I think it will be less than 1.
49. G: So you're choosing (A)?
50. S: Yes mam!
51. G: Uh-umm! Ok. Alright. One more. [Both laughed]
52. S: Ok!
53. G: Let's look at the survey number 4.
54. S: Ok!
55. G: It says that… If the… What's your feeling about the conception of infinity? Is it a process or an object? And I saw you chose it's both a process and an object.
56. S: Um! I guess I feel that way because, I think mathematically it's an object to use it in a formula, in and out. Um! It's… is not necessarily concrete. But you see infinity. You have an idea. Infinity or negative infinity, you know that it's extremely large, or extremely small. But also I think that when I first think of infinity, I think of… Yea! Kind of just like darkness been never ending. I think of it kind of visually I guess. So I can say my conception is both, that Um… when am using it in math, mathematical terms or like doing math homework, I do use it as an object. But I guess if am just thinking about infinity or Um… kind of a… like science, like astronomy, like when I think about the universe, and kind of I guess like my mind, stuff like that, I think of it more yea, it is a process. Like it's just something that continues.
57. G: Ok like something that continues? Is that a process view?
58. S: Continues without ending.
59. G: Ok. Ok let me… let's focus on the object part, the mathematical part now.
60. S: Ok. Um!

61. G: You said when you're using it in math, how do you use it, in math that makes you see it as an object?

62. S: Um! I guess, when for example we're finding the limit of something, when you write it down you draw the infinity symbol and it's there, and you know what it represent and you know what you have to do with that number. Like you know, you have to Um… it's positive, or it's negative. Like yea! I think it's an object when I draw the infinity symbol. And I feel like that is, me making it a concrete concept. But ok the number is approaching infinity, and now I know what to do with it. Where, yea! I guess that's what I mean. Like… I use it as a toy I guess. Like, it's a symbol. Not necessarily I guess for the number that it represents but for what I have to do with the problem because that symbol is there.

63. G: Oh. So it's the symbol that's just the object… part of it?

64. S: Yea! And mathematically. Yea!

65. G: Ok let me, let me do something. I want you to look at that Task (b) again [Interview protocol number 2]. Those statements before… The five statements A, B, C, D, E. You chose (A) to be less than 1.

66. S: Yes mam.

67. G: Do we take that to be an object view or a process view?

68. S: I guess it's a process. [Laughed]. Yea I guess, I guess when I think of it as an object is when I have the symbol of infinity but, this is a process, and I do think of it as infinity too. [Laughing] It's funny! [Laughing]

69. G: Yea! So you couldn't see anything that could depict it as an object?

70. S: Yes. O I guess!

71. G: Am just asking, maybe…

72. S: No! Yeah! That's true. Even, even in mathematics I guess, when I'm using it for the symbol, I guess it‘s an object. But when you're repeating something over and over again,

such as this, I guess it's a process. So, yeah! I do. Something repeatable is a process. [Whispered] I don't know. It's funny I think.

73. G: Do you see any relationship with (A) and (B)?
74. S: Um!
75. G: With Dichotomy and this!
76. S: Well, I guess! When you're Um… calculating how much distance he's travelled over all, he is kind of… is process.
77. G: Ok! Ok… Ok now I get to see that's how you… you have the one-fourth (1/4). That's how you added it. Let me see what you added there.
78. S: This is how I did it. [Laughing]
79. G Ok. So you have the 1… How… Can you explain this? Why do you have the 8s?
80. S: Ok. Um! Well. It's an addition problem.
81. G: Ok.
82. S: And Um, even though most of these are inequalities, I guess when I'm solving an inequality I just make it equal to whatever. And Um… for me to add it better, I just want them all to have the same Um… Um… denominator.
83. G: Oh! Ok.
84. S: So that way… in the end I'm just adding numbers.
85. G: Oh! Ok.
86. S: So Um… I just changed like one-fourth (1/4) to the same as Um… four-eighth (4/8).
87. G: one-half (1/2).
88. S: I mean yea! One- half (1/2) is the same as four-eighth (4/8). So one-fourth (1/4) is two-eighth (2/8).
89. G: Ok so that's just what you did?
90. S: Yes! [Laughed]
91. G: Ok. So when you got 7/8… that's when you now…
92. S: It's less than 1…

93. G: It will be less than 1!

94. S: And then I guess, I thought that if I also add the next smallest thing, that it will be one-sixteenth (1/16). And then, when you change that, it will be eight-sixteenth (8/16), two-sixteenth (2/16) whatever, and it will still be less than 1. This bottom number will just get bigger and bigger even as the top ones get bigger and bigger.

95. G: So the process continues?

96. S: Yes [Laughing]

97. G: Cool! This is just what I… want to find out about that, 'cause I'm only particular about why you chose a process and an object.

98. S: OK.

99. G: What actually do you understand by that?

100. S: Yea!

101. G: So that…

102. S: I changed my mind about it. [Both laughing]

103. G: Because you said "not mathematically but physically", so I really want to know the physical aspect, the math aspect that maybe you didn't have the chance to explain in writing.

104. S: Yea! So yea! I guess I think of it as a process with repetition. You're repeating this by adding a number, and you add a number and you add a number, and you repeat it. But when I think of it as an object, I think of it when you're using the infinity symbol. You're not, you're not denoting to repeat something over and over again. Well, kind of you are but [Laughing] I don't know. [Laughing]

105. G: O Ok! Thank you.

106. S: Thank you so much.

107. G: Thank you so much.

108. G: [Interruption-End of memory space] Do you remember the definition of infinity given you, in any of the classes you have taken?

