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# Adjoint Fuzzy Partition and Generalized Sampling Theorem

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**Abstract.** A new notion of adjoint fuzzy partition is introduced and the reconstruction of a function from its F-transform components is analyzed. An analogy with the Nyquist-Shannon-Kotelnikov sampling theorem is discussed.

**Keywords:** F-transform, adjoint fuzzy partition, sampling theorem, Nyquist-Shannon-Kotelnikov reconstruction

## 1 Introduction

We analyze the problem of whether a function can be reconstructed from a countable set of its F-transform components. We prove that if a function fulfills the same conditions as in the Nyquist-Shannon-Kotelnikov theorem (also known as a sampling theorem), see [4, 6, 12], then the above mentioned reconstruction is possible and moreover, the sampling theorem is its particular case.

Our inspiration came from the following analogy: similar to the F-transform components, signal samples can be computed on the basis of the partition generated by Dirac's delta function  $\delta$ . On the other hand, the reconstruction is performed with the help of another partition generated by the function  $\text{sinc}$ . We analyzed the interconnection between  $\delta$  and  $\text{sinc}$  and extracted a principal characteristic that we call *adjointness*. If partitions are generated by adjoint functions, they are called *adjoint* as well. Adjoint fuzzy partitions are used in the direct and newly defined inverse F-transform so that their mutually inverse correspondence is guaranteed for functions that fulfill the same conditions as in the standard sampling theorem.

The F-transform is very useful in many applications such as image and signal processing, image compression, time series prediction, etc.; see, e.g., [2, 5, 8, 9]. The initially proposed inverse F-transform [8] is lossy; i.e., except for constant functions, it produces a result that is different from an original object. This fact motivated us to modify the definition of the inverse F-transform to extend

the space of original functions, for which direct and inverse F-transforms are mutually inverse.

In the proposed contribution<sup>\*)</sup>, we give a short overview of the F-transform theory and its evolution. We discuss various fuzzy partitions and extend the notion of the inverse F-transform. We introduce a notion of an adjoint fuzzy partition and discuss its properties. Finally, we prove the main theoretical result about reconstruction from a countable set of F-transform components.

## 2 Preliminaries: Nyquist-Shannon-Kotelnikov Reconstruction

In this section, we provide a short review of the background of the sample-based reconstruction of a band-limited signal.

We assume that a digital signal is identified with a function varying in time, which is assumed to have a Fourier transform that is zero outside some bounded interval (in other words, a signal is *band-limited* to a given *bandwidth*). The sampling theorem (also known as Nyquist-Shannon-Kotelnikov theorem, see [4, 6, 12]) characterizes what is sufficient for full reconstruction of a signal from a set of its samples.

### Theorem 1 (Sampling Theorem).

Let  $x \in L_2(\mathbb{R})$  be continuous and band-limited, i.e.,  $\hat{x}(\omega) = 0$  for  $|\omega| > \Omega$  where  $\hat{x}$  is the Fourier transform of  $x$  and  $\Omega$  is some positive constant. Then,  $x$  can be determined by its values at a discrete set of points:

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k\pi}{\Omega}\right) \cdot \frac{\sin(\Omega t - k\pi)}{\Omega t - k\pi}. \quad (1)$$

We will be using the following notation:  $h = \frac{\pi}{\Omega}$ ,  $t_k = \frac{k\pi}{\Omega} = k \cdot h$  and the corresponding reconstruction formula:

$$x(t) = \sum_{k=-\infty}^{\infty} x(t_k) \cdot \operatorname{sinc}\left(\frac{t}{h} - k\right), \quad (2)$$

where

$$\operatorname{sinc}(t) \stackrel{\text{def}}{=} \frac{\sin(\pi t)}{\pi t}.$$

## 3 The F-Transform: Short Overview and Evolution

The F-transform (originally, *fuzzy transform*) is a particular integral transform whose peculiarity consists in using a *fuzzy partition* of a universe of discourse

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<sup>\*)</sup> The extended version of this contribution together with the application to the problem of function “de-noising” was submitted to [11].

(usually,  $\mathbb{R}$ ). We observe that the F-transform method was motivated by the ideas and techniques of fuzzy logic (see, e.g., [15]) and especially by the Takagi-Sugeno models [14]. In addition, the idea of a fuzzy partition was derived from observing a collection of antecedents in a fuzzy rule based system. The direct F-transform components are possible consequents in the Takagi-Sugeno model with singletons.

The F-transform has two phases: direct and inverse (see details in [8]). The direct F-transform is applied to functions from  $L_2(\mathbb{R})$  and maps them linearly onto sequences (originally finite) of numeric/functional components. The inverse F-transform smoothly approximates the original function.

Let us remark that almost all fuzzy approximation models, including Takagi-Sugeno models [14], are based on linear-like combinations of fuzzy sets with numeric or functional coefficients. The principal difference between them and the inverse F-transform is in the computation of coefficients. In the F-transform case, these coefficients are weighted orthogonal projections on subdomains, such that the best approximation in a local sense is guaranteed. In Takagi-Sugeno models, the coefficients guarantee that the corresponding approximating function is a best approximation on a whole domain in the sense of the  $L_2$  metric. Similar models have been considered in [1, 7].

### 3.1 Fuzzy partition

The notion of a fuzzy partition does not have a nonambiguous meaning in fuzzy literature. We will not go into full detail but concentrate on an evolution of this notion in connection with the F-transform (see [3, 10, 13]).

A *fuzzy partition with the Ruspini condition* was introduced in [8] as a collection of bell-shaped fuzzy sets  $A_1, \dots, A_n$  on the real interval  $[a, b]$  with continuous membership functions, such that for all  $x \in [a, b]$ ,

$$\sum_{k=1}^n A_k(x) = 1.$$

This partition can be characterized as a “partition-of-unity”.

In [10], a generalized fuzzy partition without the Ruspini condition was proposed with the purpose of obtaining a better approximation by the inverse F-transform.

Below, in Definition 1, we introduce a particular case of a generalized fuzzy partition that is determined by a generating function. We say that function  $a : \mathbb{R} \rightarrow [0, 1]$  is a *generating function of a fuzzy partition* (a *generating function*, for short), if it is non-negative, continuous, even, bell-shaped and moreover, it vanishes outside  $[-1, 1]$  and fulfills  $\int_{-1}^1 a(t) dt = 1$ . Below, we give the example of a generating function, which we call the *raised cosine*:

$$a^{cos}(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi t)), & -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Generating function  $a$  produces infinitely many *rescaled* functions  $a_H : \mathbb{R} \rightarrow [0, 1]$  such that

$$a_H(t) \stackrel{\text{def}}{=} a\left(\frac{t}{H}\right),$$

where  $H$  is a positive number called a *scale factor*.

**Definition 1.** Let  $a : \mathbb{R} \rightarrow [0, 1]$  be a generating function of a fuzzy partition, i.e.,  $a$  is non-negative, continuous, even, bell-shaped, vanishes outside  $[-1, 1]$  and fulfills  $\int_{-1}^1 a(t) dt = 1$ . Let  $h > 0$ ,  $t_k = t_0 + k \cdot h$ ,  $k \in \mathbb{Z}$ , be uniformly distributed nodes<sup>\*\*)</sup> in  $\mathbb{R}$ . Let  $H > \frac{h}{2}$  and  $a_H$  be an  $H$ -rescaled version of  $a$ . With each node  $t_k$ , we correspond the translation  $a_k(t) = a_H(t_k - t)$ . We say that the set  $\{a_k, k \in \mathbb{Z}\}$  establishes an  $(h, H)$ -uniform fuzzy partition of  $\mathbb{R}$ . Functions  $a_k$  are called *basic functions*.

By the condition  $H > \frac{h}{2}$ , each point from  $\mathbb{R}$  is “covered” by at least one basic function - by this we mean that the value of this function at this point is greater than zero. By the condition  $h > 0$ , each point from  $\mathbb{R}$  is covered by at most a finite number of basic functions.

It is easy to see that (substituting  $s = \frac{t}{H}$ )

$$\int_{-\infty}^{\infty} a_H(t) dt = \int_{-H}^H a_H(t) dt = \int_{-H}^H a\left(\frac{t}{H}\right) dt = H \cdot \int_{-1}^1 a(s) ds = H. \quad (4)$$

If  $h = H$ , then an  $(h, H)$ -uniform fuzzy partition is called an  $h$ -uniform fuzzy partition.

The following lemma will be used in the sequel.