109. S: I can't really… remember. I just… I guess I was thought infinity was just a very, very large number. In class… and that… yea! So very large number! And that is from using it in math. But when I think of… I guess infinity… not really. I don't know. I don't remember well really. So I'm telling you, what infinity was. I guess kind of…

110. G: In any of your math classes?

111. S: I don't remember. I just… I see the symbol and I know that was infinity. I don't know. I was… I was trying to remember well… really like… Ok. Well this means infinity. And infinity… since… I don't remember well. I can't.

Appendix G: Transcript for Robin

Interview with Robin, May 8, 2013 at 9:30 am in the Education Building

Duration: 23.41minutes

1. G: Hi Robin.
2. R: Hello. Good morning.
3. G: Good morning. You are welcome. Eem! Once again we want to thank you for making out time to attend to this interview. We want to really appreciate, one, your time for the survey, and also your time for this one.
4. R: Thank you. My pleasure.
5. G: You are welcome. Yeah! So I have here your survey. Your responses the other time. And before we look at that, I want you to look at the other 2 questions I have there.
6. R: Yes!
7. G: Yeah! That talks about the Zeno's Dichotomy,
8. R: Uhnm!
9. G: And then the… the second one that talks about … about 5 questions there, that you need to make a choice of it. So let's look at the dichotomy. Robin, what is your response to that?
10. R: For what? The first one?
11. G: To the first one! Yeah!
12. R: On the first one, I don't think he will actually reach the actual point. Because, if you notice, he starts off with ah! One-half the distance, and then one-fourth of the one-half the distance, then one-eighth. Then if he keeps going, that the number will keep getting smaller and smaller and smaller, smaller, he won't actually reach the actual point. Because we know like, say like one-half is point 5, and then plus point twenty five is one-fourth. The numbers will keep getting smaller and smaller. And it keeps going around towards infinity. Will be the actual final … I don't think actually hit that one spot.

13. G: So the runner will not complete the course?
14. R: No. As he keeps going. He keeps going half and half and half. No!
15. G: Oh ok. Thank you. Then, aa-mm! In line with that, let's look at your … your part 2.
16. R: Uuhm!
17. G: It talks about which of these statements is true. Is it the one that is less than, equal to… A, B, C, D, E? What's your choice
18. R: Aa-mm! I chose… Ee-mm! (A) That it will be less than 1, because of the way I think … Well, it will keep getting smaller and smaller. You can have two numbers say for instance, one and zero. There is an infinite number of points between any number. And since all we're doing is just making each values half and half and half, you won't ever really approach the actual regular…. You may get really, really close, eight point nine, nine, nine, nine… But not actually towards whatever. You keep getting smaller. It it's bigger, it'll be a different story.
19. G: Ok, so it will never reach 1?
20. R: No!
21. G: Ok. Ok. Thank you Robin. Ok now, let's come back to the… [He handed me survey copy] I made a copy. So we can…. We'll start with question 1. It says when you think of infinity, what comes to your mind? And you said something without beginning or end. Something undefined.
22. R: Yes!
23. G: You want to… throw more light on that?
24. R: Yes! The reason I said undefined is, often times I think people think infinity is actual concept because of the sign. But it represents something that is like an actual number. Say for instance we have x that is set to equal to something. However, infinity or negative infinity, you can't really set that, because it keeps going on and on. Just like numerical value, you can actually place towards infinity, someone just realize it keeps going on and on. That's infinity.

25. G: So it’s like a numerical value?
26. R: No, not numerical value. I think it just represent something that keeps going on and on. I think people would sometimes get that kind of confused. They think infinity is actual numbers but infinity is not number. It’s just keeps going and going. It’s never gonna stop.
27. G: Ok. So when you say something undefined…
28. R: Yes. Undefined because it’s not an actual number per se, so we can’t really define it. We just know it just keeps going and going. Like say for instance, I have 2 plus infinity. That number will be undefined because the infinity is not actual number we can add. That just means 2 plus what will keep going on and on. So the answer will kind of be undefined. Well, will be undefined.
29. G: Ok. Thank you Robin. Alright, turn to question 2 on the survey.
30. R: Huumm!
31. G: Talking about the cookie monster. [Both laughed] Yeah! I see your response to the (C) part. That if the process continues, will he eat the entire cookie? You said No! The remaining will continue to grow smaller.
32. R: Yeah! You never… you will see it. It just keeps getting smaller and smaller. Cause of the half of that half. Then, half of half will be one-fourth and half of that will be one-eight. Then half will be one-sixteenth. Then just because we don’t see it. It just keep going and going and going. So until everything will grow smaller. Which I think goes back to my first point which says; each number has an infinite number of numbers between each point. So I don’t think he will ever finish it because it will keep going at that rate. The number will just be smaller and smaller. Would be one over … like some large … large number that will just be getting really, really small, but you really can’t see it, it’s still there, technically speaking or mathematically speaking.
33. G: Ok. But what actually caught my interest was the “grow smaller”. So, I want to… that grow and smaller…