**Lemma 1.** Let  $a : \mathbb{R} \rightarrow [0, 1]$  be a generating function so that it is continuous, even, bell-shaped, vanishes outside  $[-1, 1]$  and fulfills  $\int_{-1}^1 a(t) dt = 1$ . Then, the following is valid:

$$\frac{1}{2} \leq \|a\|^2 \leq 1, \quad (5)$$

where  $\|a\|$  is the norm in  $L_2([-1, 1])$ .

In particular, if  $a = a^{\cos}$ , then  $\|a^{\cos}\|^2 = \frac{3}{4}$ .

### 3.2 Direct and Inverse F-transform

In this section, we review formal notions of the direct and inverse F-transforms as introduced in [8] and extend the latter.

Assume that  $x \in L_2(\mathbb{R})$  and  $\{a_k, k \in \mathbb{Z}\}$  is an  $(h, H)$ -uniform fuzzy partition of  $\mathbb{R}$ , where  $a_k(t) = a_H(t_k - t)$ ,  $a_H$  is the  $H$ -rescaled generating function  $a$ , and  $t_k = k \cdot h$ ,  $k \in \mathbb{Z}$ , are nodes. The sequence  $F[x] = \{X_k, k \in \mathbb{Z}\}$ , where

$$X_k = \frac{\int_{-\infty}^{\infty} a_k(s) \cdot x(s) ds}{\int_{-\infty}^{\infty} a_k(s) ds}, \quad (6)$$

<sup>\*\*)</sup> For simplicity of representation, we assume that  $t_0 = 0$ .

is called the (*direct*) *F-transform* of  $x$  with respect to  $\{a_k, k \in \mathbb{Z}\}$ . Real numbers  $X_k, k \in \mathbb{Z}$ , are called the *F-transform components* of  $x$ . Due to the assumption of uniformity of the partition and by (4), the representation (6) of  $X_k$  can be simplified as follows:

$$X_k = \frac{\int_{-\infty}^{\infty} a_H(t_k - s) \cdot x(s) ds}{\int_{-\infty}^{\infty} a_H(t_k - s) ds} = \frac{1}{H} \int_{-\infty}^{\infty} a_H(t_k - s) \cdot x(s) ds. \quad (7)$$

It is easy to see that if  $x, y \in L_2(\mathbb{R})$ ,  $\alpha \in \mathbb{R}$ , then

$$\begin{aligned} F[x + y] &= F[x] + F[y], \\ F[\alpha x] &= \alpha F[x]. \end{aligned} \quad (8)$$

The basic idea of the F-transform is to “capture” a local behavior of an original function and characterize it by a certain value. It follows from (6) that the F-transform can be effectively computed for a rather wide class of functions. In particular, all continuous functions on compact domains can be originals of the F-transform.

Let  $\mathbf{x} = (X_k, k \in \mathbb{Z})$  be an arbitrary sequence of reals and  $\{a_k, k \in \mathbb{Z}\}$  be an  $(h, H)$ -uniform fuzzy partition of  $\mathbb{R}$  with the  $H$ -rescaled generating function  $a$ . The following *inversion formula*

$$\hat{\mathbf{x}}^F(t) = \frac{\sum_{k=-\infty}^{\infty} X_k \cdot a_k(t)}{\sum_{k=-\infty}^{\infty} a_k(t)}, \quad t \in \mathbb{R}, \quad (9)$$

converts the sequence  $\mathbf{x}$  into the real valued function  $\hat{\mathbf{x}}^F$ . Because the parameter  $h$  in an  $(h, H)$ -uniform fuzzy partition  $\{a_k, k \in \mathbb{Z}\}$  of  $\mathbb{R}$  is greater than zero, both sums in (9) contain only a finite number of non-zero summands. Because  $H > \frac{h}{2}$ , each point from  $\mathbb{R}$  is covered by at least one basic function, so that the denominator in (9) is always non-zero. Therefore, the expression in (9) is well defined.

We say that the function  $\hat{\mathbf{x}}^F$  is the *inverse F-transform of the sequence*  $\mathbf{x} = (X_k, k \in \mathbb{Z})$  with respect to the fuzzy partition  $\{a_k, k \in \mathbb{Z}\}$ . If the sequence  $\mathbf{x}$  consists of the F-transform components of some function  $x$  with respect to  $\{a_k, k \in \mathbb{Z}\}$ , then  $\hat{\mathbf{x}}^F$  is simply called the *inverse F-transform* of  $x$ .

The inverse F-transform  $\hat{\mathbf{x}}^F$  of a continuous function  $x$  can approximate  $x$  with an arbitrary precision. The desired quality of approximation can be achieved by a special choice of a partition. This fact can be easily proved using the technique introduced in [8].

## 4 Reconstruction from the F-transform Components

The F-transform is the result of a linear correspondence between a set of functions from  $L_2(\mathbb{R})$  and a set of sequences of reals. In general, the inversion formula does not define the inverse correspondence. In [8], it has been shown that the inverse F-transform can approximate a continuous function with an arbitrary

precision. In the later publications [1, 7], other smooth approximations for functions from  $L_2(\mathbb{R})$  by the inverse F-transforms were proposed.

Below, we show even more; namely, the original function can be reconstructed from its F-transform components. Of course, this result can be established for a narrower than  $L_2(\mathbb{R})$  class of functions. Our motivation stems from the Nyquist-Shannon-Kotelnikov reconstruction theorem discussed above.

#### 4.1 Adjoint partition

If a fuzzy partition is fixed, then both direct and inverse F-transforms are uniquely determined by this partition. If we require the inverse F-transform to be coincident with the original function, we shall change its main parameter – the fuzzy partition.

**Definition 2.** Let  $\{a_k, k \in \mathbb{Z}\}$  be an  $(h, H)$ -uniform fuzzy partition of  $\mathbb{R}$ , where  $a_k(t) = a_H(t_k - t)$ ,  $a_H$  is the  $H$ -rescaled generating function  $a$  and  $t_k = k \cdot h$ ,  $k \in \mathbb{Z}$ , are uniformly distributed nodes. We say that the set of functions  $\{b_k, k \in \mathbb{Z}\}$ , establishes an adjoint  $(h, H)$ -uniform partition of  $\mathbb{R}$  (with respect to  $\{a_k, k \in \mathbb{Z}\}$ ), if  $b_k(t) = b_H(t - t_k)$  are translations of the continuous function  $b_H : \mathbb{R} \rightarrow \mathbb{R}$  with the same nodes  $t_k, k \in \mathbb{Z}$ , and  $b_H$  is determined by

$$\widehat{a_H} \cdot \widehat{b_H} = \mathbf{1}_{[-\Omega, \Omega]}, \quad (10)$$

where  $\Omega > 0$  is some positive constant,  $\mathbf{1}_{[-\Omega, \Omega]}$  is a characteristic function of  $[-\Omega, \Omega]$  and  $\widehat{a_H}, \widehat{b_H}$  are the Fourier transforms of  $a_H$  and  $b_H$ , respectively.

The lemma given below gives a necessary and sufficient condition on an  $(h, 1)$ -uniform fuzzy partition that guarantees the existence of the adjoint one.

**Lemma 2.** Let  $\{a_k, k \in \mathbb{Z}\}$ , be an  $(h, 1)$ -uniform fuzzy partition of  $\mathbb{R}$  with generating function  $a : \mathbb{R} \rightarrow [0, 1]$ , such that  $a_k(t) = a(t - t_k)$  and  $t_k = k \cdot h$ ,  $k \in \mathbb{Z}$ , are nodes. Then, the adjoint partition  $\{b_k, k \in \mathbb{Z}\}$  exists if and only if there exists  $\Omega > 0$  such that for all  $\omega \in [-\Omega, \Omega]$ ,

$$\widehat{a}(\omega) \neq 0. \quad (11)$$

Moreover, the adjoint partition  $\{b_k, k \in \mathbb{Z}\}$  is determined by  $h$ -translations of function  $b : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$b(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{e^{i\omega t}}{\widehat{a}(\omega)} d\omega. \quad (12)$$

*Remark 1.* Let  $\{a_k, k \in \mathbb{Z}\}$  be an  $(h, H)$ -uniform fuzzy partition of  $\mathbb{R}$ , where  $a_k(t) = a_H(t_k - t)$  and  $a_H$  is the  $H$ -rescaled generating function  $a$ . Let  $\{b_k, k \in \mathbb{Z}\}$ , where  $b_k(t) = b_H(t - t_k)$  be the adjoint  $(h, H)$ -uniform partition of  $\mathbb{R}$  with respect to  $\{a_k, k \in \mathbb{Z}\}$ .

In Remark 1, we discuss some particular properties of functions  $b_k, k \in \mathbb{Z}$ .