34. R: Huumm! Means, I'm sorry, not grow… I kind of word that wrong. It would become smaller, not grow smaller. Yeah!
35. G: [Laughing]
36. R: ------------- That's actually poor choice of word on my part.
37. G: Oh. Ok.
38. R: It would become smaller, not grow smaller. My apologies.
39. G: Oh Ok. Because I thought maybe… ok.
40. R: Yeah! It sounds like an oxymoron that I put that on there. Yes. It will become smaller, not grow smaller. My apologies.
41. G: Alright! Alright! Alright! That's why I just say let me understand you very well. Ok! Then your number (3) you said, it's not possible to depict infinity.
42. R: Oh! Eemm! Well. I guess you can draw aa-mm! Like a sign which we can stand for infinity. But the actual infinity you can't really draw because it's without beginning or end. It's hard to put something without beginning or end on a piece of paper.
43. G: When you say, draw the sign… Ehm! Do I…let me see the sign you are talking about?
44. R: [Drew the infinity symbol] Yes!
45. G: Oh! Oh. Ok.
46. R: O yes! That represent infinity, but I don't think we can actually draw infinity. I guess that would be kind of akin to trying to represent a four dimensional, I guess object in three dimensional way. It just wouldn't work. Something like that!
47. G: Hmmh! That's deep. [Both laughing] OK! Then number (4).
48. R: Yes!
49. G: Talks about your conception of infinity. Here we're talking about a process and object. And you said… You chose both… A process and an object. Can you explain?
50. R: Yes! I think because… Well in math, I think there's an actual infinity and potential infinity, in which with an understanding in both somehow well. Like I guess in, when we use limit we can actually use the concept of infinity to actually… Eehm! Get actual

concrete numbers. But the actual infinity is not really defined. And I guess we all still use infinity when we're trying to calculate aa-mm! Interest in something like aah! Re—ca… ahm! Compounds who move continuously, we really can't do it ourselves to actual infinity, but we use math, you know the theorem can actually state it's gonna be approaching e, which is an actual number. So this I guess is kind of like a process-object. But still, eehm! Infinity is still not something that we understand completely, because it's not really defined. Like a concept or something. We can't really say infinity is this number right here, because it's not. Because then it wouldn't be going. It wouldn't be infinity.

51. G: Ok! If… I want you to explain your view, your own understanding of the process view and then the object view. Can you do that for me?
52. R: Eehm! Well, I think we could actually use the process view to actually get concrete numbers, like I stated previously. Like we use it, when we use limits to get horizontal asymptote and also we use that with aahm! We can count continuously with actual real numbers. However, the actual infinity is not something that can be understood is--------- well… without understanding the math I know. ----------------------- It just fold over itself, you know. But I don't think we can actually define it because it just keep going and going and going. We just understand, since it doesn't stop or ends, it will just tend to infinity.
53. G: So, that's your process…?
54. R: Yes!
55. G: Then, what of your object…?
56. R: Well. I'm sorry. I think the process would have been… I really reverse that pert. Yeah!
57. G: Oh. Ok aahan!
58. R: Yeah! The object aa-mm! You can't really define, but the process you can actually use infinity to get the concrete numbers, is what am saying.

59. G: Let's… I'll like to say… [Pause] Ee-mm! Let me just ask some extra questions based on number (2) of the ...
60. R: Ok. This one!
61. G: Not this one. The other one.
62. R: Ok. Yes!
63. G: Question (2) that gives the five statements. I know you chose that it will be less than 1.
64. R: Yes!
65. G: You said as it goes smaller and smaller, it will never reach 1.
66. R: Yes! And yes, because well… I think you keep adding the numbers, it might be bigger or larger. But since the numbers are getting smaller and smaller, you never gonna actually reach 1. [Referring to (2B)]. You might get very close like point nine, nine, nine. But then if you keep adding one, and reach 1. It might be like point zero nine, nine, nine. But it won't kind of ever be 1. Now I don't think it would ever be equal or would be equal to 1, based on my reason before. [Referring to (2C)] And so, I don't think it would be less than or equal to because it's only less than or equal to that. So this one is actually wrong. And I don't think it would be greater because it's gonna get smaller not larger.
67. G: Uuhm! So it will not be equal to 1.
68. R: Yes. I don't think it can equal 1.
69. G: It can equal 1?
70. R: I don't think it can equal 1.
71. G: Ok, why don't we look at this statement now, and try to take it as process and object view. What do you think your perspective would be?
72. R: Uumh! Well, I guess it'll possibly be… what's the question? Let me see. Well while it was like, I guess we've added this already, this means seven over eight, so if we add another one, cause we know the denominator is decreasing by half. So we add another one over sixteen, and another… the number will just keep getting smaller and smaller.
73. G: Ok, based on your option on number … (A). You choose (A), which is less than 1.

74. R: Yes!
75. G: What view can we take that to be? Is it a process or an object?
76. R: Uhm! [Pause] I guess it is kind of a process because you'll… Yea! You'll keep getting smaller towards infinity, but it's not actual… you're never gonna reach actual object of infinity. If that makes sense. Yes!
77. G: There is none that can be likened to an object there?
78. R: I beg your pardon!
79. G: Is there anyone that can be likened to an object? Or how can we represent the object view? Since your view is like it's a process and also an object. So how can we…?
80. R: I guess we'll… I guess we'll... Ahm! How will I write this?
81. G: How can we represent the object view?
82. R: Well. I was just trying to think of... maybe as in limit. Or as these numbers...
83. G: Or if you have any other example to depict the object view of infinity.
84. R: Well I was just trying to think if I actually think it's an object though, because that would depend on the findings. Will… will it really be an object. Am trying to think back on that. But I guess we just know that as it keeps on going forever and ever, it can still go on towards infinity, but not ever actually reach the object infinity because it's not actually an object, it's just a concept. I think about it. Yeah!
85. G: Because I want to really listen to your response, since you choose both process and object. [Referring to Survey number 4]. So I want to see if there is any way you could give me an example...
86. R: See! This process will actually show how it's actually going towards... it keeps going on and on and on towards infinity. But it won't actually reach the actual object of infinity. Now I think about it, because it is undefined. Yes! I think ahm! Looking back, I probably would have just... I should... I think I didn't read this carefully, I should have probably be an (a)
87. G: So you think it would be an (a)?

88. R: Yeah! Because we can keep doing this and we can keep going on and on and on, but we'll never stop because we just keep increasing it by half. It would keep going towards infinity but not actual infinity.
89. G: Then if you tell me it's an (a) now, then you want to tell me that you don't think infinity is an object?
90. R: No. I think it's undefined. So I don't think you can actually grasp what infinity actually is… like we would with like another number like pi.
91. G: So, it can only be a process as far as you think?
92. R: Yeah! I think you can't actually ever get to infinity because it would... infinity by definition is a stop or an end... So, he will never actually get to that.
93. G: So, if we get to an infinity to you it is what?
94. R: I... I don't think we can actually get to that because I’ll go get some definition. Being because it keeps going on and on so, if I kept adding numbers my whole life, I don't think I can ever get close to infinity. Because even if I die, it would still keep going on and on. Regardless.
95. G: Oh! OK. So that's why even to you, this... this statement... it's not possible adding all these... fractions to...
96. R: Yeah! I think... I think all those fractions they won’t ever each… reach 1. It’s gonna reach a limit of 1. But won’t actually reach 1. I don't know how to just represent that numerically. I guess I put limit as we add all these would reach 1, but won't ever be 1.
97. G: It would reach 1?
98. R: Oh! It'll... The limit would be 1.
99. G: The limit would be 1...
100. R: Yea!
101. G: …but would never be 1.
102. R: Yes!
103. G: So, it will still be less than 1?

104. R: Yes! It might get very, very close to 1, but won't ever be 1.

105. G: Oh! OK. OK. OK! Good. I was... I was gonna think that do you mean to say it's a (C) or ...[Laughing]

106. R: Oh no, no! I don't think it would ever reach 1. I think it might get very, very close to 1, but it won't ever be 1 or greater than 1.

107. G: OK. OK. OK! So it's definitely an (a)?

108. R: Yes!

109. G: OK.

110. R: Yes mam!

111. G: That is what I wanna find out. Because, when you said a process and an object, I'm like OK. Maybe you need to explain to me... When you say undefined, I want to know your process understanding...

112. R: Hu-un!

113. G: ...and your object understanding. That maybe you didn't have enough time to really explain...

114. R: OK.

115. G: ...further to me. So...so… so our number...number (4), so do you still wanna keep it a (C)?

116. R: You know if I went back I'll probably put it as em! Let me see... Which and again? That is true, it’s gonna be … The set of natural numbers is infinite because there is no end to natural numbers too. So I guess it will kind of, could be thought of as to be both. ‘Cause it does the huh! --- If you look back to the definition of natural numbers.

117. G: Uhun!

118. R: It keeps going on and on, but a object is actual em! Natural numbers... but the process of natural numbers doesn't stop. It just keeps going on and on. So I guess it can kind of be both. If you restrict the definitions, fair enough. If you use the definition it might be only both.

119. G: OK. By the definition?

120. R: Yeah!

121. G: It will be both?

122. R: I think it might be, because set of natural numbers don't end. It keeps going. And if as an object, the set of natural numbers. Then that natural numbers are infinite. Because, they... the process... well... [Pause] Hu-un! That was really... that's a really good question. [Laughing] Yeah! Hu-un!

123. G: [Laughed] I want to say look at it very well.

124. R: [Long pause]

125. G: Just take your time, because I know that day, you know, we didn't really have much time to settle down to...

126. R: O Yes!

127. G: ...to look at it. So... And because of your response to it, I said OK. Maybe you might... you will have something to say and... [Laughed]

128. R: O yes! [Long pause] Well, natural numbers are just real numbers and they are just an infinite numbers of natural numbers. So I guess, well! They... I don't know. I think they might be… still be the process. I think em! Natural numbers are actual infinite though.

129. G: Can you represent these set of natural numbers? Can you give an example?

130. R: Oh! Natural numbers are 1, 2, 3, 4, and on and on and on. That's na... They are non... They are non-imaginary numbers. In other words.

131. G: OK! A set of natural numbers.

132. R: Cause, yeah! The set of natural numbers is infinite. It's not gonna stop. 'Cause it's gonna be getting larger and larger. And if you think about them, it is kind of a natural number. Something that goes on and on. So that's why I kind of like, I'll see it in both of those. Because object too is as natural numbers 1, 2, 3. But the actual natural numbers keeps going on and on, so might be a process too. I guess that's why I chose em! (a) Oh!