- (i) The function  $b_H$  is a rescaled version of a certain function  $b : \mathbb{R} \rightarrow \mathbb{R}$  in both vertical and horizontal directions. Specifically,

$$b_H(t) = \frac{1}{H^2} \cdot b\left(\frac{t}{H}\right), \quad (13)$$

where  $b$  is determined as follows:

$$\widehat{a} \cdot \widehat{b} = \mathbf{1}_{[-H\Omega, H\Omega]}. \quad (14)$$

Indeed, equality (13) easily follows from (14) and the scaling property of the Fourier transform applied to the function  $a$ :

$$\widehat{a_H}(\omega) = H\widehat{a}(H\omega).$$

- (ii) The explicit representation of a particular function  $b_k$ ,  $k \in \mathbb{Z}$  as a translation and rescaling of the function  $b$  is as follows:

$$b_k(t) = b_H(t - t_k) = \frac{1}{H^2} \cdot b\left(\frac{t - t_k}{H}\right). \quad (15)$$

This representation justifies the name “partition”, assigned to the set  $\{b_k, k \in \mathbb{Z}\}$ . Moreover, as we see in Lemma 3 below, the generating function  $b$  fulfills the extended Ruspini condition (16).

We call  $b$  a *generating function of the adjoint  $(h, H)$ -uniform partition*  $\{b_k, k \in \mathbb{Z}\}$ ,<sup>\*\*\*</sup> which corresponds to the  $(h, H)$ -uniform fuzzy partition  $\{a_k, k \in \mathbb{Z}\}$ , determined by  $a$ . If  $h = H$ , we simply call both partitions as  $h$ -uniform.

As the following result shows, the set of translations (without rescaling) of a generating function of an adjoint  $H$ -uniform partition establishes the Ruspini partition. This is an additional argument in favor of using the word “partition” in the notion of adjoint partition.

**Lemma 3.** *Let  $a : \mathbb{R} \rightarrow [0, 1]$  be a generating function such that for all  $\omega \in [-\Omega, \Omega]$ ,  $\widehat{a}(\omega) \neq 0$ , where  $\Omega > 0$  is some positive constant. Let  $H = \frac{\pi}{\Omega}$  and  $\{a_k, k \in \mathbb{Z}\}$ , be an  $H$ -uniform fuzzy partition such that  $a_k(t) = a_H(t - t_k)$ ,  $a_H$  is the  $H$ -rescaled generating function  $a$  and  $t_k = k \cdot H$ ,  $k \in \mathbb{Z}$ . Let  $\{b_k, k \in \mathbb{Z}\}$ , where  $b_k(t) = b_H(t - t_k)$ , be the adjoint  $H$ -uniform partition of  $\mathbb{R}$  with respect*

<sup>\*\*\*</sup> We distinguish between a generating function of an adjoint partition (in this paper, denoted by  $b$ ) and a generating function of a fuzzy partition (in this paper, denoted by  $a$ ). The latter is characterized in Definition 1, while the former is associated with an adjoint partition and can have values outside the interval  $[0, 1]$ .



to  $\{a_k, k \in \mathbb{Z}\}$  with the generating function  $b$ . Then, for all  $t \in \mathbb{R}$ ,

$$\sum_{k=-\infty}^{\infty} b\left(\frac{t}{H} - k\right) = 1, \quad (16)$$

$$\sum_{k=-\infty}^{\infty} b_k(t) = \frac{1}{H^2}, \quad (17)$$

$$\sum_{k=-\infty}^{\infty} b^2\left(\frac{t}{H} - k\right) = \|b\|^2 < \infty, \quad (18)$$

where  $\|\cdot\|$  is the norm in  $L_2(\mathbb{R})$ .

At the end of this subsection, we give a particular example of an  $h$ -uniform partition of  $\mathbb{R}$  and its adjoint where the latter has an analytic representation.