(c). [Referring to Survey number 4]. I just want to know that infinity isn’t actually something really hard to find.

133. G: OK! Um! Now. Let me... Let me get you right. So you are questioning the (b) now, from being a process?

134. R: Hunh!

135. G: You're questioning the (b) from being a... being a process?

136. R: Yes! Because, well! The object of that actual number is infinite. That's true. And then the process is something that goes on and on. Well! Am not sh! Not think about it. Am not really sure if... how that goes on and on. Like in … Yes! Yeah! Let’s try this in a lot of good question. Good falsie question.

137. G: OK. So, left to you. Will change (a) and (b). Both, is a process from the definition.

138. R: Yes I think you might be able to argue for both. But probably reflecting on this, ahm! View the object as natural numbers you can't really represent all of them. So, that's only defined. Cause, to define all natural numbers, you have to write them all down. So you write them all down, and you can't do that. I think that is why you just use... represent using that symbol right there. It goes on and on and on.

139. G: [Pause] Hu-umh! I'm thinking of something now. [Pause to write] If you have something like this [Writing A= {1, 2, 3, 4 }]

140. R: Hu-umh!

141. G: What will you call that?

142. R: Oh may I see real quick?

143. G: Yeah!

144. R: See! What... we define A as 1, 2, 3, and 4. I guess A is just 1 through 4 of natural numbers I believe.

145. G: Is what?

146. R: We can define A as natural numbers 1 through 4. O well! Yeah! You could define it right there from 1 to 4.

147. G: OK! So it's infinite?

148. R: Yeah! That one is finite, for sure.

149. G: It's finite?

150. R: Yes!

151. G: OK!

152. R: So you define them as those numbers. Yes!

153. G: So, if I put it this way... [Writing A= {1, 2, 3, ……}]

154. R: Oh! OK. Good! Yes. I think A now would be... It's... Well! It's rather one number that just keeps going on. It won't stop. Hence, ta-ta-ta. I think A would... is one number that will approach infinity.

155. G: OK! So A itself is what?

156. R: A is itself is... Ahm ... I will... I guess you can say, approach infinity but won't be infinity. Because, yeah! Infinity is not something defined yet. So I guess you can also represent that as I… I guess. Moving on, put a domain in every… from 1 to infinity like that [1, ∞) ----- Meanwhile, 1 is defined but infinity won't be defined.

157. G: OK. So that's also a process?

158. R: Yeah! Kind of like your domain right there.

159. G: Yeah! Because it continues.

160. R: Yes!

161. G: OK. OK! Alright now I get you.

162. R: Then, that was... I think... am... Does that make sense kind of?

163. G: Yeah! Yeah!

164. R: ...with my response?

165. G: That's what I'm trying to... to get from you. That's what... And with the example you gave. This [1, ∞) actually says the same thing with this [A= {1, 2, 3 …}].

166. R: Uhum!

167. G: With what I wrote. So, OK Robin. I think I'm good.

168. R: OK. Awesome!

169. G: Yeah! I think I'm good. Am clear with your undefined!

170. R: OK. Yes! [Laughing]

171. G: Thank you so very much.

172. R: You're very welcome! Thank you very much for your time.

[Further questioning]

173. G: Ok. Ahm! You mentioned something about actual and potential infinity?

174. R: Yes! Oh ok! Yes mam!

175. G: Do you want to…

176. R: O well I see here in mathematics you can understand, like there's an actual thing… well… potential thing going on. Oh! Oh! I shouldn't compare that. Because it has something to do with philosophy today, I was reading about, I guess it's ----- I can't recall the definition at this point.

177. G: Aw! Aw!

178. R: And that's error on my part, I'm sorry.

179. G: Aw! Ok. Because like…

180. R: 'Cause I think actual infinity doesn't exist. But there's a very good possibility that actual infinity we can actually define one possibly. But potential infinity does exist because it keeps going on and on.

181. G: Actual does not exist?

182. R: Well, it's not defined.

183. G: It's not defined!

184. R: Yes.

185. G: O ok! Aah! I just… It just occurred to me that ah! You mentioned something about potential, you must know so much about it. [Laughing]. Ahm, how, how long have you known about infinity?

186. R: I was thinking about this because ah ... before in philosophy as well. And I guess because I took cosmetology before I began University and stuff … and I guess there's some theory that was stating that ah… the university is finite… and was infinity in the past and I kind of got interested, but I guess mathematics just disproof that.

187. G: Oh o o!

188. R: Yeah! Because like ahm… [inaudible] Lincoln, currently to meet this theorem states any University is been expanding on average has a finite past the beginning can… they can be going after. That's why I was really interested in that too.

189. G: O ok.

190. R: Yes.

191. G: Why I ask is that I want to know, is it that you leant it in class, maybe one of your math class?

192. R: Oh it was just something I did personally. It was personal reading.

193. G: Oh! Oh ok! So it's not that, it's in any of your math class that you know!

194. R: Oh ooh! Math class! No. No mam!

195. G: But have you come across the word infinity in any of your math class?

196. R: I beg your pardon! The what infinity?

197. G: Infinity. The word infinity.

198. R: Huunh! O well yeah we use it all time in calculus, like we use infinity when we're taking limits, calculating horizontal asymptote, and then we also use those ehm! As we… well the definition for compound continuously depends on the limit of infinity. That's why we use a… number "e".