*Example 1.* We consider an  $h$ -uniform partition  $\{\delta_k, k \in \mathbb{Z}\}$  of  $\mathbb{R}$ , where  $\delta_k(t) = \delta(t - t_k)$ ,  $t_k = k \cdot h$  and  $\delta$  is the Dirac's delta function<sup>†</sup>). Although this partition is not fuzzy (it is generated by the non-bounded delta function), it fulfills all the assumptions of Lemma 2, including the main condition (11). The latter is because for all  $\omega \in \mathbb{R}$ ,  $\widehat{\delta}(\omega) = 1$ , so that we can choose an arbitrary bounded interval  $[-\Omega, \Omega]$  where this condition is fulfilled. We choose  $\Omega = \pi$  and apply the proof of Lemma 2 to the partition  $\{\delta_k, k \in \mathbb{Z}\}$ . After substitution into (12), we easily obtain the generating function sinc of the adjoint to  $\{\delta_k, k \in \mathbb{Z}\}$  partition, so that

$$b(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega t} d\omega = \frac{1}{\pi t} \sin(\pi t) = \text{sinc}(t). \quad (19)$$

The resulting adjoint  $h$ -uniform partition is given by the set of functions  $\{\text{sinc}_k, k \in \mathbb{Z}\}$ , where  $\text{sinc}_k(t) = \text{sinc}(t - t_k)$ , so that sinc is its generating function.

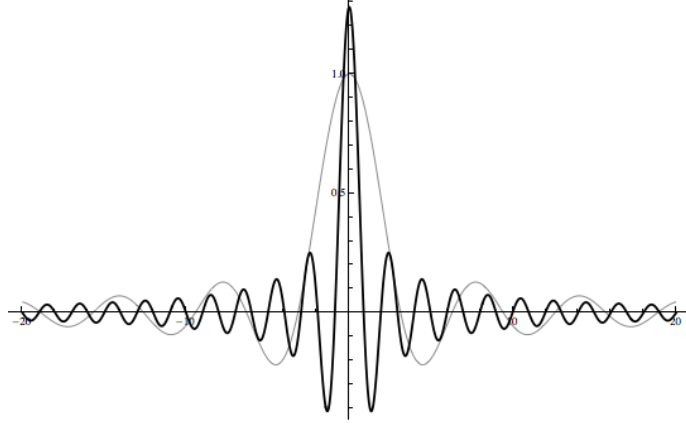
In Figure 1, we demonstrate graphs of generating functions of the two adjoint uniform partitions of  $\mathbb{R}$  with respect to two uniform partitions with the following generating functions:  $\delta$  (Dirac's delta) and  $a^{\cos}$  (raised cosine). The latter is given by (3), and it is of the fuzzy type.

In almost all cases, a computation of a generating function  $b$  of an adjoint partition cannot be performed analytically. It is a matter of a numeric computation on the basis of the expression (12). The example given Figure 1, has been numerically computed as well.

## 4.2 Main result

In this subsection, we show that a function that fulfills the same conditions as in the Nyquist-Shannon-Kotelnikov theorem (also known as a sampling theo-

<sup>†</sup>) Strictly speaking, the Dirac's delta is not a function, but a generalized function or a linear functional. Therefore, it makes sense to use it only if it appears inside an integral. In our paper, we always follow this restriction.



**Fig. 1.** Generating functions of the two adjoint uniform partitions of  $\mathbb{R}$  with respect to uniform partitions with generating functions  $\delta$  (in gray) and the raised cosine  $a^{\cos}$  (in black).

rem) can be reconstructed from a countable set of its F-transform components. Moreover, we obtain the sampling theorem as a particular case.

**Theorem 2 (Reconstruction from the F-transform).**

Let function  $x \in L_2(\mathbb{R})$  be continuous and band-limited, i.e.,  $\hat{x}(\omega) = 0$  for  $|\omega| > \Omega$ , where  $\Omega$  is some positive constant. Let  $h = \frac{\pi}{\Omega}$ ,  $H > h/2$  and  $a_H$  an  $H$ -rescaled version of the generating function  $a$ , such that for all  $\omega \in [-\Omega, \Omega]$ ,  $\hat{a}_H(\omega) \neq 0$ .

Let  $\{b_k, k \in \mathbb{Z}\}$  be the adjoint  $(h, H)$ -uniform partition of  $\mathbb{R}$  with respect to that given by  $\{a_k, k \in \mathbb{Z}\}$ , where  $a_k(s) = a_H(t_k - s)$  and  $t_k = k \cdot h$ ,  $k \in \mathbb{Z}$ .

Finally, let the sequence  $\{X_k, k \in \mathbb{Z}\}$  consist of the F-transform components of  $x$  with respect to the fuzzy partition  $\{a_k, k \in \mathbb{Z}\}$ .