199. G: Oh! OK. Ok, ok! Not that infinity was actually thought as maybe defined for you…

200. R: O no! Actually our professor actually stated it more than once, so it’s not defined. Yeah! She’s told us that before. I think she presents that problem together. Two miles to infinity, that would be undefined.

201. G: In which class is that?

202. R: Ah! Cal 1.

203. G: In Cal 1. Oh!

204. R: I forgot why that came up, but she brought that up for some reason. I think it was something to do with limits, if I’m not mistaken.

205. G: Oh! Ok. Ok. Now, this your study about infinity, was it after the… you took the survey or before you took the survey?

206. R: On which one?

207. G: The aw! Your…the study on the actual and potential infinity.

208. R: Well I thinking… that was before. I was just reading some stuff about cosmetology. I was just trying to…

209. G: Oh that was times ago. Ahm! Sometimes ago.

210. R: It was time I already think about it before. Yes.

211. G: Aah! Ok. So you’ve already had aah… understanding of …

212. R: Yeah! I was just trying to understand it more or less. Right.

213. G: Ok. Ok. I want to think maybe it was one of the classes you took that …

214. R: O no, mam. Nothing like… well just like everything we did in class was just what I told you now. Just when she brought that example of where infinity is undefined. When we got to the infinity minus 2.

215. G: Oh. Ok. Ok. Thank you Robin.

216. R: No problem.

Appendix H: Transcript for Emma

Interview with Emma, April 29, 2013 at 1:00 pm in the Education Building. Duration: 11.32minutes.

1. G: So, Emma, I want you to look at the question 1 of the survey.
2. E: Uh-hum!
3. G: It talks about when you think of infinity, what comes to your mind. And I like your response, saying “infinity is forever, an amount that cannot be reached or counted”. I want you to really throw more light into that for me.
4. E: Well if you count to numbers like an hundred and that sort of thing. And if you keep going, you can actually count to a million, but you can never really get to infinity because it just goes on and on and on. So in that way you can never count to infinity, you can only count to a million. But even then you’ll be really tired. So in that sense, infinity goes on forever because you’ll never be able to count it, because by the time you get there, you might be dead. [Both laughing]
5. G: Interesting! [Laughing] Ok, so when you say an amount, so does it mean that when you’re given a certain amount, you can’t reach it or what?
6. E: Yes! Like if I have 5 something that have 5, that I can count to 5. But if someone says am going to give you infinite number of apples, then you can’t ever have that many apples because it’s too many.
7. G: Good. Thank you. Ok! Let’s look at 2(c). It says, ah! That’s talking about the cookie monster.
8. E: Uh-hum!
9. G: It says if the process continues of eating half and half and half of the cookie, that will he ever eat the entire cookie? And you said “No, because as time progresses, the part of the cookie left to eat will become smaller until it is nonexistent”. Can you throw more light on that too?

10. E: Well if you have a cookie and you break in half, then you have half a cookie. But as you keep breaking in half you're having smaller and smaller cookies, until eventually all you have left is a speck of cookie. And that's really not an entire cookie because the speck can be… it is the same size as a speck of dust, so it just floats away. So you can't really ever eat the whole cookie, by eating half.
11. G: Ah! Ok! So because of the speck that will be left,
12. E: Uh-hum!
13. G: Ah! Ok. [Flipping pages] I have another one. Actually before we look at number (4) I want, I want us to look at these problems. I have 2 problems here, the Zeno's Dichotomy and then the second one, I have about 5 problems there which I want you to choose which one do you think, which one or which ones are true, and I would like an explanation on that. So first look at this. The Zeno's. You want to read it and tell what you think about it.
14. E: [Long pause. Reading the problem]
15. G: Ok. So Emma, on Um! the dichotomy, what's your explanation?
16. E: I believe that the runner will never complete the course because as you keep running and running, you'll still going to be adding distance, so as you keep reaching smaller and smaller number, the smaller the distance but you're still running it. So, I think that you're never going to finish the course, 'cause you're still adding more and more distance, as you continue along the fractions.
17. G: Oh ok. So as the pace continues, it would still continue. It doesn't reach an end?
18. E: Yes!
19. G: Ok. Alright then. Go the (b) part, number (2). Those math problems,
20. E: I believe that (B) and (C) are true because as you keep adding the fractions, you eventually get very close to 1 if not equal to 1. And if you keep adding the fractions, then you eventually get a number that's greater than 1. Even if it's just a little bit, it will still be bigger than 1.
21. G: Ok! Wait. Let's take it one at a time. The (B)?

22. E: Uh-hum!
23. G: What do you say?
24. E: I think it's true, because if you keep adding the fractions you will eventually get somewhere, very close, if equal to 1.
25. G: That will be equal to 1?
26. E: Uh-hum!
27. G: Is there any way you can do that, on the paper?
28. E: Well not very close, but I'll run out of fractions.
29. G: You'll run out of fractions?
30. E: Yea! I can't even figure it out any more. I got up to about point ninety seven (0.975). But I believe that if you keep going and going eventually you'll get to 1.
31. G: To one?
32. E: Yes!
33. G: Ok! So that's for (B)? Then what about the (C)? You said (C) also.
34. E: I think that once you get to 1, you'll still have fractions to add, even if it would be like one more millionth. But it would still be another number that you can add, so it'll be number bigger than 1.
35. G: Did you say bigger than 1?
36. E: Uh-hum!
37. G: Would the summation be bigger than 1, from what we have there?
38. E: Oh, I looked at the symbol there the wrong way. I mean, I meant to say (E) instead.
39. G: (E)! Ok, ok! Ok. You see that's why I want … [Both laughed] Ok. So you have (B) and (E)?
40. E: Yes!
41. G: So the summation of this fraction will either give you 1?
42. E: Uh-hum!
43. G: Or the (E). It will be bigger than 1?