Then,  $x$  can be uniquely determined by its F-transform components, so that the following representation holds:

$$x(t) = \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_k(t). \quad (20)$$

*Proof.* Let the assumptions be fulfilled, and consider the real function  $X : \mathbb{R} \rightarrow \mathbb{R}$ , represented by the expression similar to (7)

$$X(t) = \frac{\int_{-\infty}^{\infty} a_H(t-s) \cdot x(s) ds}{H}, \quad t \in \mathbb{R}.$$

At the fixed nodes  $t_k = k \cdot h$ ,  $k \in \mathbb{Z}$ , the values of  $X$  coincide with the corresponding F-transform components of  $x$ , i.e.,  $X(t_k) = X_k$ ,  $k \in \mathbb{Z}$ . We observe

that the function  $X$  can also be represented by

$$X = \frac{1}{H}(a_H * x),$$

where  $a_H * x$  is a convolution of  $a_H$  and  $x$ . We observe that  $(a_H * x) \in L_2(\mathbb{R})$  and thus,  $X \in L_2(\mathbb{R})$ . Moreover,  $X$  is continuous. Therefore, by the properties of the Fourier transform,  $X$  can be represented by the inversion formula

$$X(t) = \text{l.i.m.}_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-n}^n \widehat{X}(\omega) e^{i\omega t} d\omega, \quad (\dagger) \quad (21)$$

where (by convolution-to-product theorems)

$$\widehat{X}(\omega) = \frac{\widehat{x}(\omega) \cdot \widehat{a_H}(\omega)}{H}. \quad (22)$$

It follows that  $\widehat{X}$  is band-limited and  $\widehat{X}(\omega) = 0$  for  $|\omega| > \Omega$ . Therefore, by (21) and continuity of  $X$ , we have the exact representation

$$X(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \widehat{X}(\omega) e^{i\omega t} d\omega. \quad (23)$$

Because  $\widehat{X} \in L_2[-\Omega, \Omega]$ , it can be expanded in a Fourier series

$$\widehat{X}(\omega) = \sum_{k=-\infty}^{\infty} g_k e^{-ik\pi\omega/\Omega} \quad (24)$$

where

$$g_k = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \widehat{X}(\omega) e^{ik\pi\omega/\Omega} d\omega.$$

By (23),

$$g_k = \frac{\pi}{\Omega} X\left(\frac{k\pi}{\Omega}\right) = \frac{\pi}{\Omega} X(t_k) = \frac{\pi}{\Omega} X_k,$$

where  $X_k$  is the F-transform component of  $X$  with respect to  $\{a_k, k \in \mathbb{Z}\}$ . Substituting  $g_k$  into (24), we get

$$\widehat{X}(\omega) = \frac{\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k e^{-it_k\omega}. \quad (25)$$

Because the function  $x$  fulfills the assumptions of Theorem 1, we can express it with the help of the Fourier inversion formula

$$x(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \widehat{x}(\omega) e^{i\omega t} d\omega,$$

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<sup>‡</sup>) Here “l.i.m.” indicates that the convergence is in  $L_2(\mathbb{R})$  or in the quadratic mean sense.

where by (22),

$$\widehat{x}(\omega) = \frac{H \cdot \widehat{X}(\omega)}{\widehat{a_H}(\omega)}.$$

Hence, we have

$$x(t) = \frac{H}{2\pi} \int_{-\Omega}^{\Omega} \frac{\widehat{X}(\omega)}{\widehat{a_H}(\omega)} e^{i\omega t} d\omega,$$

and after substituting  $\widehat{X}(\omega)$  from (25)

$$x(t) = \frac{H}{2\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot \int_{-\Omega}^{\Omega} \frac{1}{\widehat{a_H}(\omega)} e^{i\omega(t-t_k)} d\omega. \quad (26)$$

Because  $\widehat{a_H}$  is continuous (as the inverse Fourier transform of a function from  $L_1(\mathbb{R})$ ) and  $\widehat{a_H}(\omega) \neq 0$  in  $[-\Omega, \Omega]$ , the integral in the right-hand side of (26) exists for all  $k \in \mathbb{Z}$ . Therefore, equality (26) proves that  $x$  can be determined by the set of F-transform components.

To prove (20), we observe that by Lemma 2, the  $(h, H)$ -uniform fuzzy partition  $\{a_k, k \in \mathbb{Z}\}$ , where  $a_k(s) = a_H(t_k - s)$ , has the adjoint  $(h, H)$ -uniform partition  $\{b_k, k \in \mathbb{Z}\}$  such that  $b_k(t) = b_H(t - t_k)$  and

$$b_H(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \frac{e^{i\omega t}}{\widehat{a_H}(\omega)} d\omega.$$

Therefore, the right-hand side of (26) can be easily rewritten into (20), i.e.,

$$x(t) = \frac{H\pi}{\Omega} \sum_{k=-\infty}^{\infty} X_k \cdot b_k(t).$$

□

Below, we give another expression for reconstruction formula (20) in terms of generating function  $b$  of partition  $\{b_k, k \in \mathbb{Z}\}$ .

**Corollary 1.** *Let function  $x$  fulfill the assumptions of Theorem 2. Then,  $x$  can be reconstructed from its F-transform components so that*

$$x(t) = \frac{h}{H} \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{H}\right), \quad (27)$$

where  $b \in L_2(\mathbb{R})$  is a generating function of the adjoint  $(h, H)$ -uniform partition  $\{b_k, k \in \mathbb{Z}\}$ .

*Remark 2.* If in (27), we assume that  $H = h$  (in other words,  $\{a_k, k \in \mathbb{Z}\}$  is an  $h$ -uniform fuzzy partition of  $\mathbb{R}$ ), then the reconstruction from the F-transform components takes the form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t - t_k}{h}\right) = \sum_{k=-\infty}^{\infty} X_k \cdot b\left(\frac{t}{h} - k\right), \quad (28)$$

where  $b \in L_2(\mathbb{R})$  is the function whose Fourier transform is equal to

$$\widehat{b}(\omega) = \frac{\mathbf{1}_{[-\pi, \pi]}}{\widehat{a}(\omega)}. \quad (29)$$

Reconstruction (28) is similar to the Nyquist-Shannon-Kotelnikov formula (2).

In the next corollary, we extend the range of applicability of Theorem 2 to the  $h$ -uniform partition  $\{\delta_k, k \in \mathbb{Z}\}$  introduced in the Example 1. By this, we obtain the Nyquist-Shannon-Kotelnikov reconstruction in the form of (2).

**Corollary 2.** *Let the assumptions of Theorem 2 be fulfilled and the Dirac's delta  $\delta$  and sinc be chosen as generating function of an  $h$ -uniform partition  $\{\delta_k, k \in \mathbb{Z}\}$  and the corresponding adjoint  $h$ -uniform partition  $\{\text{sinc}_k, k \in \mathbb{Z}\}$ . Then, after respective substitutions the reconstruction formula (28) becomes equivalent with the Nyquist-Shannon-Kotelnikov reconstruction in the form of (2).*

*Proof.* In the Example 1, we characterize the adjoint  $h$ -uniform partition of  $\mathbb{R}$  with respect to the  $h$ -uniform partition  $\{\delta_k, k \in \mathbb{Z}\}$ . According to (19), this adjoint partition is given by the set of translations  $\{\text{sinc}_k, k \in \mathbb{Z}\}$  of the generating function sinc. Let us substitute sinc for  $b$  in (28) and obtain

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot \text{sinc}\left(\frac{t - t_k}{h}\right). \quad (30)$$

By (6),

$$X_k = \frac{\int_{-\infty}^{\infty} \delta_k(s) \cdot x(s) ds}{\int_{-\infty}^{\infty} \delta_k(s) ds} = x(t_k), \quad k \in \mathbb{Z},$$

so that we can substitute  $x(t_k)$  for  $X_k$  in (30) and see that the latter becomes equivalent with (2).  $\square$

*Remark 3.* The principal difference between the Nyquist-Shannon-Kotelnikov and the proposed reconstruction is that the former one works as an interpolating technique, while the latter one is able to perform reconstruction even from averaged values of a given function.

## 5 Conclusion

We discussed the problem of reconstruction from a set of F-transform components. We introduced the adjoint fuzzy partition and the inversion formula and proved that a function can be reconstructed from its F-transform components. Moreover, we showed that if the Dirac's delta  $\delta$  is chosen as generating function of an  $h$ -uniform partition, then the reconstruction from the F-transform components becomes equivalent with the Nyquist-Shannon-Kotelnikov reconstruction.

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