44. E: Yes!

45. G: So what does that symbol tells us? It says what?

46. E: It will be greater than or equal to 1.

47. G: Greater than or equal to? So there are two ways?

48. E: Uh-hum!

49. G: It's either it's 1 or it's greater than 1?

50. E: Yes!

51. G: Ok! Now, do you look at (D) from what you said now?

52. E: (D) could be true because it's in between. It's greater than, but it's not equal to. So it could be true.

53. G: I want you to look at everything very well. [Both laughed] It's because of what you said, you know. It's because of what you said in (E). So am thinking, do you mean that inclusive or not? So let's know. Which are your options? Which ones do you think would be true?

54. E: (B), (D) and (E).

55. G: (B), (D),

56. E: And (E).

57. G: (E). Oh ok! So it will be equal to 1, it will be bigger than 1, and it will be bigger or equal to 1 like you said?

58. E: Uh-hum!

59. G: Oh Ok! So, that's the explanation there! Ok! Thank you. Alright. Let's look at… One I like what you wrote about your drawing; because I saw you draw a line for number (3).

60. E: Uh-hum!

61. G: And I saw the arrows going in both directions. Right? Then it says… Your explanation says "It's a line than never reaches a destination, but, much like the actual infinity". What do you mean by that?

62. E: Because, when you're adding arrow to a line like this, it means the arrow just keeps going and it has no self-stop. So I believe it's just like infinity because there is no real end to infinity, it just keeps going.
63. G: Infinity just keeps going? Then what do you mean by actual infinity? Let me just know your understanding about actual infinity.
64. E: I think what I meant by actual infinity is just the, thought of infinity going on forever, and never reaching an end.
65. G: Ok. Going on forever?
66. E: Uh-hum!
67. G: Never reaching an end! Thank you. Ok! Now lastly, let's look at (4). I feel that my conception of infinity is as… you chose (c). Both a process and as an object. Can you explain the process aspect and the object aspect?
68. E: I feel the process aspect… I really think infinity just keeps going and going, and there would be no end because, it, we can never reach it. And then, when I selected the object, I believe that you can keep counting forever and ever and ever, but you know that you'll never get to the end of all the numbers that can ever exist. Because a million is a big number, but there's a number bigger than that. And you can count those, but you can't count those forever. Like infinity, it just reaches a point where it doesn't go on, or it continues to go on. [Whispers] It's what I meant.
69. G: It reaches a point where?
70. E: It continues to go on, and then you can't count anymore because it's just too much.
71. G: So that's your object view?
72. E: Uh-hum!
73. G: of it. Ok now. Let me look at your (1) again. [Flipping the survey to page 1] Ok, so that's where you have the, an amount that never reaches. Ok, so your object view is that amount or what do you think?
74. E: Yes!

75. G: It's the amount that…

76. E: that you can never reach.

77. G: That you can never reach! That's your object view. Oh! Ok. Ok. Ok! I get you. [Both laughed] I get you. Because when you said actual infinity, I'm like, what do you mean by actual infinity? Then when you said both a process and an object, I'm like. Ok! Process! Object! I see you draw an arrow. So am trying to, let me not, assume, you know. [Both laughed]. So, ok now, I see your object view and your process view. [Pause] I'm just, let me, I'm just curious. Let's go to (2c). Going back to the cookie monster. It says if the process continues will he eat the entire cookie and you said, "No". Because there will be a tiny one

78. E: Uh-hum!

79. G: that'll be left, so he will not eat the entire cookie. What view do you have regarding that?

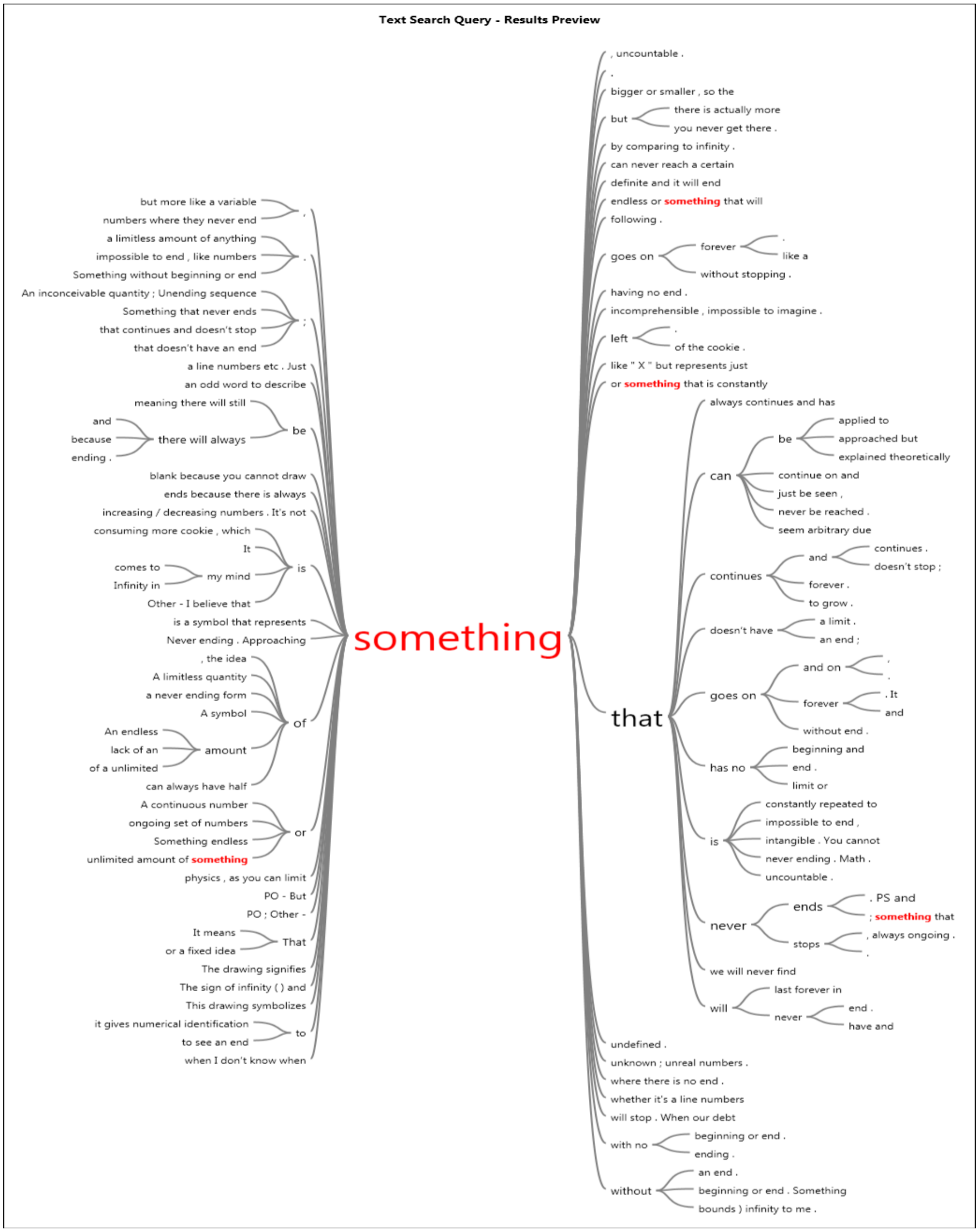
80. E: I think that will be more of the object view because it's sort of a, it's more of a number. Saying Oh I have half a cookie, and then a fourth of a cookie. So, it's more of, it's a set, you have half a cookie. So it's more of an object view because you know you have half a cookie, and then you can have a fourth, because you keep breaking it up. And then, the process view is more of, just like, you can keep eating the cookie forever and ever and ever. Ok, I'll just view like you have half a cookie and then you just keep breaking it up. So that's why I think it's the (C).

81. G: Oh ok, so that means from number (2) also, you could…

82. E: Uh-hum!

83. G: you could see that it's both an object and a process view? Interesting! Interesting!! That's why I'm happy that you are here today [Both laughed]. Am happy that you're here, to give this clarification. Thank you so much Emma. We really appreciate your time.

Appendix I: Text Search Query of the word "something"



Vita

Grace Olutayo Babarinsa-Ochiedike attended the University of Abuja, Nigeria, where she obtained her Bachelor of Science degree in Mathematics in 1995 and Post-Graduate Diploma in Computer Science in 1998. She was immediately employed as a Data Processing Manager at a Computer Institute. Her great passion for teaching made her joined Funtaj International School in Abuja, in 1999 as a mathematics teacher. She was a recipient of the Teacher of the Year Award in 2001. By 2005 and 2008 respectively, she obtained her Master of Science in Mathematics and Master of Education in Instructional Specialist from UTEP.

Dr. Babarinsa-Ochiedike joined the doctoral program in Teaching, Learning and Culture at UTEP in 2009. While pursuing her degree, she worked in several departments at UTEP as a Teaching and Research Assistant, as Adjunct Faculty in the Departments of Mathematical Sciences and Teacher Education and at the University of Phoenix as Mathematics Facilitator. She also volunteered several hours as math tutor at her local church – RCCG Living Word Center.

Dr. Babarinsa-Ochiedike has presented her research at International conferences including the Psychology of Mathematics Education-North America Conference (PME-NA), William Glasser Association International and Regional Conferences. She published her research in a book titled *"Development of creative abilities of students: In the context of implementation of federal state educational standards of new generation"* published by The National Book Center, Kazan, Russia, the Proceedings of PME-NA and the International Journal of Choice Theory and Reality Therapy. She was a recipient of UTEP Graduate School Travel Grant for research presentation in 2013. Her dissertation, *Challenges in Assessing College Students' Conception of Duality: The Case of Infinity*, was supervised by Dr. Mourat Tchoshanov.

Her post-graduation plan is to teach college mathematics, math methods courses and prepare quality mathematics teachers for quality schools, while continue research on the advanced mathematics concepts development and creating quality mathematics learning environment using Choice theory.

Permanent address: 12469 Flora Alba

El Paso, Texas, 79928

This dissertation was typed by Grace Olutayo Babarinsa-Ochiedike.